

Midterm Exam April 13, 2014.

- Duration 2h. No notes and electronic devices allowed.
Exercises are ordered in somewhat increasing difficulty order. -

- **Exercise 1** - We are given the following linear program P :

$$\begin{array}{llllll} \text{Maximize} & -3x_1 & +2x_2 & -2x_3 & -x_4 & \\ \text{Subject to} & 4x_1 & -2x_2 & +x_3 & -x_4 & \leq -2 \\ & -x_1 & +x_2 & -x_3 & & \leq -10 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

- a. Solve P with the two phased simplex algorithm (no other method allowed).
- b. Certify the optimality of your solution.

- **Exercise 2** - We consider the following linear program P :

$$\begin{array}{llllll} \text{Maximize} & 2x_1 & +4x_2 & +3x_3 & +x_4 & \\ \text{Subject to} & 3x_1 & +x_2 & +x_3 & +4x_4 & \leq 12 \\ & x_1 & -3x_2 & +2x_3 & +3x_4 & \leq 7 \\ & 2x_1 & +x_2 & +3x_3 & -x_4 & \leq 10 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

- a. Show that $x_1 = 0$, $x_2 = \frac{52}{5}$, $x_3 = 0$, $x_4 = \frac{2}{5}$ is an optimal solution of P .
- b. Assume now that the first constraint of P is replaced by $3x_1 + x_2 + x_3 + 4x_4 \leq 13$. Find an optimal solution of P . Certify the optimality of your solution.

- **Exercise 3** - We are given a family of n axis-parallel rectangles R_i in the plane, where $i = 1, \dots, n$. Each rectangle R_i is characterized by a 4-uple of reals (a_i, b_i, c_i, d_i) , where $a_i \leq b_i$ and $c_i \leq d_i$, such that the four corners of R_i are respectively the points (a_i, c_i) ; (a_i, d_i) ; (b_i, c_i) ; (b_i, d_i) .

A *rectangle stabbing* is a set of vertical and horizontal lines such that each rectangle is traversed by at least one line. The goal is to find a rectangle stabbing with a minimum number of lines. Formally we want to minimize $j + k$ such that there exists real coordinates $x_1, \dots, x_j$ and $y_1, \dots, y_k$ for which :

For every rectangle R_i , there exists some x_ℓ such that $a_i \leq x_\ell \leq b_i$ or there exists some y_ℓ such that $c_i \leq y_\ell \leq d_i$.

- a. Show that we can restrict to a finite set of possible choices for the coordinates of the lines in the rectangle stabbing problem.
- b. Propose an LP-relaxation of the problem.
- c. We denote by OPT the optimal value of the rectangle stabbing problem, and by OPT^* the optimal value of your relaxation. Show an example where OPT^* is strictly less than OPT . Try to make OPT/OPT^* as large as possible.

- d. ** (if you have finished and get bored) What about the computational complexity of the rectangle stabbing problem?

- **Exercise 4** - We are given a directed graph $D = (V, A)$ with one source s and one sink t .

- Propose a linear program P which corresponds to the fractional relaxation of the shortest directed path problem from s to t .
- Write the dual D of P .
- Propose a sensible interpretation of the integer solutions of D (hence providing lower bounds for the shortest path problem).

- **Exercise 5** - Alice and Bob are playing a generalization of the Paper/Scissor/Stone game with n objects instead of three. Each game issue is either a 1 point victory for Alice or for Bob, or in the case in which they have chosen the same object, the result is a draw, valued 0.

Formally, they play on a $n \times n$ skew-symmetric matrix A (i.e. $a_{ij} = -a_{ji}$, or $A^T = -A$) which entries are 0 on the diagonal and +1 or -1 otherwise. Alice chooses a column j , Bob chooses a row i , and the payoff for Alice is a_{ij} .

In the case of the usual Paper/Scissor/Stone game, the optimal mixed strategy is $(1/3, 1/3, 1/3)$. This optimal strategy is unique, and the goal of this exercise is to show that every generalized Paper/Scissor/Stone game has a unique strategy.

In the context of symmetric games, we recall that x is an *optimal mixed strategy* if $A \cdot x \geq 0$, where x is a probability distribution, i.e. $x \geq 0$ and $\mathbf{1}^T \cdot x = 1$.

The questions in this exercise are not in increasing order of difficulty, feel free to admit the result of a question and go to the next one.

- Show that if the i^{th} coordinate of an optimal mixed strategy x is non zero, then the i^{th} coordinate of $A \cdot x$ is equal to 0.
- Show that if n is odd, the determinant of A is 0. *Hint : compute the determinant of A^T in two ways.*
- Show that if n is even, the determinant of A is odd. *Hint : Show that changing the sign of the entries of A does not change the parity of the determinant. Then change all the -1 of A into +1 to conclude.*
- Show that the set of solutions of $A \cdot x = 0$ is at most 1-dimensional.
- Assume now that x and y are optimal strategies for Alice. For every α with $0 < \alpha < 1$, we let $z_\alpha = \alpha x + (1 - \alpha)y$. Show that z_α is an optimal strategy.
- Let S be the set of coordinates in which z_α is non zero. Let A' be the sub-matrix of A restricted to the entries a_{ij} with $i, j \in S$. Similarly, let z'_α be the vector z_α restricted to its non zero coordinates. Show that $A' \cdot z'_\alpha = 0$.
- Conclude that Alice has a unique optimal strategy x , which moreover has an odd number of non zero coordinates.