Exam 7 January 2025

- Exercise 1 - Two phase simplex. We want to solve the following linear program (P) by the two-phase simplex (and only by this method).

$$\begin{array}{lrrrrcr}
\text{Maximize} & -x_1 & +3x_2 & +x_3 & +x_4 & & \\
\text{Subject to} & 2x_1 & +x_2 & & -x_4 & \leq & 4 \\
& -2x_1 & & +x_3 & +x_4 & \leq & -2 \\
& & 2x_2 & +2x_3 & & \leq & 3 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

1. Why two phases are needed here ?
2. Perform the first phase of simplex. On which vertex of the domain does it terminates ?
3. Perform the second phase of simplex. What can you tell about (P) ?
4. Write the dual (D) of (P). What can be said about (D) ?

- Exercise 2 - Interval cover (again). We are given a collection of m intervals $I_1 = [a_1, b_1], \ldots, I_m = [a_m, b_m]$ where $a_i \leq b_i$ are integers chosen in $[n] := \{1, \ldots, n\}$. Each $j \in [n]$ comes with an integer *demand* $d_j \geq 1$. Our problem is to find a subfamily of intervals $\mathcal{F} = \{I_i : i \in S \subseteq [m]\}$ such that every $j \in [n]$ is contained in at least d_j intervals of $\mathcal{F}$. We moreover want the size of $\mathcal{F}$ (that is $|S|$) to be as small as possible.

1. Write the LP-relaxation (P) of this problem and its dual (D).
2. Since the matrix of (P) is TU, both (P) and (D) have integer solutions. Deduce from (D) a certificate of optimality accessible to a non-optimizer person.

- Exercise 3 - In the MAX-2-SAT problem we are given clauses $C_1, \ldots, C_m$ of size 1 or 2 with respective positive weights $w_1, \ldots, w_m$. The goal is to set the boolean variables $x_1, \ldots, x_n$ in order to satisfy a subset of these clauses with maximum total weight.

a. Propose a fractional relaxation of this problem.

b. Show that randomized rounding applied to this relaxation gives a 3/4-approximation of MAX-2-SAT.

- Exercise 4 - We consider the following linear program P :

$$\begin{array}{lrrrrcr}
\text{Maximize} & 2x_1 & +4x_2 & +3x_3 & +x_4 & & \\
\text{Subject to} & 3x_1 & +x_2 & +x_3 & +4x_4 & \leq & 12 \\
& x_1 & -3x_2 & +2x_3 & +3x_4 & \leq & 7 \\
& 2x_1 & +x_2 & +3x_3 & -x_4 & \leq & 10 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

a. Show that $x_1 = 0$, $x_2 = \frac{52}{5}$, $x_3 = 0$, $x_4 = \frac{2}{5}$ is an optimal solution of P.

b. Assume now that the first constraint of P is replaced by $3x_1+x_2+x_3+4x_4 \leq 13$. Find an optimal solution of P. Certify the optimality of your solution.