Category Theory 101
Graph Transformations
Discrete Structures Day

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Introduction

- High level approach to programming: graph rewriting based on category theory.
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- Much more difficult than term rewriting (which are just trees).
- Simulation of biological phenomenons.
- Simulation of chemical reactions.
- Study of cloning:

  $\Longrightarrow$ Typically to produce a web site one starts to copy an existing one, then one modifies it accordingly to its will.

  $\Longrightarrow$ Social Data Anonymization techniques rely on finely tuned cloning operations.

Plan

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Category Theory

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$\Longrightarrow$ What are the fundamental structures of those two fields ?

- Results much more general than thought at first.
- Set theory is just a special case of category (Lawvere).
- In computer science E. Moggi was able to capture ideas previously thought to be outside of reach with the monads.
- In logic J.-Y. Girard and the linear logic.
- etc.

Category Theory

Definition

A category $\mathcal{C}$ is made of

- A collection of object : $Obj(\mathcal{C})$
- $\forall x, y \in Obj(\mathcal{C})$ a set $Hom_{\mathcal{C}}(x, y)$
- $\forall x \in Obj(\mathcal{C})$ there is $id_x \in Hom_{\mathcal{C}}(x, x)$
- $\forall x, y, z \in Obj(\mathcal{C})$ a function $\circ : Hom_{\mathcal{C}}(x, y) \times Hom_{\mathcal{C}}(y, z) \to Hom_{\mathcal{C}}(y, z)$

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  $\circ : Hom_{\mathcal{C}}(x, y) \times Hom_{\mathcal{C}}(y, z) \rightarrow Hom_{\mathcal{C}}(y, z)$

such that

1. Identity: $f \circ id = id \circ f = f$
2. Associativity: $(h \circ g) \circ f = h \circ (g \circ f)$

Example: Category of graphs

- Objects: $G = (V, E, s, t)$ with $s, t : E \to V$
- Morphisms: $f : G \to H$ must respect source and target functions, ie:

$$\forall e \in E. f(s(e)) = s(f(e))$$
$$\forall e \in E. f(t(e)) = t(f(e))$$

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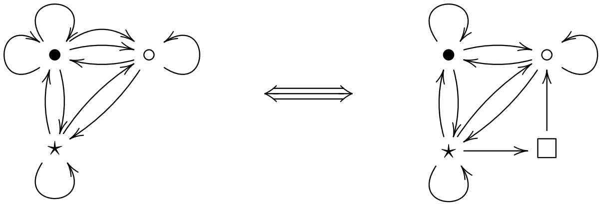
- Exemples:



Pullback

- Lets have : $f : X \to Z$ and $g : Y \to Z$
- Fiber product: $X \times_Z Y := \{(x, w, y) \mid f(x) = w = g(y)\}$

$$
\begin{tikzcd}
A \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left, "\forall t"] \arrow[ddr, bend right, "\forall u"'] & & \\
& X \times_Z Y \arrow[r, dashed, "\pi_1"] \arrow[d, dashed, "\pi_2"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

Pushout

- Co-construction of the pullback.
- Lets have : $f : X \to Z$ and $g : Y \to Z$
- disjoint sum with gluing: $X +_Z Y := X + Y + Z/ \sim$
- With $\sim$ generated by $f(z) \sim z \sim g(z)$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, dashed, "i_1"] \arrow[ddr, bend left, "\forall u"] & \\
Y \arrow[r, dashed, "i_2"'] \arrow[drr, bend right, "\forall t"'] & X +_Z Y \arrow[dr, dashed, "\exists!"] & \\
 & & A
\end{tikzcd}$$

Plan

Rule based transformations

- Rule-based term rewriting is easy: replace a tree by another one.
- Much more difficult with graphs (multiple incident edges).
- Categorical frameworks make it clean to express graph transformations systematically.

PB	PO
clone	merge
delete	add
comatch	match
global	local

AGREE extended rule

Extension of a framework proposed by A. Corradini, D. Duval, R. Echahed, F. Prost and L. Ribeiro [ICGT15].

Definition (AGREE rules and matches)

- A *rule* is

$$\begin{tikzcd} L & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "t"] & R \\ T_L & T_K \arrow[l, "l'"'] & \end{tikzcd}$$

- A *match* of such a rule is composed of a mono $L \overset{m}{\rightarrowtail} G$ and a typing morphism $G \xrightarrow{\overline{m}} T_L$ and is such that $l' \circ t = (\overline{m} \circ m) \circ l$.

AGREE rewrite step

Definition (AGREE rewriting)

Given $\rho = (K \xrightarrow{l} L, K \xrightarrow{r} R, K \overset{t}{\rightarrowtail} T_K, T_K \xrightarrow{l'} T_L)$ and a match $L \overset{m}{\rightarrowtail} G, G \xrightarrow{\overline{m}} T_L$: $G \Rightarrow_{\rho,m} H$ is computed as follows:

1. Span $G \xleftarrow{g} D \xrightarrow{n'} T_K$ is the pullback of $G \xrightarrow{\overline{m}} T(L) \xleftarrow{l'} T_K$. Since $l' \circ t = \eta_L \circ l$ there is a unique $K \overset{n}{\rightarrowtail} D$.
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$$
\begin{tikzcd}
L \arrow[d, "m"] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "n"'] \arrow[dd, bend left, "t"] \arrow[dr, phantom, "PO\ (3)"] & R \arrow[d, "p"] \\
G \arrow[d, "\overline{m}"] \arrow[dr, phantom, "PB\ (1)"] & D \arrow[l, "g"'] \arrow[r, "h"] \arrow[d, "n'"] \arrow[ul, phantom, "(2)"] & H \\
T(L) & T_K \arrow[l, "l'"] &
\end{tikzcd}
$$

Example : copy of web pages

- The structure of a web site typically as two kind of links :
  - Internal links: file hierarchy (indirect link)
  - External links: references pointing outside of the site.

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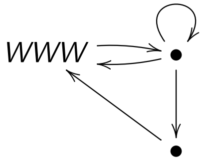
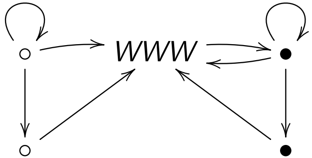
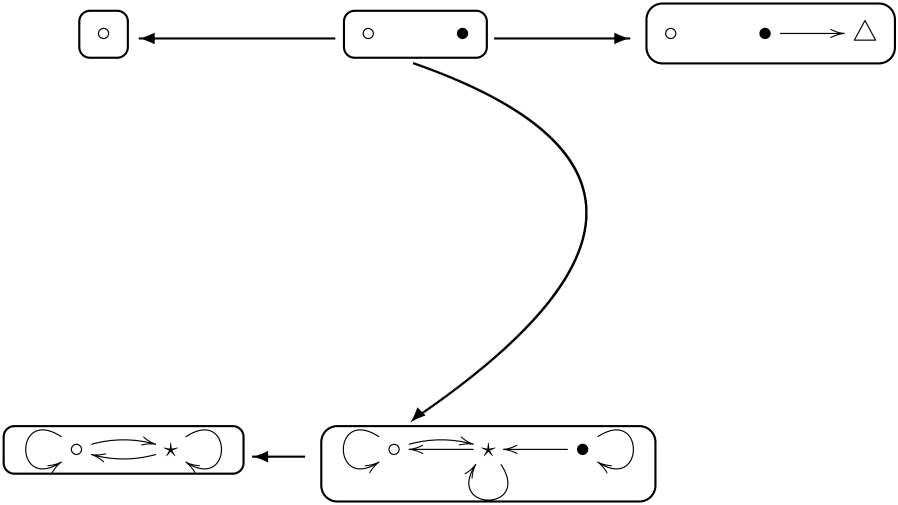
- The structure of a web site typically as two kind of links :
  - Internal links: file hierarchy (indirect link)
  - External links: references pointing outside of the site.
- The cloning of a web site consists in duplicating all local files and keeping external links shared between the two copies.



should be cloned as follows



Web copy with AGREE rewriting



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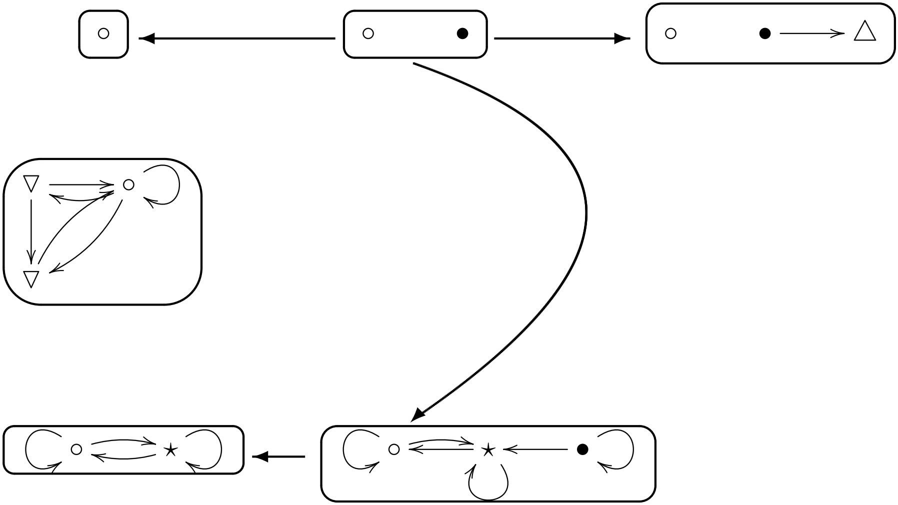

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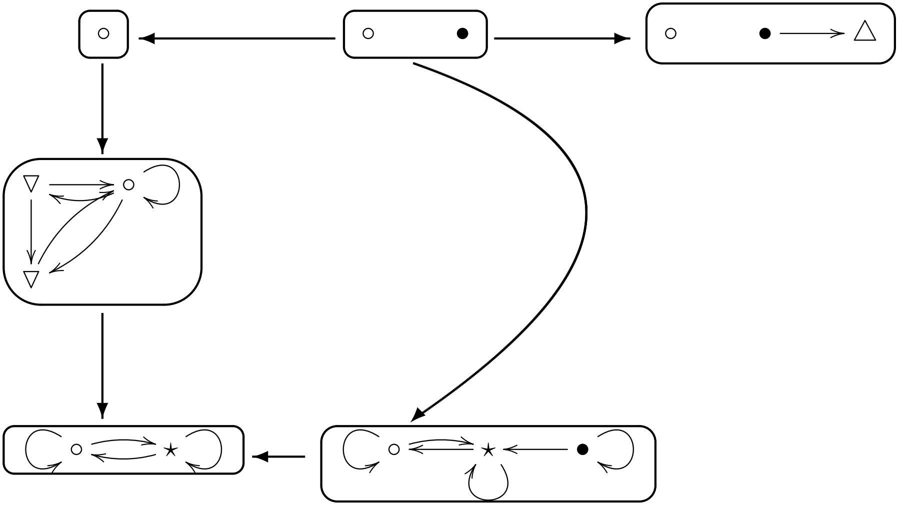

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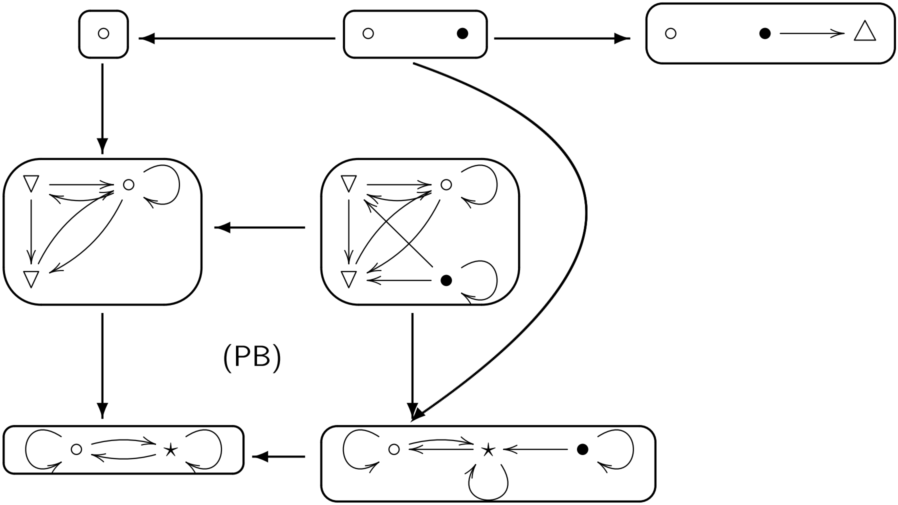

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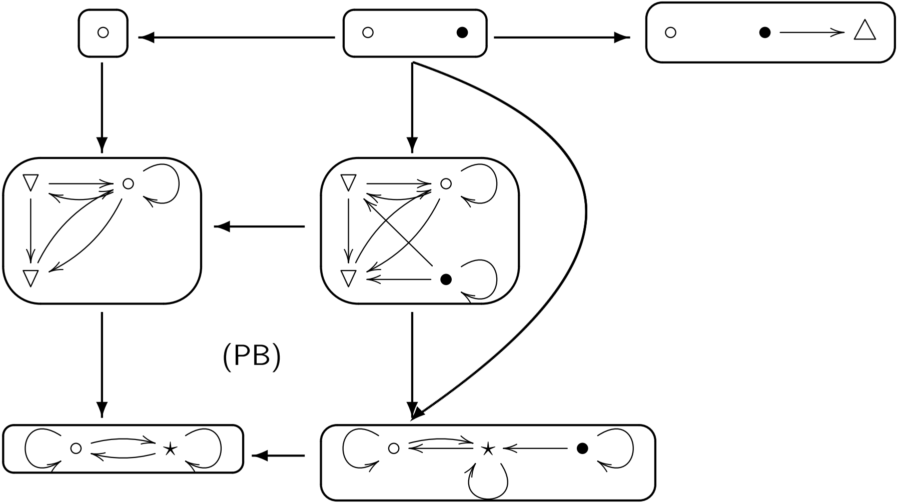

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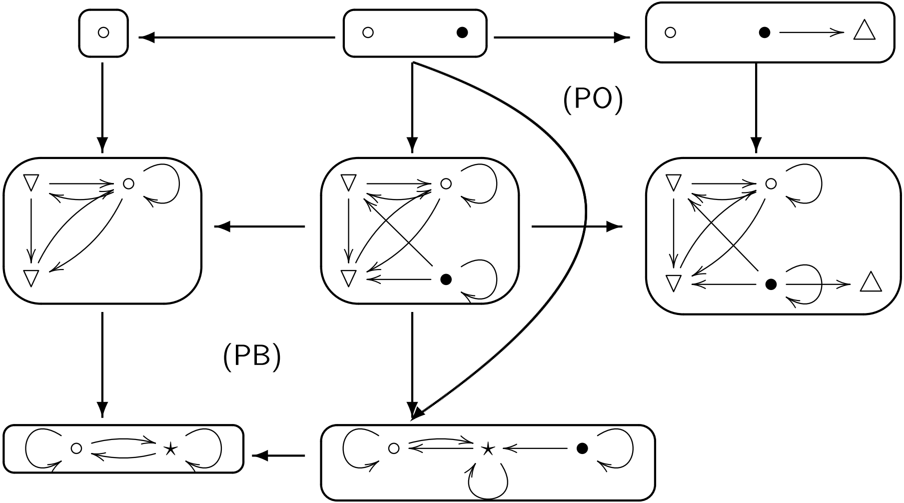

Plan

Social Data Anonymization: concepts and challenges

- Big economical issue: more or less the backbone of the business models of internet giants (Google, Facebook, Yahoo etc.).
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- Raw problem: given a graph G we would like to produce G' such that
  - $Stat(G) \simeq Stat(G')$
  - It is not possible to reidentify nodes (or edges) of G from knowing G' (and some extra informations...).
- Naïve approach doesn't work : Netflix [NarayanShmatikov06].
- Anonymization is an active research field ... rather artistic at the time: approaches validated through experiments.

Social Data Anonymization: Dimensions and Principles

- Problem more down to the earth than non-interference:
  - Partial knowledge of the graph by the opponent.
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  - Object of interests vary from one data set to another.
- Hence three important points to consider:
  1. Background Knowledge: What does the opponent know ? Model of the opponent.
  2. Privacity: what is attacked ?
  3. Usage: How the data is going to be analyzed ?

$\Longrightarrow$ Anonymizing techniques

Social Data Anonymization: Techniques

- Two families:
  - Clustering: group together edges and nodes.
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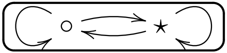
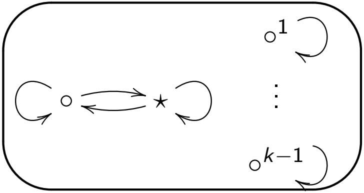
- It is NP-hard to find graph transformations minimizing the editing distance between a graph and a k-isomorphic graph.
- One solution: select $1/k$ nodes randomly, create k clones, link the clones together easy to program with *AGREE* approach.

Using *AGREE* for k-anonymity

- Progaming with types !
- L is just a cloud of nodes, and K is made of k clones of L.
- Standard T_L is :



- Simplest T_K is :



Types and structural graph properties

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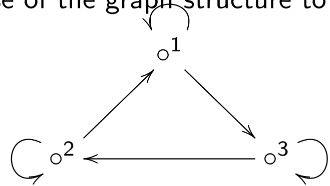

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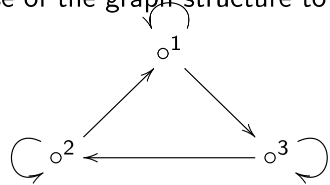

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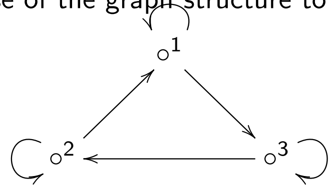
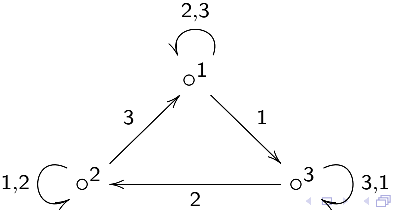
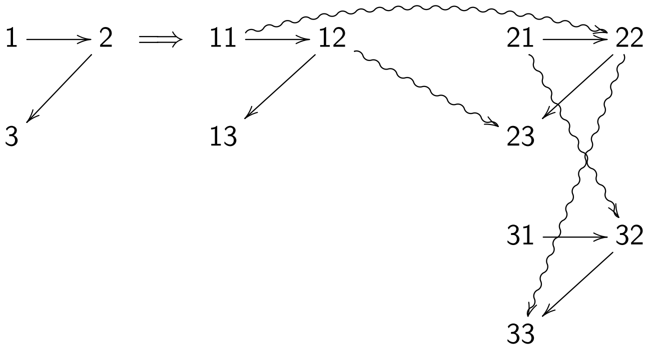


- Degree problems (nodes of degree 1).
  One possibility is to type differently the edges, eg:

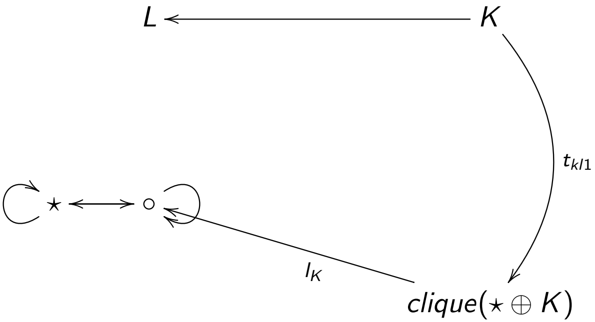


Self-similar graphs

- Every vertex is replaced by a copy of the graph.
- Interconnections between copies of the original “mimic” the ones in the target graph.



Implementation in the AGREE Framework



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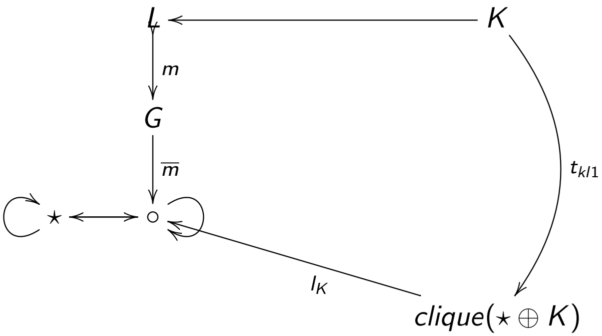

$$
\begin{tikzcd}
 & L \arrow[d, "m"] & & K \arrow[ll] \arrow[ddd, bend left, "t_{kl1}"] \\
 & G \arrow[d, "\overline{m}"] & & \\
\star \arrow[loop left] \arrow[r, leftrightarrow] & \circ \arrow[loop right] & & \\
 & & & clique(\star \oplus K) \arrow[ull, "l_K"]
\end{tikzcd}
$$

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 & G \arrow[d, "\overline{m}"] & \\
\star \arrow[loop left] \arrow[r, leftrightarrow] & \circ \arrow[loop right] & TL_1 \arrow[l, "\overline{m}\circ l_1"'] \arrow[ul, "l_1"'] \arrow[d, "kl_1"] \arrow[l, phantom, shift right=4, "PB(1)"] \\
 & & clique(\star \oplus K) \arrow[ul, "l_K"]
\end{tikzcd}$$

Implementation in the AGREE Framework

$$
\begin{tikzcd}
 & L \arrow[d, "m"] & & K \arrow[ll] \arrow[dd, "t_k"] \arrow[ddd, bend left] & \\
 & G \arrow[d, "\overline{m}"] & & & D \arrow[lll] \arrow[dl, "t_{kl1}"] \\
\star \arrow[loop left] \arrow[r, leftrightarrow] & \circ \arrow[loop right] & PB(2) & TL_1 \arrow[ull, "l_1"'] \arrow[ll, "\overline{m} \circ l_1"] \arrow[d, "kl_1"] & \\
 & & PB(1) & clique(\star \oplus K) \arrow[ull, "l_K"] &
\end{tikzcd}
$$

Plan

Conclusion

- Categorical frameworks allow simple and mathematically workable definition of complex transformations.
- Only basic constructs are needed: pushouts, pullbacks.
- An very generic implementation is scheduled.
- Open questions:
  - matching ? (random match does not lead to scale-free networks)
  - What statistics can be interesting (Ramsey-like theory) ?
  - What kind of certificate can be produced ?