Midterm Exam 4/5/2013.

- Duration 2h. No notes and electronic devices allowed. Exercises are ordered in somewhat increasing difficulty order. -

- Exercise 1 - Solve the following linear program with the two phase simplex algorithm. Certify the optimality of your solution.

$$\begin{array}{rrrl} \text{Maximize} & 3x_1 & +x_2 & \\ \text{Subject to} & x_1 & -x_2 & \leq -1 \\ & -x_1 & -x_2 & \leq -3 \\ & 2x_1 & +x_2 & \leq 4 \\ & x_1, & x_2 & \geq 0 \end{array}$$

- Exercise 2 - Show that $x_1 = \frac{1}{5}$, $x_2 = 0$, $x_3 = 0$, $x_4 = \frac{7}{5}$ is an optimal solution of the following linear program.

$$\begin{array}{rrrrrcl} \text{Maximize} & x_1 & +x_2 & +x_3 & +x_4 & & \\ \text{Subject to} & 3x_1 & +6x_2 & +3x_3 & +x_4 & \leq & 2 \\ & 2x_1 & +3x_2 & +3x_3 & +x_4 & \leq & 4 \\ & x_1 & +2x_2 & +5x_3 & +2x_4 & \leq & 3 \\ & 4x_1 & +3x_2 & +4x_3 & +x_4 & \leq & 5 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

- Exercise 3 - We are given a family of closed intervals $I_1, \ldots, I_n$ covering the $[0,1]$ segment with respective weights $w_1, \ldots, w_n$. The goal is to find a subfamily with minimum weight covering $[0,1]$.

a. Show that a 0/1 matrix $A = (a_{i,j})$ where the 1's of each line are consecutive (i.e. such that $j < k$ and $a_{i,j} = a_{i,k} = 1$ implies $a_{i,j+1} = 1$) is totally unimodular. Do not use the Network Matrix argument for this proof.

b. Show that a minimum weight cover can be obtained via the LP relaxation of the problem.

c. Can the same approach be used for a collection of weighted rectangles covering the unit square ? Show a proof or provide a counterexample.

- Exercise 4 - The goal is to prove that for every oriented graph $G = (V, E)$, there exists a weight function w from V into the non-negative reals such that $w(v^-) \geq w(v^+)$ for every $v \in V$, where v^- is the set of vertices u such that $uv \in E$ and v^+ is the set of vertices u such that $vu \in E$. Here $w(X)$ is the sum of the $w(x)$ where $x \in X$. We also want the extra condition that $w(V) = 1$. For convenience, we will assume that $v^+ \cap v^-$ is empty (which corresponds to forbidding cycles of length 2).

a. Express this problem as a linear program (P). Since this is an existence problem only, one can simply consider the constant objective function equal to 0, and discuss whether (P) is empty or not.

b. Write the dual of (P).

c. Deduce that such a function w always exists.

- Exercise 5 - Let $E = \{e_1, \ldots, e_n\}$ be a finite set and $\leq$ be a partial order on E. An *antichain* A of $(E, \leq)$ is a subset of pairwise incomparable elements of E, i.e. we do not have $a \leq a'$ when a, a' are distinct elements of A. To any antichain A, we associate a 0/1 vector $v_A = (a_1, \ldots, a_n)$ such that $a_i = 1$ if $e_i \in A$ and $a_i = 0$ otherwise. The *antichain polytope* of $E, \leq$ is the convex hull of all v_A's when A ranges over all antichains of $E, \leq$.

a. Give a set of linear inequalities that defines the facets of the antichain polytope of $E, \leq$.