Manipuler les réseaux euclidiens

Damien Stehlé

ÉNS de Lyon – équipe ARIC

SIESTE, Lyon, Octobre 2012

L'équipe ARIC (Arithmetic and Computing)

- 5 enseignants-chercheurs
- 7 chercheurs
- 2 post-doctorants
- 8 doctorants

Objectif : Améliorer les calculs, en se focalisant sur l'arithmétique

- Arithmétique flottante
  (opérateurs, fonctions élémentaires, FPGA, validation)
- Calcul formel (algèbre linéaire flottante et exacte)
- **Réseaux euclidiens (algorithmes et cryptographie)**
- Théorie des nombres

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Goals and plan of the talk

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- An introduction to the computational aspects of lattices
- An example of how floating-point arithmetic can be used to accelerate an algebraic computation

Plan of the talk:

1. **Euclidean lattices**
2. Applications of euclidean lattices
3. The LLL algorithm and its accelerations

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1. **Euclidean lattices**
2. Applications of euclidean lattices
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Euclidean lattices

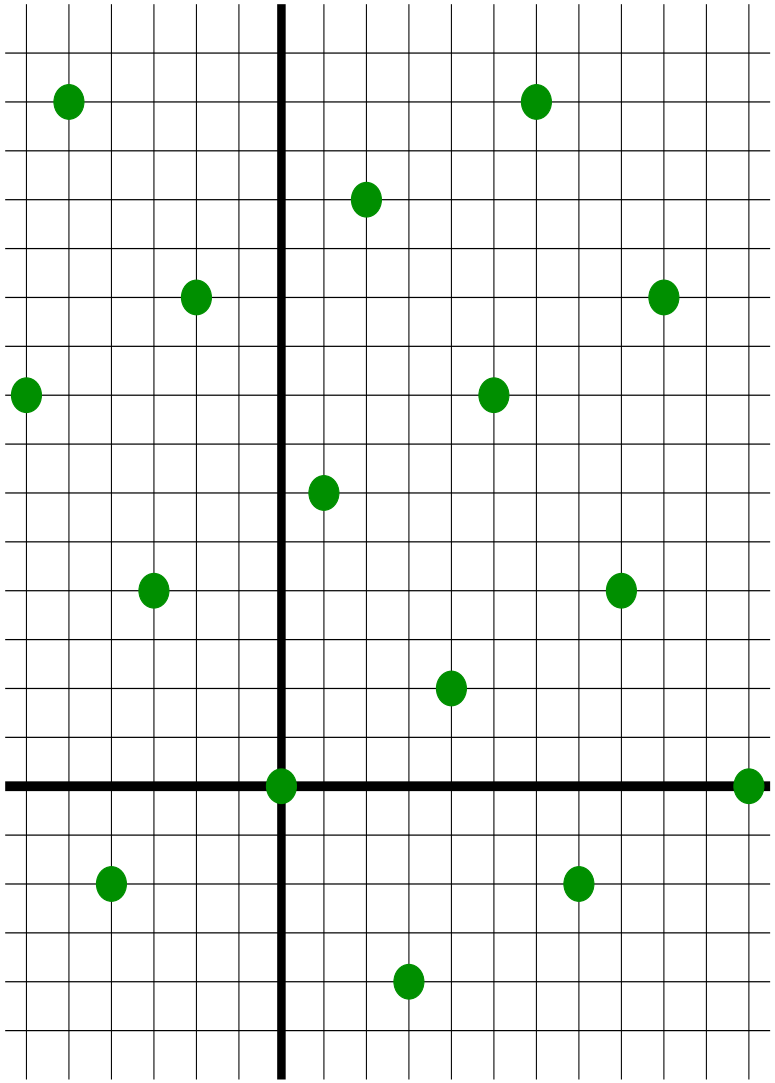
Lattice $\equiv$ discrete subgroup of $\mathbb{R}^n$

$\equiv \{\sum_{i \leq n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\}$

If the $\mathbf{b}_i$'s are linearly independent, they are called a **basis**.

Bases are not unique, but they can be obtained from each other by integer transforms of determinant ± 1:

$$\begin{bmatrix} -2 & 1 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$



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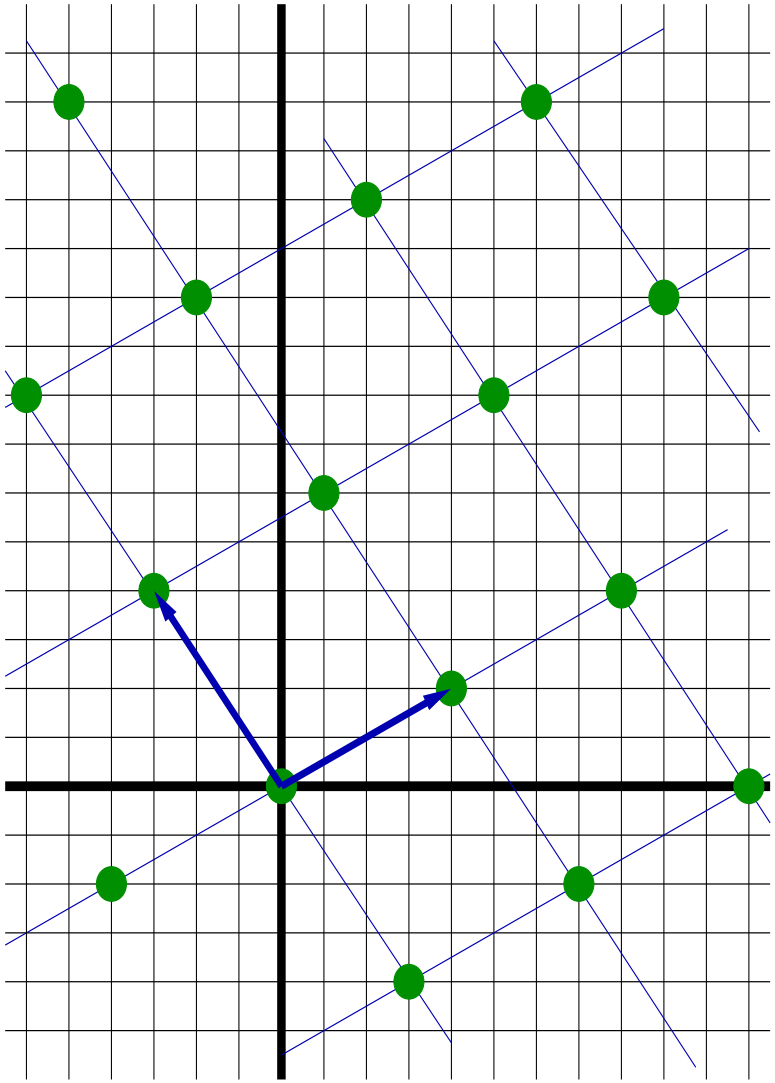

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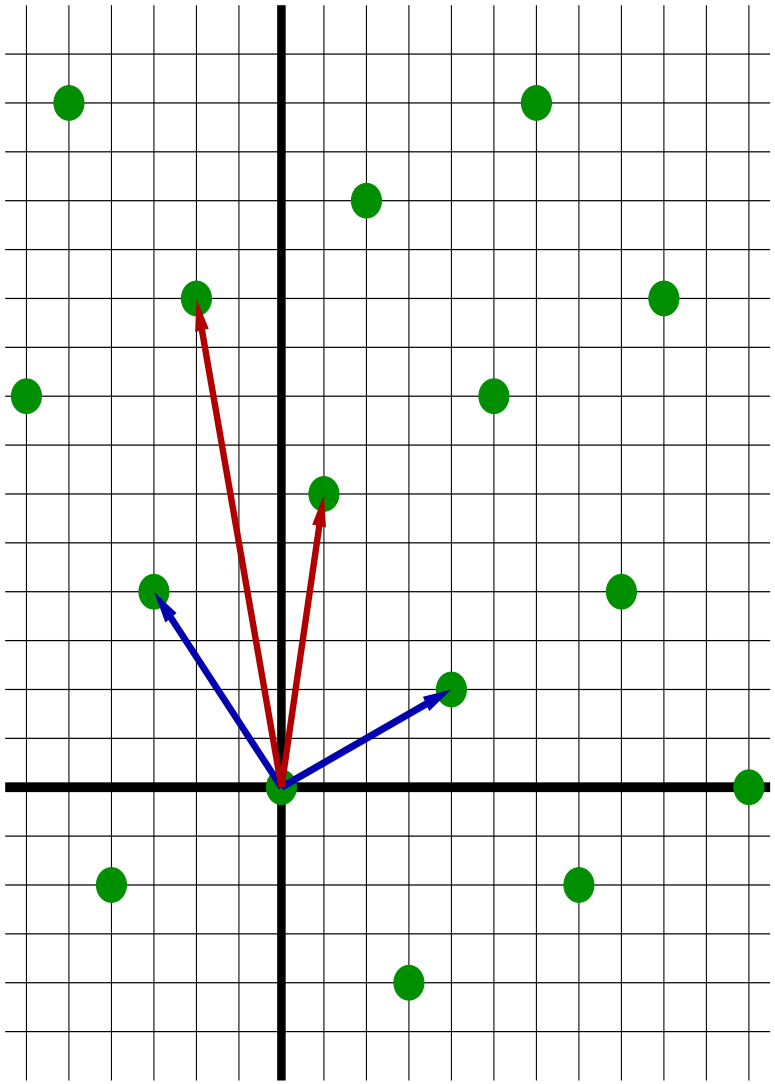


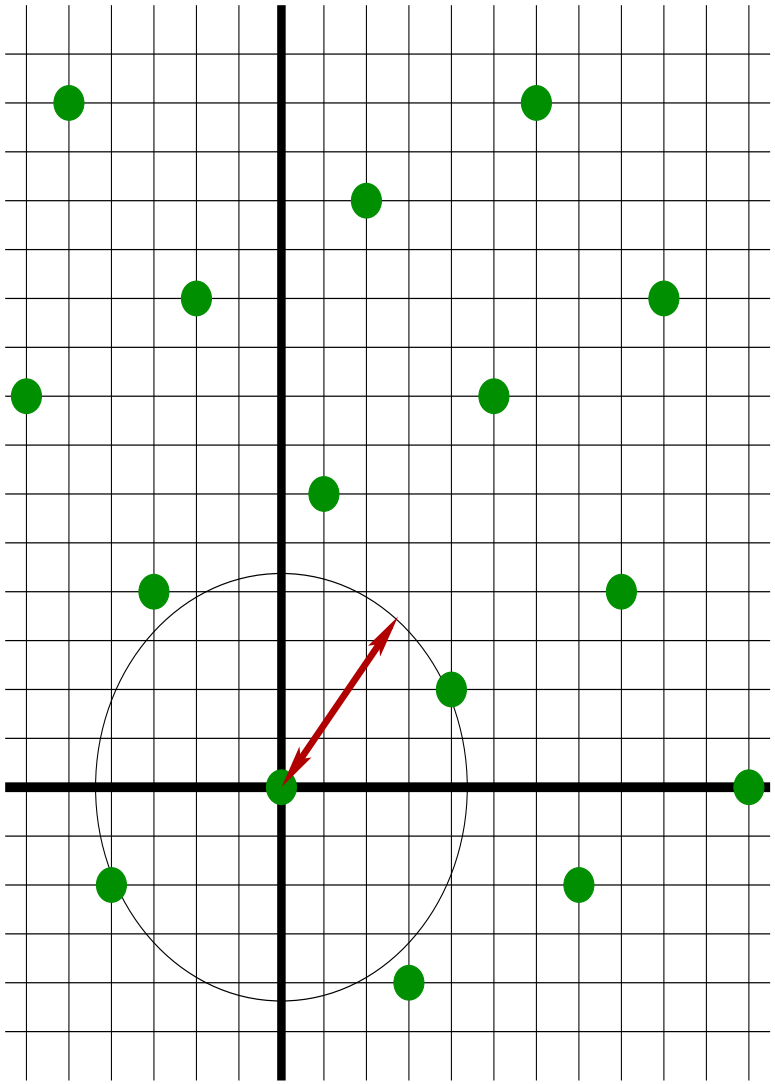
Lattice invariants

First minimum:
$\lambda(L) = \min(\|\mathbf{b}\| : \mathbf{b} \in L \setminus \mathbf{0})$.

Determinant:
$\det L = |\det(\mathbf{b}_i)_i|$, for any basis

Minkowski theorem:
$\lambda(L) \leq \sqrt{n} \cdot (\det L)^{1/n}$.

Algorithmic approach: lattice reduction
Start from a basis, and progressively improve its norm/orthogonality.



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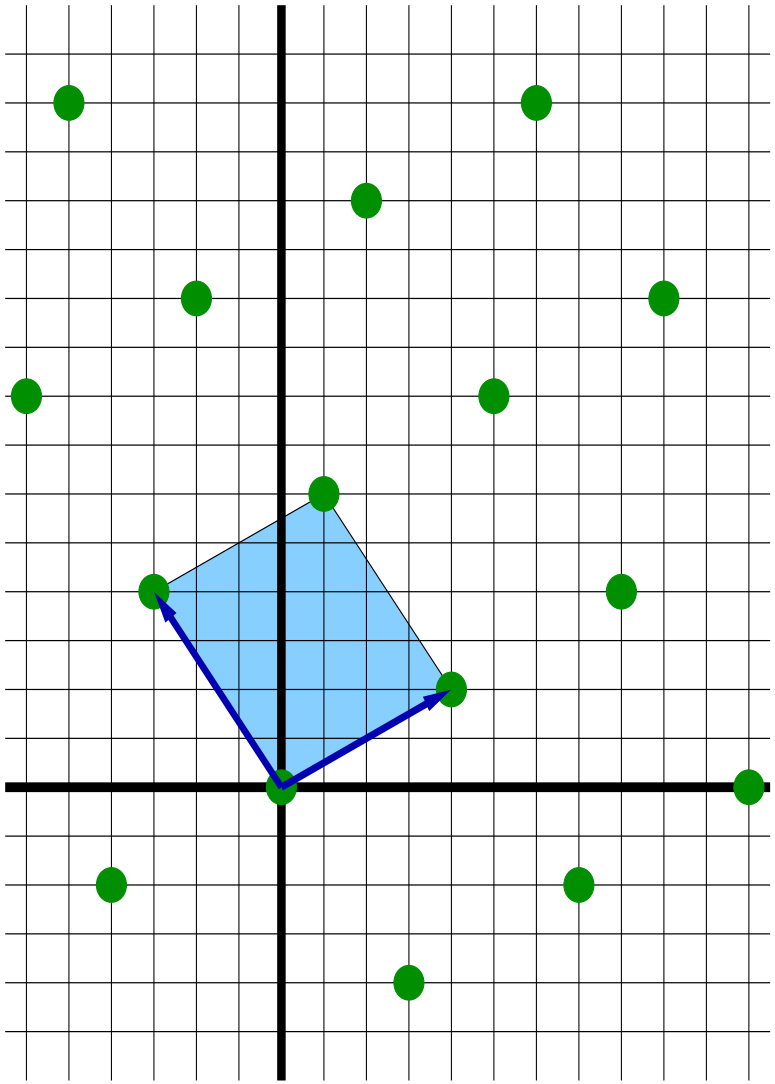

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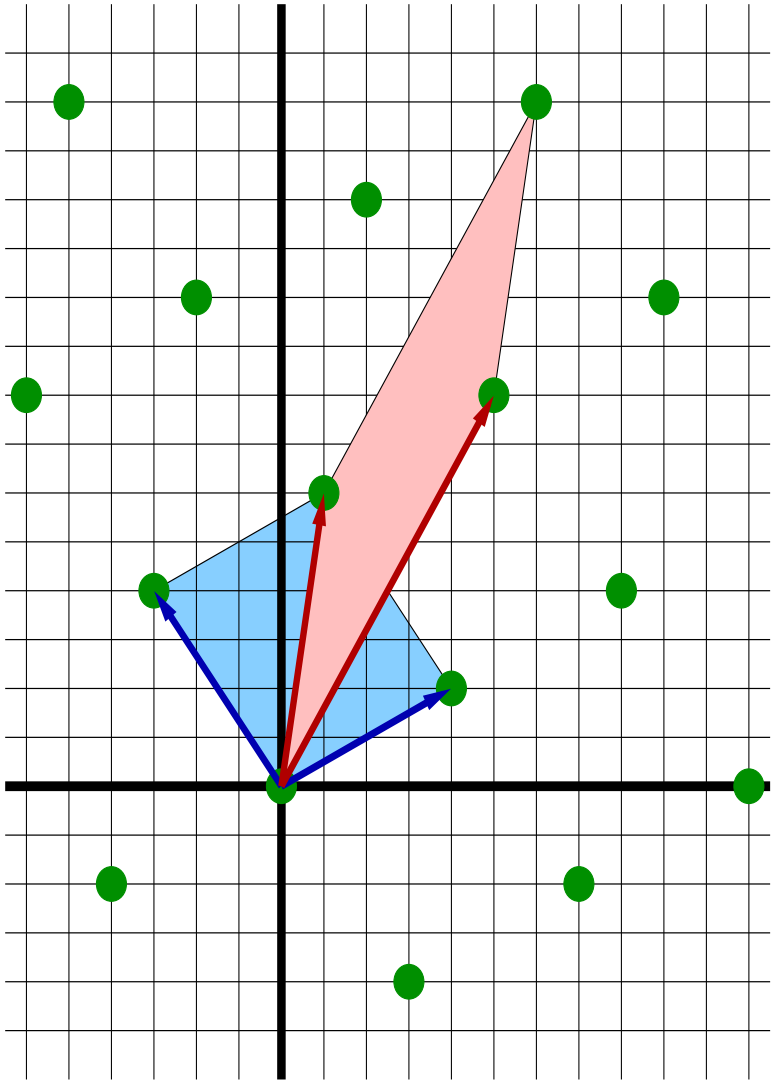

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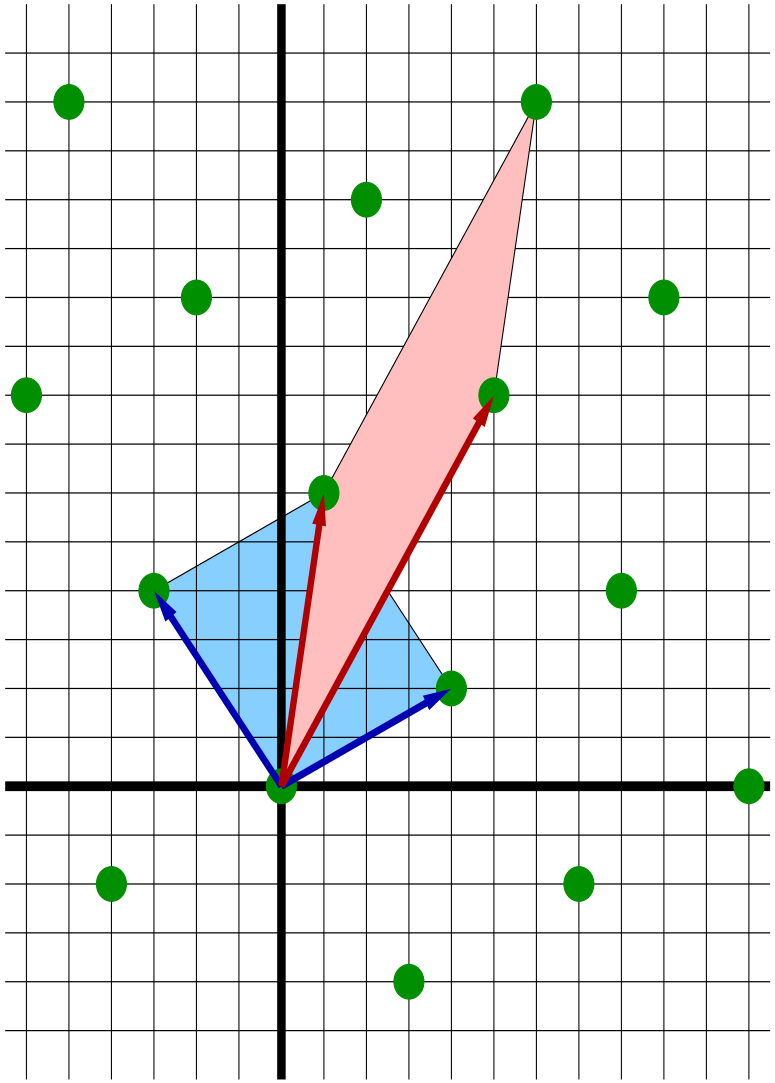

Why do we care about lattices?

- Computer algebra: factorisation of rational polynomials.
- Cryptanalysis of variants of RSA.
- Lattice-based cryptography.
- Communications theory: MIMO, GPS.
- Combinatorial optimisation, algorithmic group theory, algorithmic number theory, computer arithmetic, etc.

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Main computational problem: SVP

- SVP_γ: Given a basis of L, find $\mathbf{b} \in L$ with

$$0 < \|\mathbf{b}\| \leq \gamma \cdot \lambda(L).$$

- $\mathrm{Dec\text{-}SVP}_\gamma$: Given a basis of L, and $t > 0$, reply:

$$\text{YES if } \lambda(L) \leq t \text{ and NO if } \lambda(L) > \gamma \cdot t.$$

- NP-hard for any $\gamma \leq \mathcal{O}(1)$, under randomized reductions.
- Most likely not NP-hard for $\gamma \geq \sqrt{n}$.
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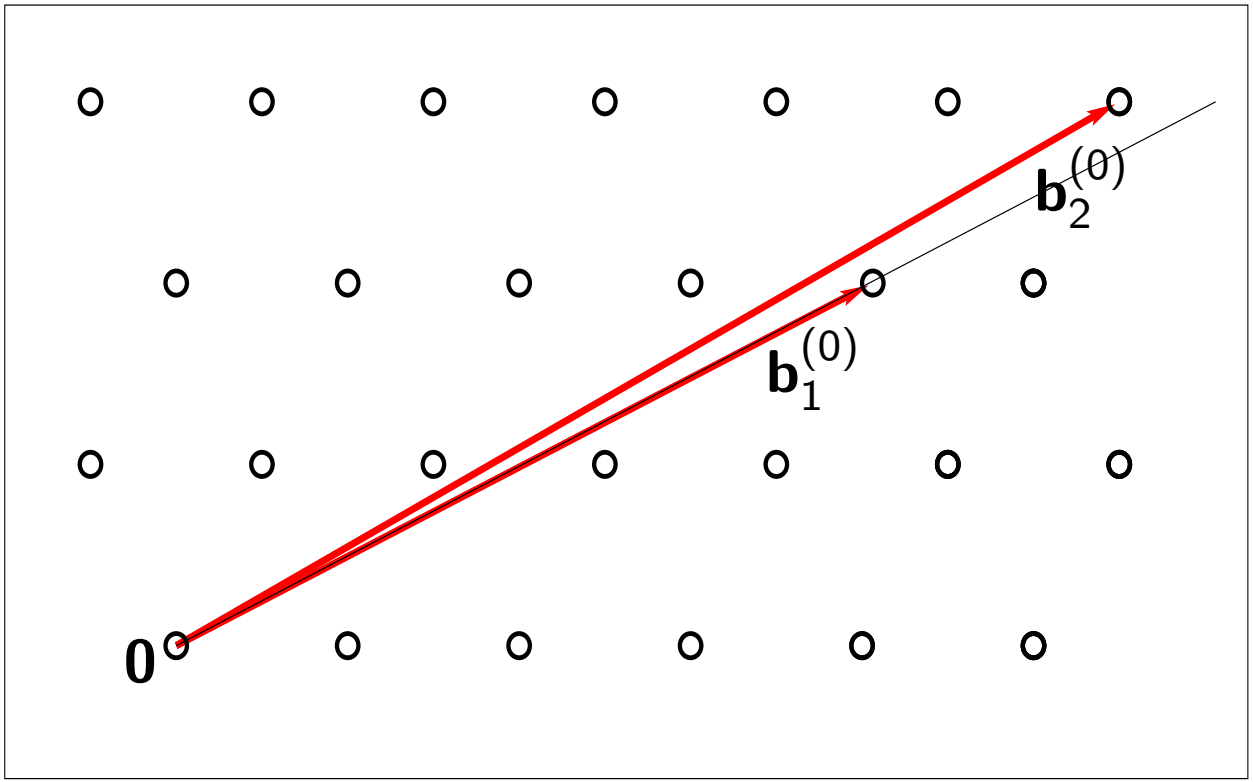
SVP is easy in small dimensions!



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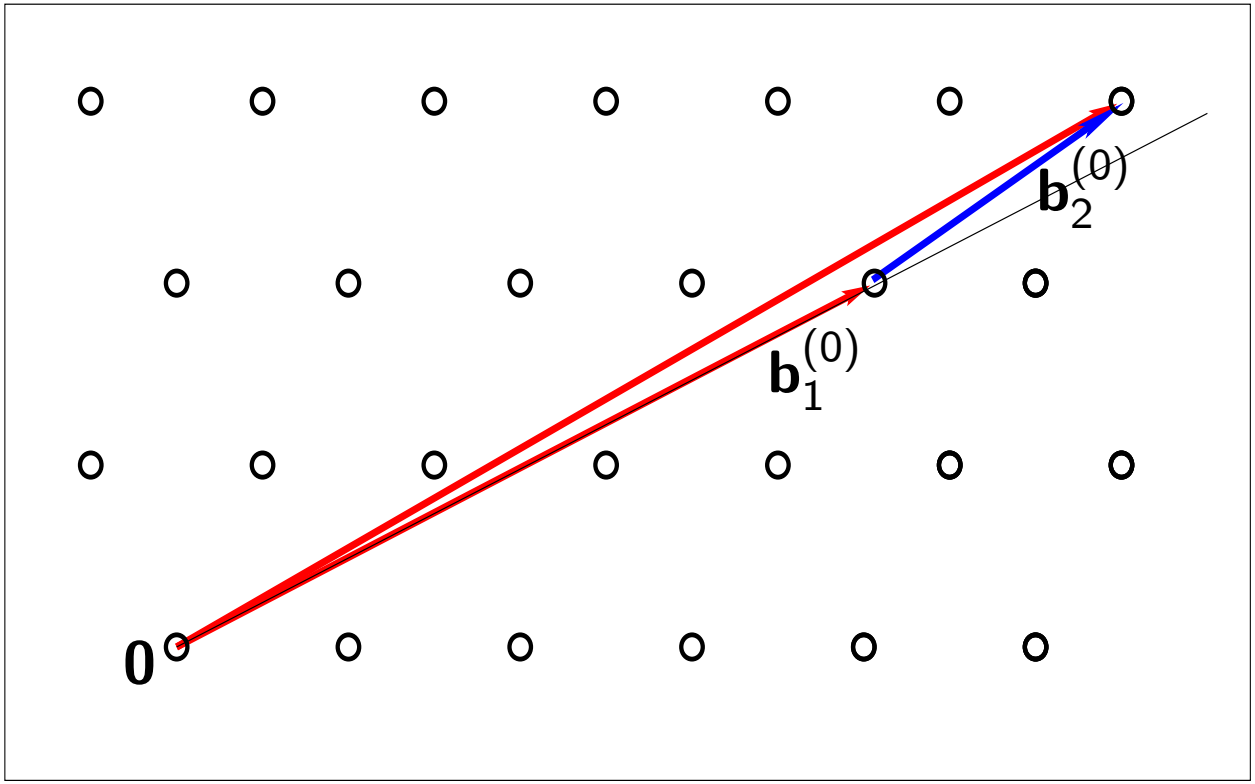

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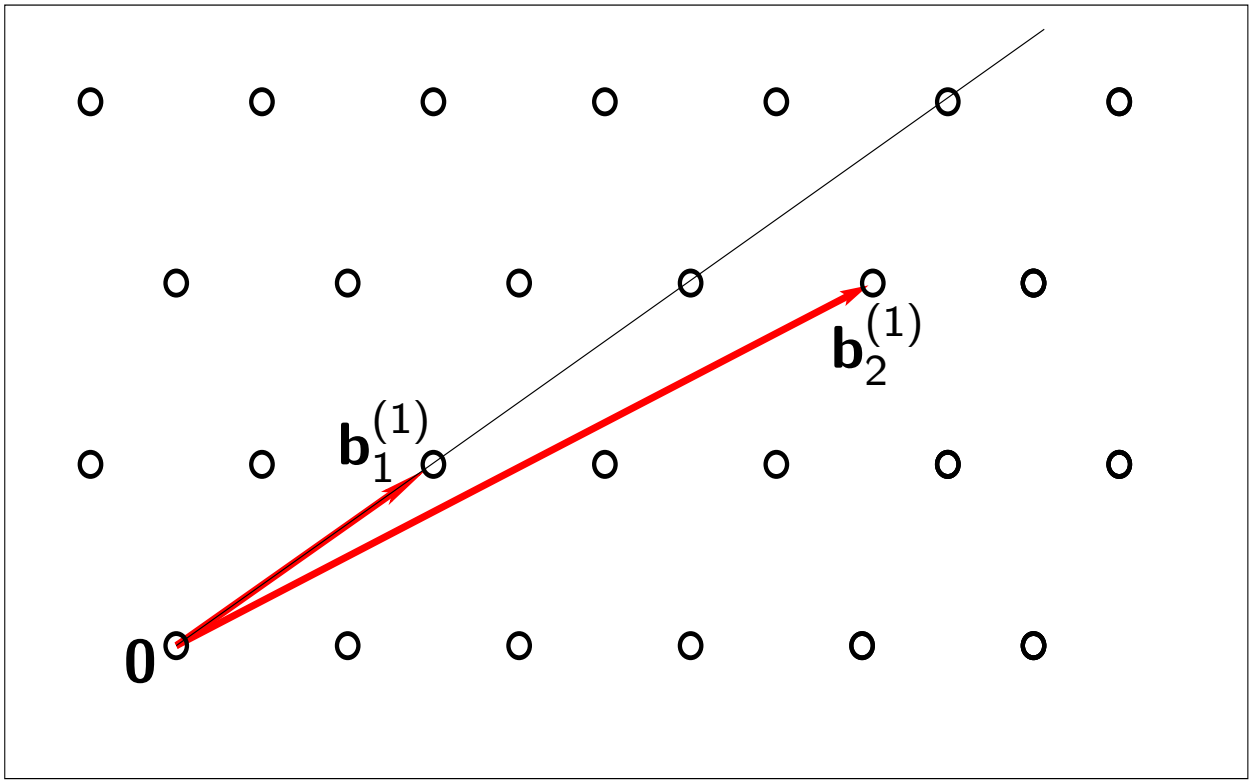

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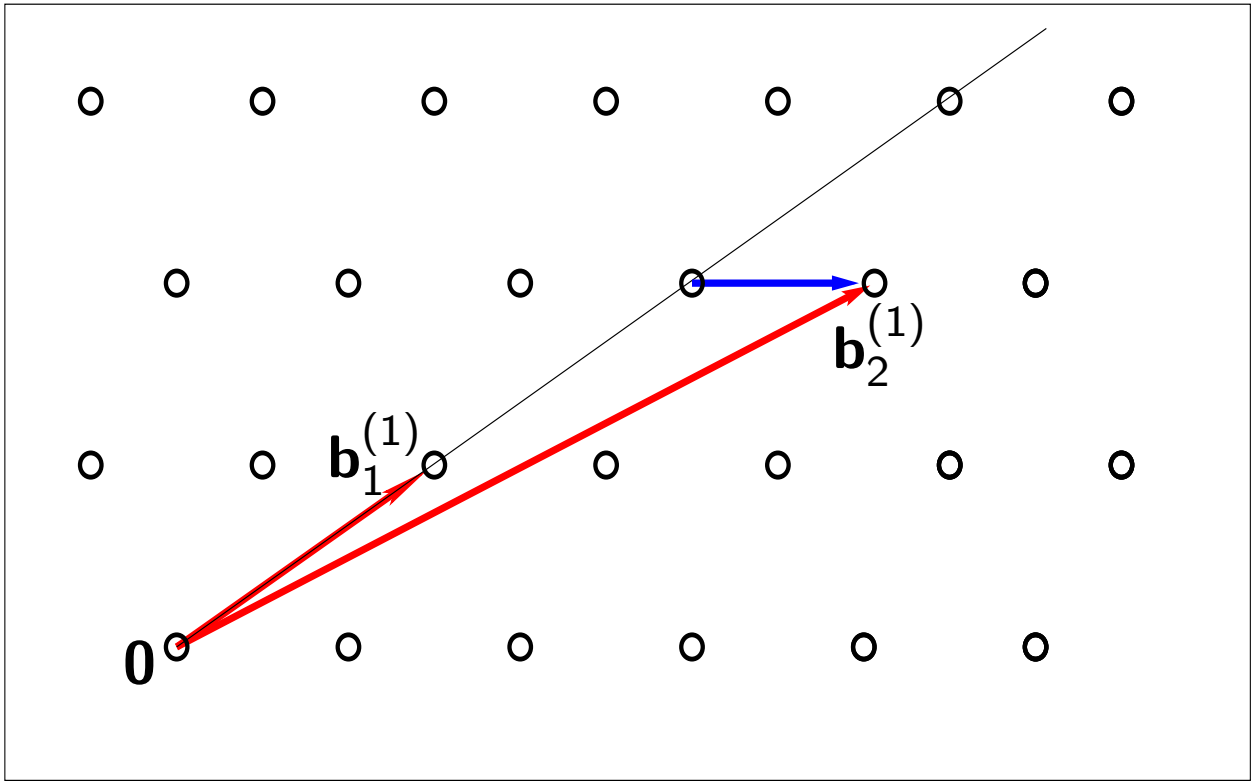

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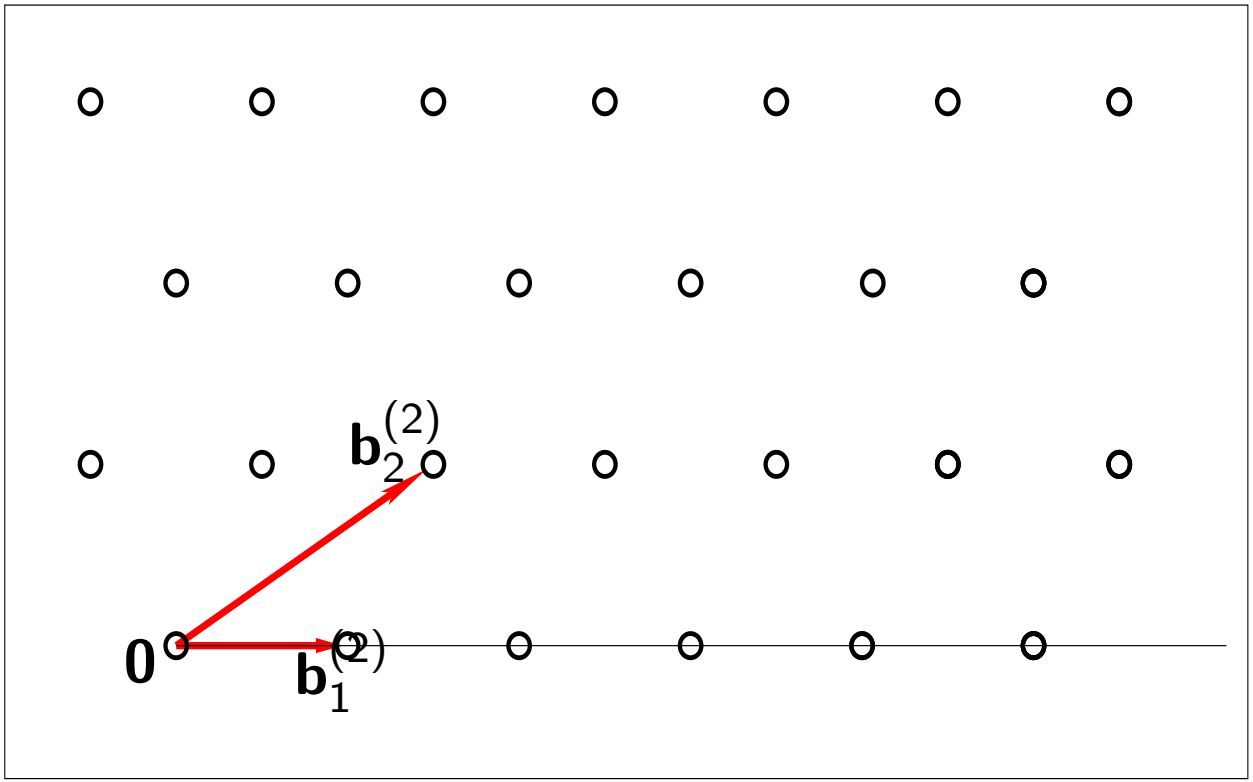

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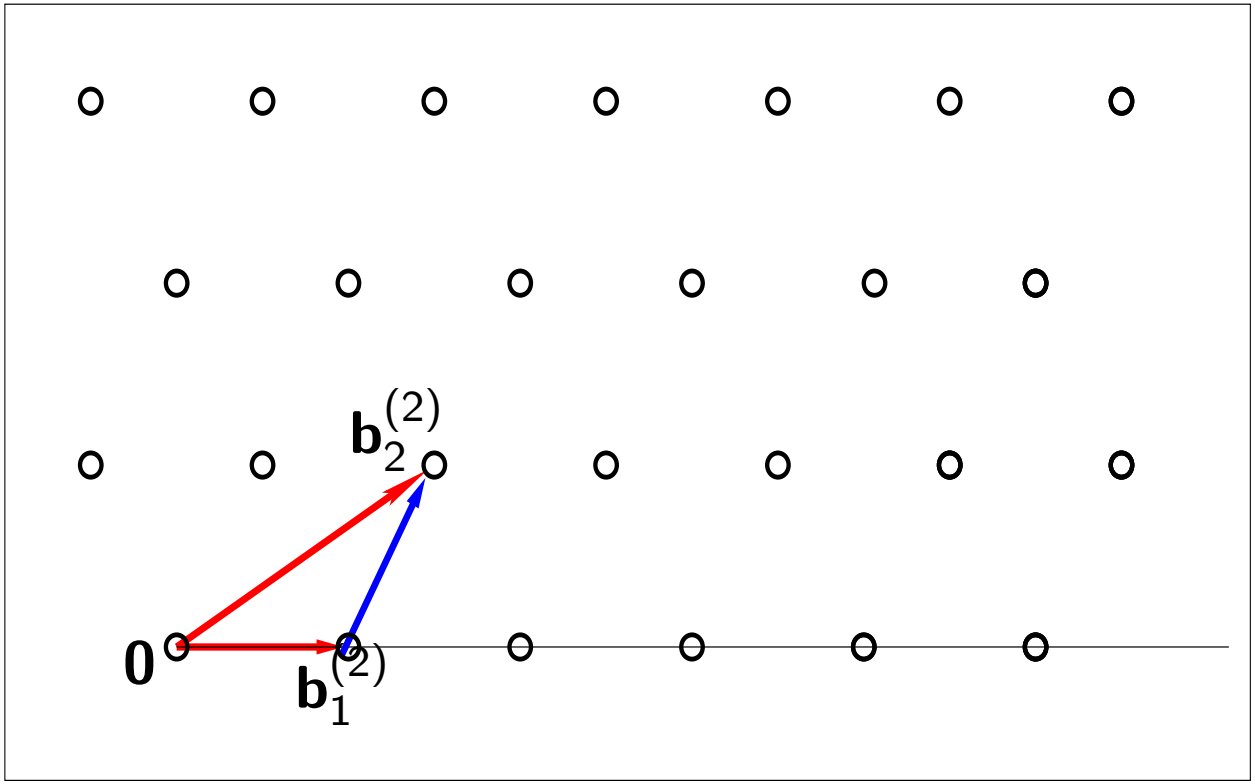

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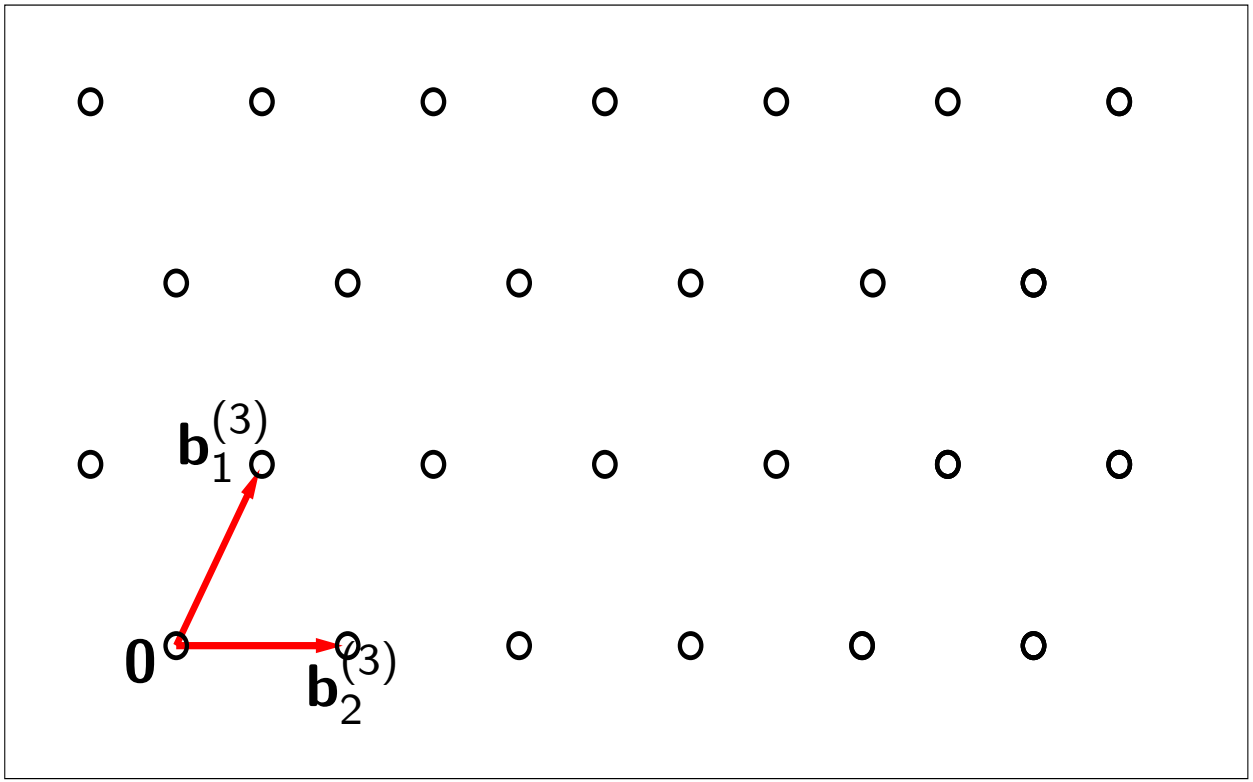

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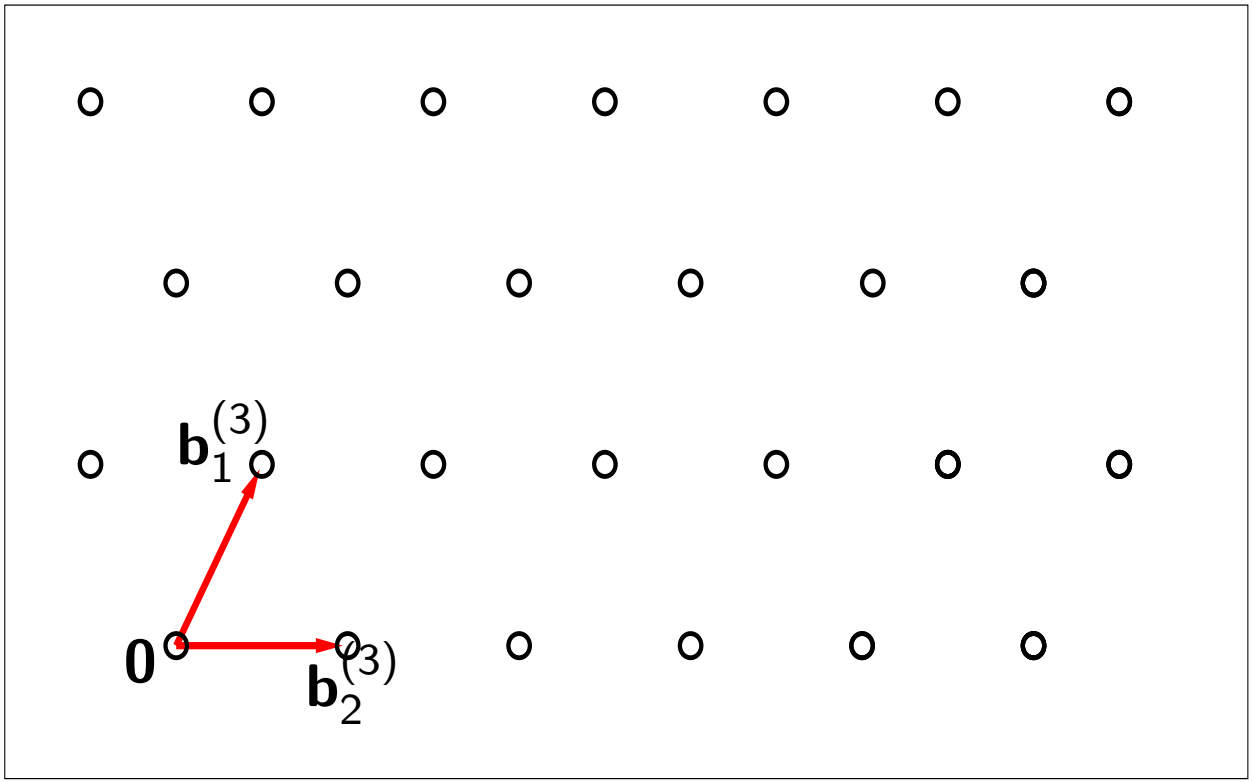

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1. Euclidean lattices
2. **Applications of euclidean lattices**
3. The LLL algorithm and its accelerations

1. Integer relation detection
2. Polynomial factorisation
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Finding small integer relations between real numbers

Lattices may be used to find the Bailey-Borwein-Plouffe formula:

$$\pi = \sum_{i \geq 0} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right).$$

Interesting to compute a given hexadecimal digit of π

Assume we search a small integer relation between $y_1, \ldots, y_d \in \mathbb{R}$.

Consider $L := L[(\mathbf{b}_i)_i]$, with $B = \begin{bmatrix} y_1 & y_2 & \cdots & y_d \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$

A relation $\mathbf{x} \in \mathbb{Z}^{d+1}$ leads to a small vector $(0, x_1, \ldots, x_d)^T$.

Using a large C makes it a shortest of L.

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Factoring integer polynomials

This idea may be used to factor polynomials over $\mathbb{Z}[x]$.

Factoring polynomials with rational coefficients,
A. K. Lenstra, H. W. Lenstra Jr. and L. Lovász. Math. Ann., 1982

$\Rightarrow$ Cited 2544 times!

- Step 0: If $\deg P \leq 1$, then stop
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Cryptanalysis of RSA with small decryption exponent

RSA:

- $N = p \cdot q$ with $p, q \approx \sqrt{N}$ prime
- e, d with $e \cdot d = 1 \bmod (p-1) \cdot (q-1)$
- $PK = (N, e)$, $sk = d$

RSA with small decryption exponent: $d \ll N^{1/4}$, $e \approx N$.

Some orders of magnitude:

$$\begin{array}{rcl} e \approx N & \Rightarrow & k \approx d \ll N^{1/4} \\ ed = 1 + k\phi(N) & \Rightarrow & |ed - kN| \ll N^{3/4} \end{array}$$

Cryptanalysis: Consider $L(\mathbf{b}_1, \mathbf{b}_2)$ with $[\mathbf{b}_1 | \mathbf{b}_2] = \begin{bmatrix} e & N \\ \sqrt{N} & 0 \end{bmatrix}$.

$\lambda(L)$ expected to be $\approx \sqrt{\det(L)} = N^{3/4}$.
But $\|d\mathbf{b}_1 - k\mathbf{b}_2\| \ll N^{3/4}$ ...

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Cryptographic design

Lattice-based cryptography:

- Secret key: very short basis of a lattice
- Public key: long basis of the same lattice
- Relies on the assumed hardness of SVP

Very popular research topic, because LBC is:

- More secure: post-quantum
- More efficient: no modular exponentiation
- More flexible: fully homomorphic encryption

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Gram-Schmidt orthogonalization (GSO)

$(\mathbf{b}_i)_i$ linearly independent.

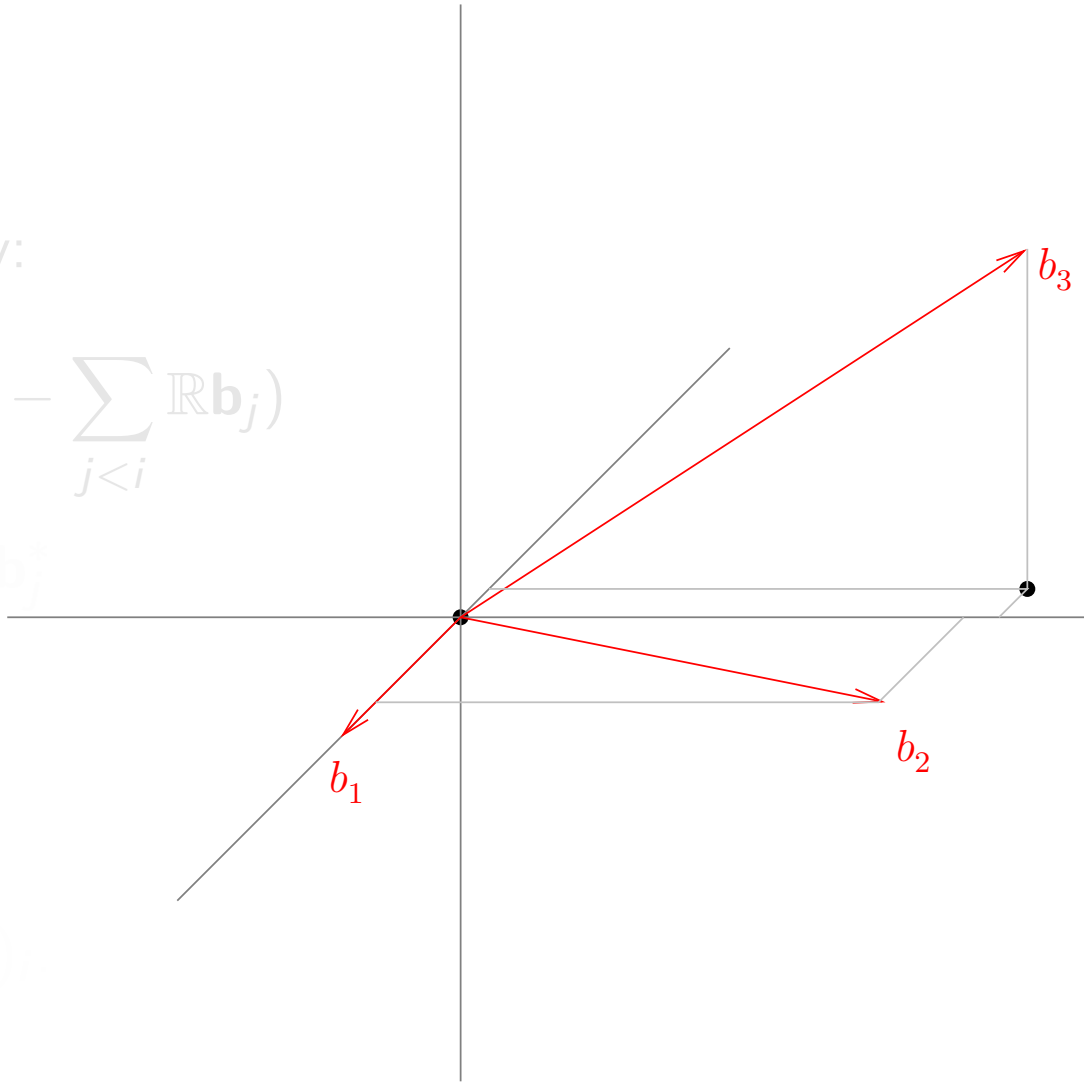
The GSO $(\mathbf{b}_i^*)_i$ is defined by:

$$\forall i, \quad \mathbf{b}_i^* = \text{argmin}_{\|\cdot\|}(\mathbf{b}_i - \sum_{j<i} \mathbb{R}\mathbf{b}_j)$$

$$= \mathbf{b}_i - \sum_{j<i} \mu_{i,j}\mathbf{b}_j^*$$

$$\forall i > j, \mu_{i,j} = \frac{(\mathbf{b}_i, \mathbf{b}_j^*)}{\|\mathbf{b}_j^*\|^2}.$$

Triangularization of B : $(\mathbf{b}_i)_i$.



Exercise: For any basis $(\mathbf{b}_i)_i$ of L, we have $\lambda(L) \geq \min_i \|\mathbf{b}_i^*\|$.

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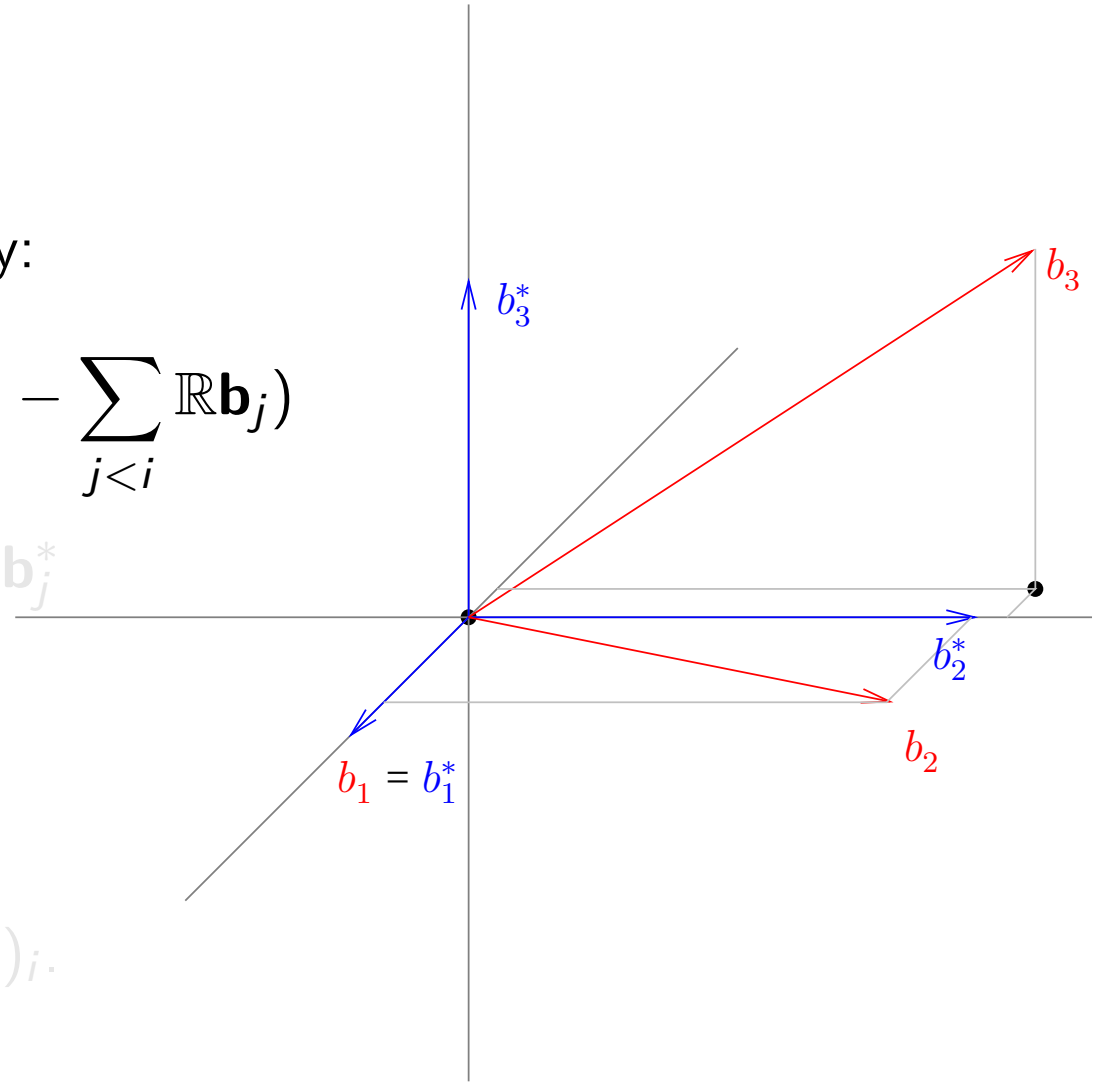

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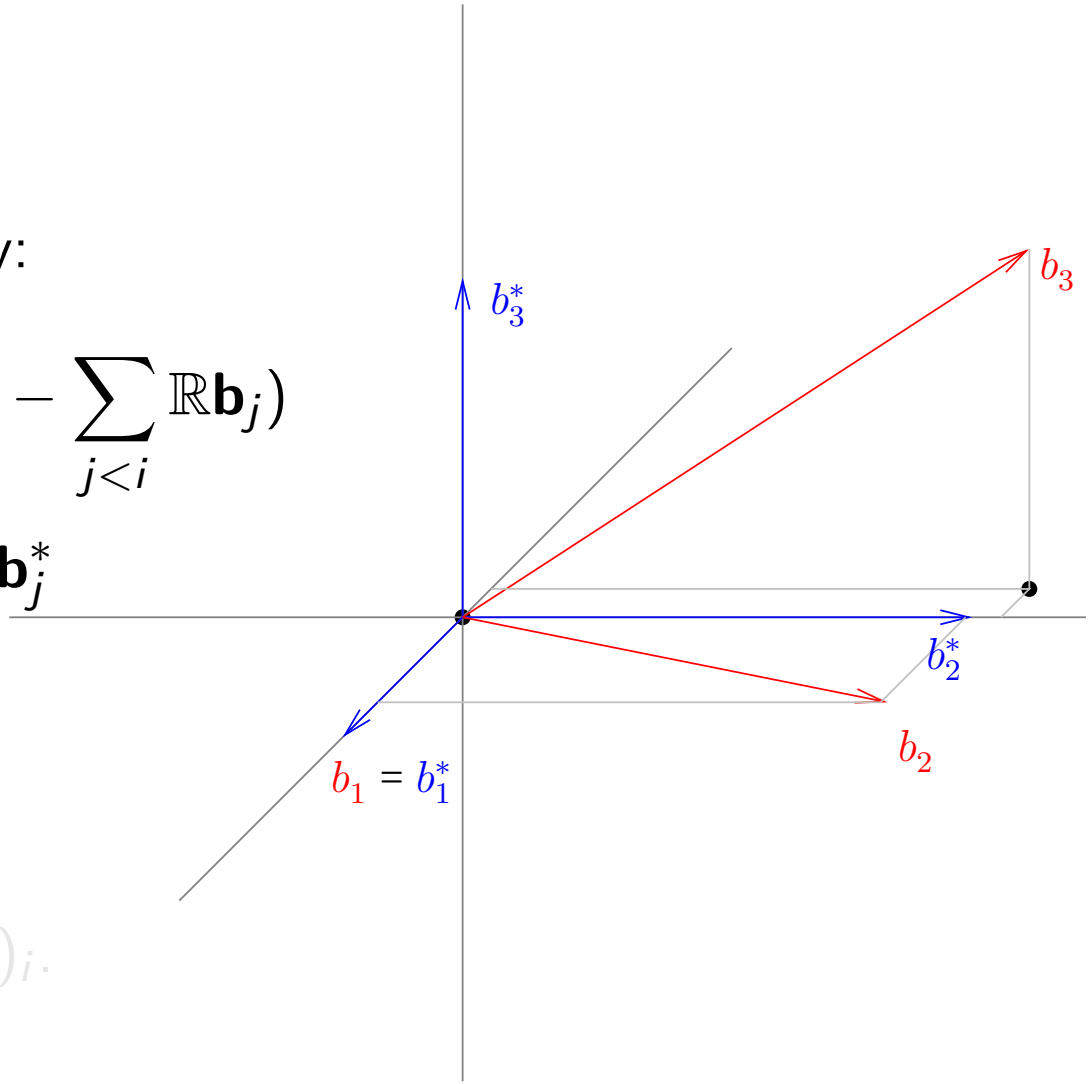

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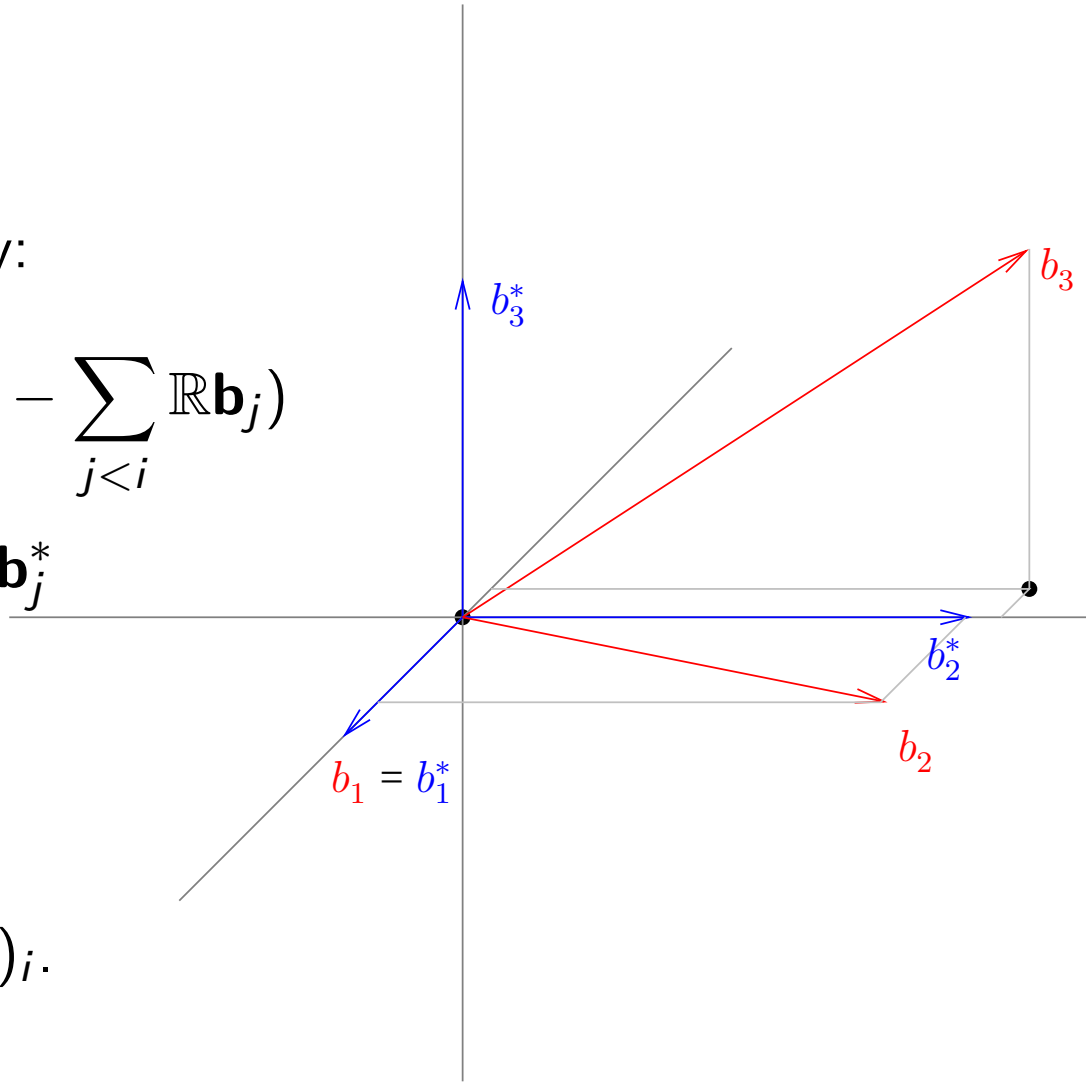

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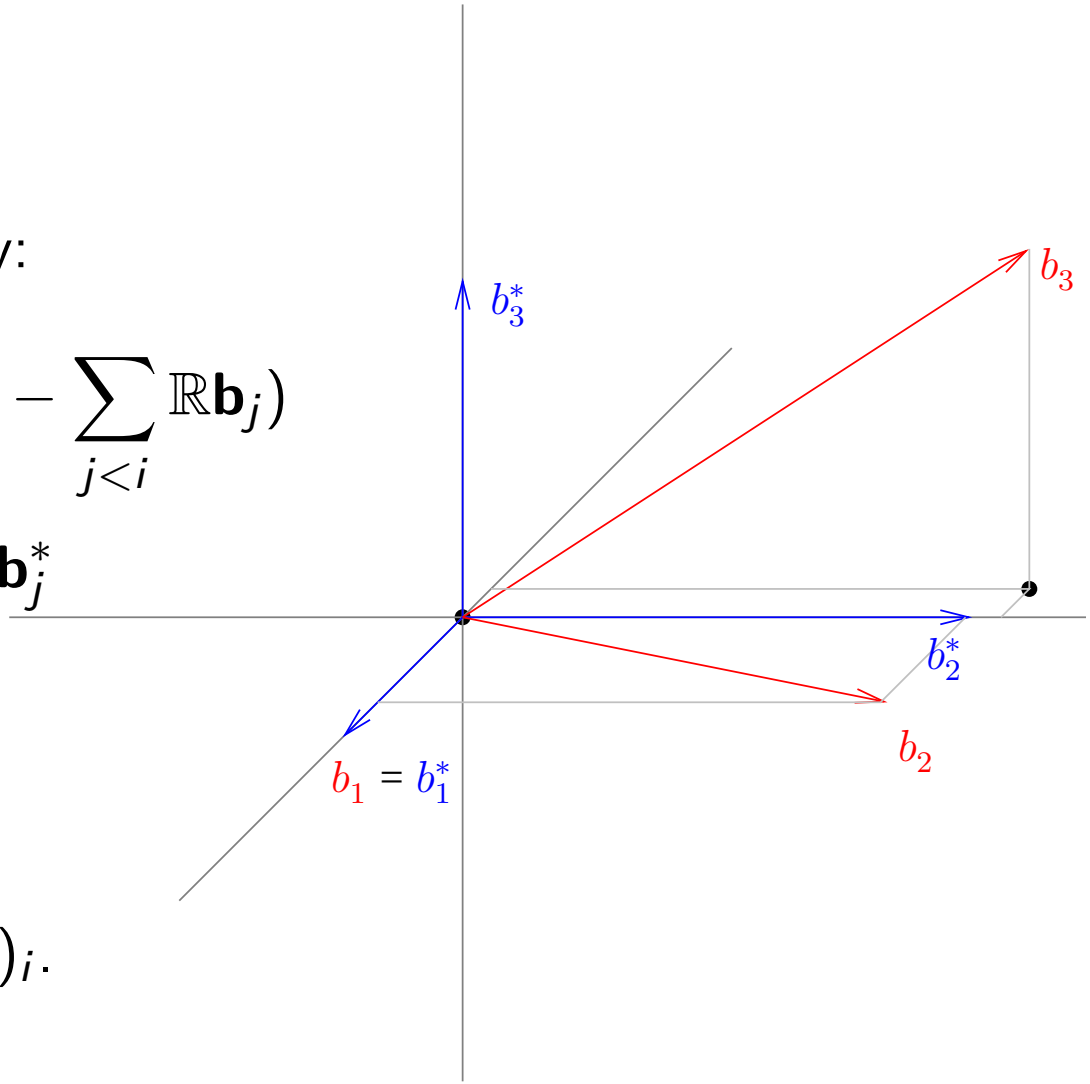

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Let $\delta \in (1/4, 1)$. A basis $B = (\mathbf{b}_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ is said LLL-reduced if:

- $\forall i, j : \ |\mu_{i,j}| \leq 1/2$ [size-reduction]
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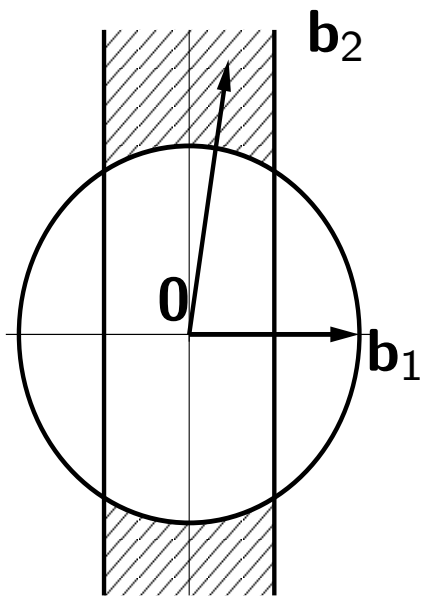
$\Rightarrow \lambda(L) \leq \|\mathbf{b}_1\| \leq 2^{\mathcal{O}(n)} \cdot \lambda(L)$

$\delta < 1$ is crucial to get a polynomial complexity.



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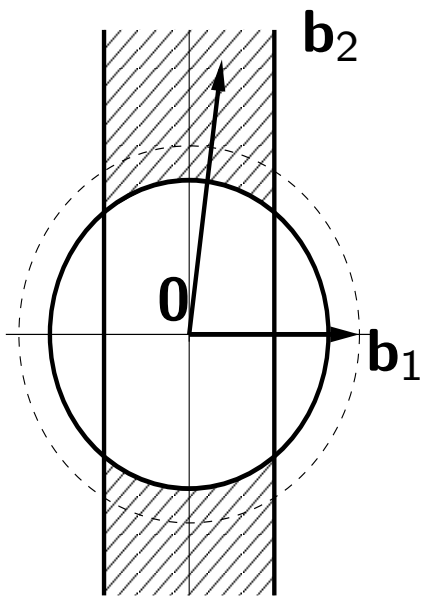

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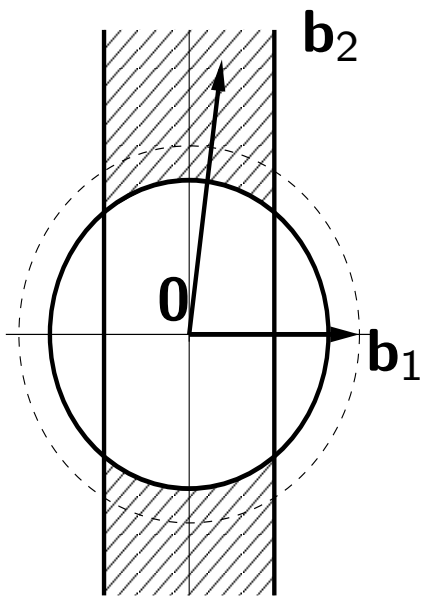

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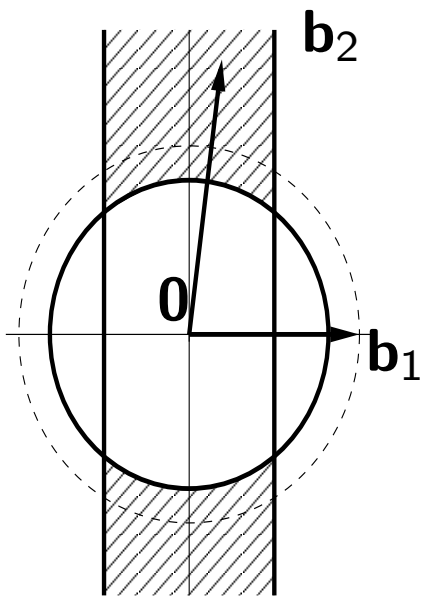

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1. Enforce size-reduction, using a modified Gaussian elimination
2. If there is an i with $\delta \cdot \|\mathbf{b}_i^*\|^2 > \|\mathbf{b}_{i+1}^*\|^2 + \mu_{i+1,i}^2 \|\mathbf{b}_i^*\|^2$, swap $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$, then go to Step 1
3. Return the current basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$

$\Rightarrow$ Correctness is trivial

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Complexity of LLL and practical run-time

[LLL82,Kaltofen83]

The LLL algorithm terminates within $\mathcal{O}(n^4\beta^2(n+\beta))$ bit operations, with $\beta = \log \max_i \|\mathbf{b}_i^{init}\|$.

With MAGMA V2.16:

```
> n := 25; B := RMatrixSpace(Integers(),n,n)!0;
> for i:=1 to 25 do
>    B[i][i]:=1; B[i][1]:=RandomBits(2000);
> end for;
> time C := LLL(B:Method:=``Integral'');
```

Time: 11.700

```
> time C := LLL(B);
```

Time: 0.240

LLL in practice: the numeric-symbolic approach

The Gram-Schmidt computations dominate the cost

Odlyzko's hybrid approach

Replace the rational computations on the GSO by floating-point approximations, but keep the basis operations exact

Floating-point numbers: $x_1.x_2x_3\ldots x_p \cdot B^e$, where:

- p is the precision
- B is the base, and $x_i \in \{0, \ldots, B-1\}$ for all i
- $e \in \mathbb{Z}$ is the exponent
- $x_1.x_2\ldots x_p$ is called the mantissa

Arithmetic: $fl(a\ op\ b)$ is a nearest floating-point number to $a\ op\ b$, for any $op \in \{+, -, /, \times\}$.

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Making the numeric-symbolic approach rigorous

Underlying mathematical phenomenon

Any LLL-reduced basis is well-conditioned with respect to GSO computations

- Well-conditioned? The GSO computed in small precision fp arithmetic is close to the true GSO

$\Rightarrow$ We'd like to rely on LLL-reduced bases as much as we can

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Small precision? $\mathcal{O}(n)$ bits of precision suffice for correctness.
Bit-complexity:

$$\mathcal{O}(n^2\beta) \cdot \mathcal{O}(n^2) \cdot \left[\mathcal{O}(n\beta) + \mathcal{O}(n^2)\right] = \mathcal{O}(n^5\beta(n+\beta)).$$

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The totally numeric approach

LLL can be accelerated further by using approximations for the bases too, in a recursive manner! $\Rightarrow \widetilde{\mathcal{O}}(n^5\beta)$ bit operations.

The totally numeric approach with blocking

Restrict to sub-matrices of the GSO, and use fast matrix multiplication $\Rightarrow \widetilde{\mathcal{O}}(n^4\beta)$ bit operations.

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Plan of the talk

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1. Euclidean lattices
2. Applications of euclidean lattices
3. The LLL algorithm and its accelerations
4. **Conclusion**

Research topics

- Lowering the cost of LLL-reduction
- Algorithms computing shorter vectors than LLL

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- Hardness proofs

  NP hardness of SVP under deterministic reductions?
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Further readings

L. Lovász: *An Algorithmic Theory of Numbers, Graphs, and Convexity*

D. Micciancio & S. Goldwasser: *Complexity of Lattice Problems, A Cryptographic Perspective*

I. Morel, D. Stehlé & G. Villard: *Analyse numérique et réduction de réseaux*

D. Micciancio: *The Geometry of Lattice Cryptography*

F. Eisenbrand: *Integer Programming and Algorithmic Geometry of Numbers*