Exam.

- Duration 2h. -

- Exercise 1 -
Solve the following linear program by the two-phase simplex algorithm (and only by this method).

$$\begin{array}{lrrl} \text{Maximize} & 3x_1 & +x_2 & \\ \text{Subject to} & x_1 & -x_2 & \leq -1 \\ & -x_1 & -x_2 & \leq -3 \\ & 2x_1 & +x_2 & \leq 4 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Write the dual of (P) and solve it using your previous simplex.

- Exercise 2 -
We want to approximate a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the plane by a line. The least square method consists of finding a, b in order to minimize $(y_1 - ax_1 - b)^2 + (y_2 - ax_2 - b)^2 + \cdots + (y_n - ax_n - b)^2$ (but exceptional points are overcounted due to the square). Propose an efficient way of minimizing $|y_1 - ax_1 - b| + |y_2 - ax_2 - b| + \cdots + |y_n - ax_n - b|$.

- Exercise 3 - Branch and bound. To compute an optimal integer solution of some maximization (standard) linear program (P), one can apply the following branching algorithm:

First of all, solve (P) fractionally, and get an optimal solution $x = (x_1, \ldots, x_n)$. Then, unless x is already integer, consider a non-integer x_i, chosen as non integer as possible (i.e. minimize $|1/2 - x_i + \lfloor x_i \rfloor)|$). Now create two new linear programs: (P_1) obtained from (P) by adding the constraint $x_i \leq \lfloor x_i \rfloor$ and (P_2) by adding the constraint $x_i \geq \lceil x_i \rceil$...

a. Finish the description of the algorithm and show that it provides an optimal integer solution. Provide a decent heuristic.

b. Apply your algorithm to the linear program (here using simplex is not mandatory):

$$\begin{array}{lrrl} \text{Maximize} & 2x_1 & +3x_2 & \\ \text{Subject to} & x_1 & +2x_2 & \leq 3 \\ & 6x_1 & +8x_2 & \leq 15 \\ & x_1, & x_2 & \geq 0 \end{array}$$

- Exercise 4 - TU matrices. Let A be a totally unimodular matrix. Show that the set of columns of A can be partitioned into *blue* columns and *red* columns in such a way that on each row, the sum of blue values and the sum of red values differ at most by one. Hint: propose a fractional relaxation of the problem, and use TU property.

- Exercise 5 - In the MAX-(≤ 2)-SAT problem we are given clauses $C_1, \ldots, C_m$ of size 1 or 2 with respective positive weights $w_1, \ldots, w_m$. The goal is to assign the boolean variables $x_1, \ldots, x_n$ in order to satisfy a subset of these clauses with maximum total weight.

a. Propose a fractional relaxation of this problem.

b. Show that randomized rounding applied to this relaxation gives a 3/4-approximation of MAX-2-SAT.