Optimization and approximation
Final exam 03/01/2023

Exercise 1 [4pt] We consider the following linear program (P).

$$\begin{array}{rlll} \text{Minimize} & 5x_1 & +3x_2 & \\ \text{Subject to} & x_1 & +x_2 & \geq 5 \\ & 3x_1 & +x_2 & \geq 7 \\ & x_1 & +2x_2 & \geq 6 \\ & x_1, & x_2 & \geq 0 \end{array}$$

1. Write (P) in standard form.
2. Solve (P) by the two-phase simplex algorithm (no other method admitted). Carefully choose pivots to avoid long computations.
3. Write the dual (D) of (P).
4. Certify the optimality of your solution using (D).

Exercise 2 [3pt] We are given a family of closed intervals $I_1, \ldots, I_n$ of $[0,1]$ which union is equal to $[0,1]$. These intervals have respective (non negative) weights $w_1, \ldots, w_n$. The goal is to find a subfamily of intervals with minimum total weight which union covers $[0,1]$.

1. Model the fractional relaxation of this problem as a linear program (P).
2. How the optimal solution OPT^* of (P) compares with the optimal solution OPT of the original problem?

Exercise 3 [3pt] We are given a directed graph $D = (V, A)$ with one source s and one sink t. Each arc $a \in A$ has a length $\ell_a \geq 0$. The goal is to find a shortest directed path from s to t.

1. Model the fractional relaxation of this problem as a linear program (P).
2. How the optimal solution OPT^* of (P) compares with the optimal solution OPT of the original problem?

Exercise 4 [1pt] Consider the method $x_{k+1} = x_k + \alpha_k d_k$ for the unconstrained minimization of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. State which of the following statements are true and which are false. Justify your answers.

1. If $d_k = -\nabla f(x_k)$ and α_k is such that $f(x_{k+1}) < f(x_k)$ for all k, every limit point of the generated sequence $\{x_k\}$ is stationary.

2. If $d_k = -\nabla f(x_k)$, α_k is chosen satisfying the Armijo and Wolfe rules, and the function f has the form $f(x_1, x_2) = x_1^2 + x_2^2 + x_1$, the generated sequence $\{x_k\}$ converges to a global minimum of f.

Exercise 5 [2.5pt] Consider the 2-dimensional function $f(x, y) = (y - x^2)^2 - x^2$.

- Show that f has only one stationary point, which is neither a local maximum nor a local minimum.
- Consider the minimization of f subject to no constraint on x and the constraint $-1 \leq y \leq 1$ on y. Find all global minima.

Exercise 6 [6.5pt] Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Suppose that x^* is a local minimum such that, for $m > 0$:

$$m\|d\|^2 \leq d^T H(x)d, \ \forall d \in \mathbb{R}^n,$$

for all x in an open ball $B_r(x^*)$ ($H(x)$ is the hessian matrix of f in x). Show that for every $x \in B_r(x^*)$ we have:

1. $\|x - x^*\| \leq \frac{\|\nabla f(x)\|}{m}$
2. $F(t) := f(x^* + t(x - x^*))$ for any scalar t is such that F' is an increasing function.
3. $f(x) - f(x^*) \leq \frac{\|\nabla f(x)\|^2}{m}$

Hint: use the following relation to prove 1. :

$$\nabla f(y) = \nabla f(x) + \int_0^1 H(x + t(y - x))(y - x)\, dt.$$

Use 1. and 2. to prove 3.