

Midterm Exam April 3, 2015.

- Duration 2h. No notes and electronic devices allowed except a manuscript "cheat sheet".

- **Exercise 1** - We consider two power plants, each of them producing 800 megawatt. These plants are connected to three cities whose respective power demands are 700, 400 and 500 megawatt. Each plant can freely split its supply to any of the cities.

The respective transportation costs (per megawatt) in the network are :

	Plant 1	Plant 2
City 1	20	25
City 2	15	10
City 3	10	15

The problem is to minimize the total cost while providing power supply to the cities.

- Model by a linear program.
- Propose a (handmade) optimal solution of this problem.
- Show the optimality of your solution via a dual certificate. Carefully explain your method.

- **Exercise 2** -

We are given the following linear program (P) :

$$\begin{array}{llllll}
 \text{Maximize} & 3x_1 & -2x_2 & +2x_3 & -4x_4 & \\
 \text{Subject to} & 2x_1 & +x_2 & +2x_3 & -x_4 & \leq -2 \\
 & & -x_2 & -x_3 & & \leq -2 \\
 & & & & x_1, x_2, x_3, x_4 & \geq 0
 \end{array}$$

- Solve (P) with the two-phase simplex algorithm.
- Write the dual (D) of (P).
- Solve (D) geometrically. Compare your solution with the one you found from your simplex algorithm.

- **Exercise 3** - Is $x_1 = 2/13$, $x_2 = 0$, $x_3 = 8/13$, $x_4 = 0$ an optimal solution of the following problem? Justify.

$$\begin{array}{llllll}
 \text{Maximize} & x_1 & +x_2 & +x_3 & +x_4 & \\
 \text{Subject to} & 5x_1 & +6x_2 & +2x_3 & +3x_4 & \leq 2 \\
 & & x_1 & +3x_2 & +3x_3 & +5x_4 \leq 2 \\
 & & 2x_1 & +6x_2 & +4x_3 & +2x_4 \leq 3 \\
 & & 6x_1 & +5x_2 & +4x_3 & +x_4 \leq 6 \\
 & & & & x_1, x_2, x_3, x_4 & \geq 0
 \end{array}$$

- **Exercise 4 - Fraction** We want to maximize the following fraction :

$$\frac{3 + 2x_1 + 3x_2 + x_3}{1 + 3x_1 + x_2 + 4x_3}$$

subject to the constraints $5x_1 + x_2 + 6x_3 \leq 10$ and $x_1 + 2x_2 + x_3 \leq 2$ and non negative x_i . Model with a linear program.

- **Exercise 5 -** Consider an oriented graph $G = (V, E)$. We denote by v^- the set of vertices u such that $uv \in E$ and by v^+ the set of vertices u such that $vu \in E$. We assume that $v^+ \cap v^-$ is empty (which corresponds to forbidding cycles of length 2).

The goal is to prove that there exists a weight function w from V into the non-negative reals such that $w(v^-) \geq w(v^+)$ for every $v \in V$, where $w(X)$ is the sum of the $w(x)$ where $x \in X$, with the additional property that $w(V) = 1$.

- a. Express this problem as a linear program (P). Since this is an existence problem only, one can simply consider the constant objective function equal to 0, and discuss whether (P) is empty or not.
- b. Write the dual of (P).
- c. Deduce that such a function w always exists.
- d. Here is an open problem from Paul Seymour : For every vertex v , the first neighborhood v^+ of v is the set of out-neighbors of v (all w 's for which an arc vw exists). The second neighborhood v^{++} is the set of vertices at oriented distance 2 from v (all w 's not in v^+ for which there is a path vuw). The conjecture is the existence of a vertex v such that v^{++} has at least as many elements as v^+ . Show that this conjecture is equivalent to its weighted version : for every non-negative weight function w on the vertices, there is a vertex v such that $w(v^{++}) \geq w(v^+)$.
- e. Show that this conjecture is in turn equivalent to : for every oriented graph, there is a weight function w such that $w(v^{++}) \geq w(v^+)$ for all vertices v .