Graphs for Relation Algebra

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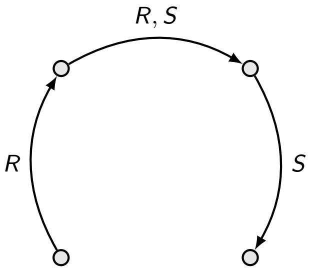
Villemanzy, 17.12.2o15

Binary relations

$R, S \in \mathcal{P}(X^2)$



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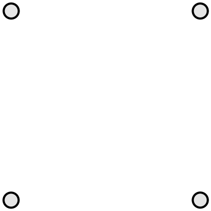

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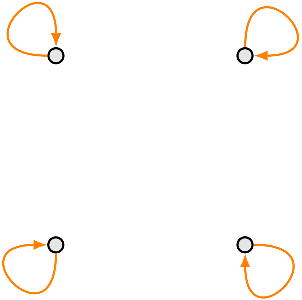

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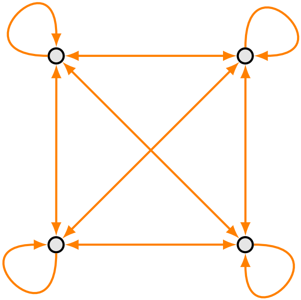

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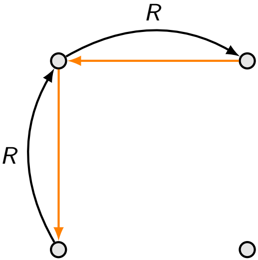

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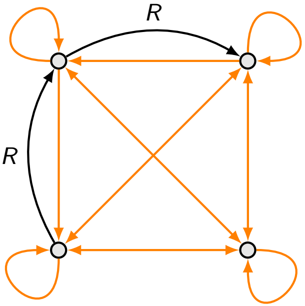

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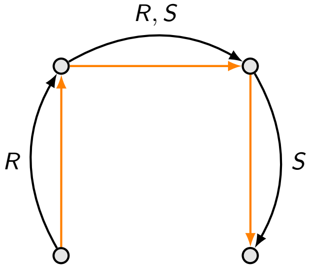

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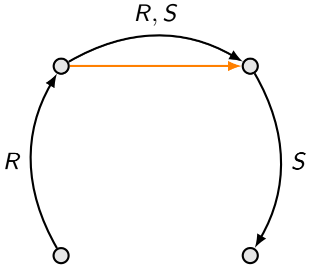

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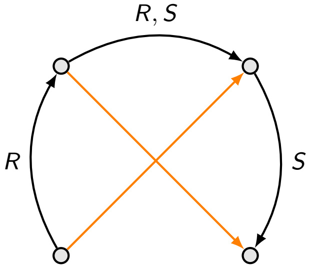

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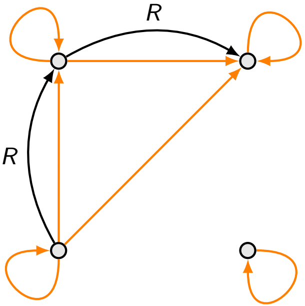

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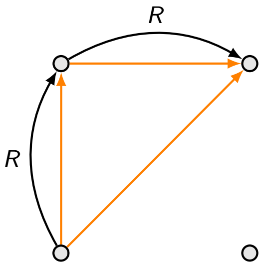

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Abstract properties

Many properties of relations can be expressed algebraically, without mentioning points:

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$R \cdot R \subseteq R$	transitivity
$R^{+} \cap 1 = 0$	acyclicity
$R^{\circ} \cdot R \subseteq R^{\star} \cdot R^{\circ\star}$	local confluence

Laws

Many (in)equations hold universally:

- Intuitive ones:

$$R \cup S = S \cup R$$
$$R \cdot (S \cdot P) = (R \cdot S) \cdot P$$
$$\dots$$

$\langle \cup, \cap, \cdot^c, 0, \top \rangle$ form a Boolean lattice
$\langle \cdot, \cup, 1, 0 \rangle$ form an idempotent semiring

- Stranger ones:

$$R{\cdot}S \,\cap\, P \subseteq (R \,\cap\, P{\cdot}S^\circ){\cdot}(S \,\cap\, R^\circ{\cdot}P)$$
$$(R \,\cap\, S{\cdot}\top){\cdot}P = R{\cdot}P \,\cap\, S{\cdot}\top$$
$$(R \cup S)^\star = R^\star{\cdot}(S{\cdot}R^\star)^\star$$
$$\dots$$

Relation algebra

1. Can we decide the validity of an equation?
2. At which cost?

With complement

1. No
2. -

Merci Gödel

Kleene algebra

Consider the fragment with operations $\cdot$, $\cup$, $\cdot^\star$, 1, 0,

i.e., regular expressions

Theorem. *An equation is valid iff the corresponding regular expressions recognise the same language.*

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$\hookrightarrow$ The problem is thus PSPACE-complete in this fragment

Allegories

Now consider the fragment with operations $\cdot$, $\cap$, $\cdot^\circ$, 1, $\top$,

called allegories in the litterature

The previous theorem fails

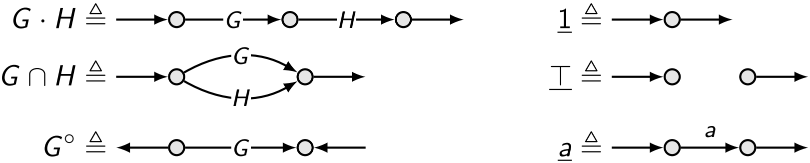
Valid laws include

$$a\cdot b \cap c \subset (a \cap c\cdot b^\circ)\cdot b$$
$$a\cdot(b \cap c) \subset a\cdot b \cap a\cdot c$$
$$(a \cap b\cdot\top)\cdot c = a\cdot c \cap b\cdot\top$$

Graphs, at last

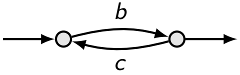
Call graph a directed edge-labelled graph with two distinguished vertices called input and output

Define the following operations on such graphs

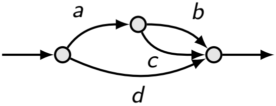


Graphs of expressions

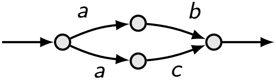
$G(b \cap c^\circ)$



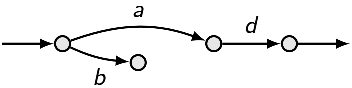
$G\,(a{\cdot}(b \cap c) \cap d)$



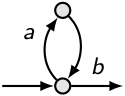
$G\,(a{\cdot}b \cap a{\cdot}c)$



$G\,((a \cap b{\cdot}\top){\cdot}d)$



$G\,(a{\cdot}b \cap 1)$



Graph homomorphisms

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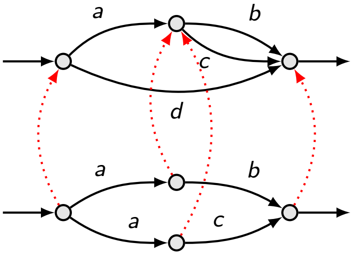
$a{\cdot}b \cap a{\cdot}c$



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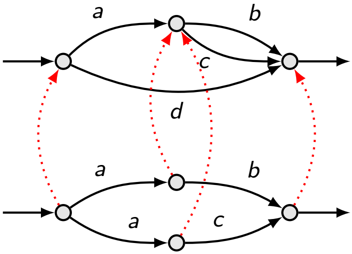

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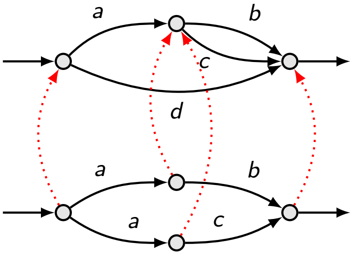

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- is it NP-hard?
- is it in P?

Kleene allegories

Kleene algebras	Allegories
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language	graph

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Theorem. *An equation is valid in Kleene allegories iff the corresponding languages of graphs coincide.*

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Theorem. *An equation is valid in Kleene allegories iff the corresponding languages of graphs coincide.*

- “Petri automata” for recognising those languages
- Comparison of such automata is EXPSPACE-complete

. . . in a fragment where graphs are series-parallel