

Branchwidth of graphic matroids.

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Abstract

Answering a question of Geelen, Gerards, Robertson and Whittle [1], we prove that the branchwidth of a bridgeless graph is equal to the branchwidth of its cycle matroid. Our proof is based on branch-decompositions of hypergraphs. By matroid duality, a direct corollary of this result is that the branchwidth of a bridgeless planar graph is equal to the branchwidth of its planar dual. This consequence was a direct corollary of a result by Seymour and Thomas [4].

1 Introduction.

The notion of branchwidth was introduced by Robertson and Seymour in their seminal paper Graph Minors X [3]. Very roughly speaking, the goal is to decompose a structure S along a tree \mathcal{T} in such a way that subsets of S corresponding to disjoint branches of \mathcal{T} are pairwise as disjoint as possible. One can define the branchwidth of various structures like graphs, hypergraphs, matroids, submodular functions... Our goal in this paper is to prove that the definitions of branchwidth for graphs and matroids coincide in the sense that the branchwidth of a bridgeless graph is equal to the branchwidth of its cycle matroid. Let us now define properly what these definitions are.

Let $H = (V, E)$ be a graph, or a hypergraph, and (E_1, E_2) be a partition of E . The *border* of (E_1, E_2) is the set of vertices $\delta(E_1, E_2)$ which belong to both an edge of E_1 and an edge of E_2 . We write it $\delta(E_1, E_2)$, or simply $\delta(E_1)$.

A *branch-decomposition* \mathcal{T} of H is a ternary tree \mathcal{T} and a bijection from the set of leaves of \mathcal{T} into the set of edges of H . Practically, we simply identify the leaves of \mathcal{T} with the edges of H . Observe that every edge e of \mathcal{T} partitions $\mathcal{T} \setminus e$ into two subtrees, and thus corresponds to a bipartition of E , called *e -separation*. More generally, a *\mathcal{T} -separation* is an e -separation for some edge e of \mathcal{T} . We will often identify the edge e of \mathcal{T} with the e -separation, allowing us to write, for instance, $\delta(e)$ instead of $\delta(E_1, E_2)$, where (E_1, E_2) is the e -separation. Let \mathcal{T} be a branch-decomposition of H . The *width* of \mathcal{T} , denoted by $w(\mathcal{T})$, is the maximum value of $|\delta(e)|$ for all edges e of \mathcal{T} . The *branchwidth* of H , denoted by $\text{bw}(H)$, is the minimum width of a branch-decomposition of H . A branch-decomposition achieving $\text{bw}(H)$ is *optimal*.

Let us now turn to matroids. Let M be a matroid on base set E with rank function r . The *width* of every non-trivial partition (E_1, E_2) of E is $w_m(E_1, E_2) := r(E_1) + r(E_2) - r(E) + 1$. When \mathcal{T} is a branch-decomposition of M , i.e. a ternary tree whose leaves are labelled by E , the *width* $w_m(\mathcal{T})$ of \mathcal{T} is the maximum width of a \mathcal{T} -separation. Again, the *branchwidth* $\text{bw}_m(M)$ of M is the minimum width of a branch-decomposition of M . One nice fact about branchwidth is that it is invariant under matroid duality (recall that the bases of the dual matroid M^* of M are the complement of the bases of M). Indeed, since $r_{M^*}(U) = |U| + r_M(E \setminus U) - r_M(E)$, the width of any partition (E_1, E_2) is the same in M and in M^* . Remark that since branchwidth is a measure of how complicated the matroid is, it is a good fact that

M and M^* have the same branchwidth.

Having defined both the branchwidth of a graph and of a matroid, a very natural question is to compare them when the matroid M is precisely the cycle matroid of a graph G , i.e. the matroid M_G which base set is the set of edges of G and which independent sets are the acyclic subsets of edges. A first observation is that they differ, for instance the branchwidth of the path of length three is 2 whereas the branchwidth of its cycle matroid is 1. The inequality $\text{bw}(M_G) \leq \text{bw}(G)$ always holds, and simply comes from the fact that $w_m(E_1, E_2) \leq |\delta(E_1, E_2)|$ for every partition of E which has a nonempty border. To see this, define, when $H = (V, E)$ is a hypergraph, a *component* of E to be a minimum nonempty subset $C \subseteq E$ such that $\delta(C) = \emptyset$. Let F be a subset of E . We denote by $c(F)$ the number of components of the subhypergraph of H spanned by F , i.e. the hypergraph $(V(F), F)$. The hypergraph H is *connected* if $c(E) = 1$ and is moreover *bridgeless* if $c(E \setminus e) = 1$ for all $e \in E$ (since our definition is based on edges, we may have vertices with degree 0 or 1 in a connected bridgeless hypergraph). Observe now that:

$$w_m(E_1, E_2) = r(E_1) + r(E_2) - r(E) + 1 = n_1 - c(E_1) + n_2 - c(E_2) - n + c(E) + 1,$$

where n_1, n_2, n are the number of vertices respectively spanned by E_1, E_2, E . In particular,

$$w_m(E_1, E_2) = |\delta(E_1, E_2)| + c(E) + 1 - c(E_1) - c(E_2) \leq |\delta(E_1, E_2)|,$$

since $c(E) + 1 - c(E_1) - c(E_2) \leq 0$ when $\delta(E_1, E_2)$ is not empty.

Let us define a new branchwidth, the *matroid branchwidth* $\text{bw}_m(H)$ of a hypergraph H in which the separations (E_1, E_2) are evaluated with the function $w_m(E_1, E_2) = |\delta(E_1, E_2)| + 1 + c(E) - c(E_1) - c(E_2)$ instead of the function $|\delta(E_1, E_2)|$. We also write $w_m(E_1)$ instead of $w_m(E_1, E_2)$. In particular, when G is a graph, we have $\text{bw}_m(G) = \text{bw}_m(M_G)$. The main result of this paper is to prove that when H is connected and bridgeless, there exists a branch-decomposition \mathcal{T} of H achieving $\text{bw}_m(H)$ such that every \mathcal{T} -separation (E_1, E_2) is such that $c(E_1) = c(E_2) = 1$.

Thus we have $w(\mathcal{T}) = w_m(\mathcal{T})$, and since $\text{bw}_m(H) \leq \text{bw}(H)$ and \mathcal{T} is optimal, we have $\text{bw}_m(H) = \text{bw}(H)$. This implies in particular that the branchwidth of a bridgeless graph is equal to the branchwidth of its cycle matroid. Moreover, the case $\text{bw}(G) > \text{bw}_m(M_G)$ happens if and only if the graph G has a bridge, $\text{bw}_m(M_G) = 1$ and $\text{bw}(G) = 2$. In other words G is a tree which is not a star.

Another consequence of our result concerns planar graphs. The key-fact here is that planar duality corresponds to matroid duality, in other words, when G is planar and G^* is the planar dual of G , we have $(M_G)^* = M_{G^*}$. Therefore, when G is a planar bridgeless graph, we derive:

$$\text{bw}(G) = \text{bw}_m(M_G) = \text{bw}_m((M_G)^*) = \text{bw}_m(M_{G^*}) = \text{bw}(G^*).$$

Which is a new proof of the fact that for bridgeless graphs, the branchwidth is invariant under taking planar duality. The first proof of this result was from Seymour and Thomas in [4].

The paper is organized as follows. In Section 2 and 3, we analyze the properties of a possible minimal counterexample H of our main theorem. We get more and more structure, step by step. At the end of Section 3, the hypergraph H is very

constrained, tripartite, triangle-free, etc..., but no further step seems to conclude. The contradiction is achieved via a particular separation of H . The existence of such a separation relies on a (technical) partition lemma on multigraphs, the proof of which is postponed in Section 4.

Unless stated otherwise, we always assume that \mathcal{T} is a branch-decomposition of a hypergraph $H = (V, E)$. Also, when speaking about width, branchwidth, etc, we implicitly mean the matroid one.

2 Faithful branch-decompositions.

Let (E_1, E_2) be a \mathcal{T} -separation. The decomposition \mathcal{T} is *faithful* to E_1 if for every component C of E_1 , the partition $(C, E \setminus C)$ is a \mathcal{T} -separation. The *border graph* $G_{\mathcal{T}}$ has vertex set V and contains all edges xy for which there exists a \mathcal{T} -separation e such that $\{x, y\} \subseteq \delta(e)$. A branch-decomposition \mathcal{T}' is *tighter* than \mathcal{T} if $w_m(\mathcal{T}') < w_m(\mathcal{T})$ or if $w_m(\mathcal{T}) = w_m(\mathcal{T}')$ and $G_{\mathcal{T}'}$ is a subgraph of $G_{\mathcal{T}}$. Moreover, \mathcal{T}' is *strictly tighter* than \mathcal{T} if \mathcal{T}' is tighter than \mathcal{T} , and \mathcal{T} is not tighter than \mathcal{T}' . Finally, \mathcal{T} is *tight* if no \mathcal{T}' is strictly tighter than \mathcal{T} .

Lemma 1 *Let (E_1, E_2) be a partition of E . For any union E'_1 of connected components of E_1 and E_2 , we both have $\delta(E'_1) \subseteq \delta(E_1)$ and $w_m(E'_1) \leq w_m(E_1)$.*

Proof. Clearly, $\delta(E'_1) \subseteq \delta(E_1)$. Moreover, every vertex of $\delta(E_1)$ belongs to one component of E_1 and one component of E_2 . Therefore, if C is a component of E'_1 which is the union of k components of E_1 and E_2 , there are at least $k - 1$ vertices of $C \setminus \delta(C)$ which belong to $\delta(E_1)$. In all, the weight of the separation increased by $k - 1$ since we merge k components into one, but it also decreased by at least $k - 1$ since we lose at least that many vertices on the border. Since this is the case for every component of E'_1 or of $E \setminus E'_1$, we have $w_m(E'_1) \leq w_m(E_1)$. ■

Lemma 2 *Let (E_1, E_2) be an e -separation of \mathcal{T} . Let \mathcal{T}_1 be the subtree of $\mathcal{T} \setminus e$ with set of leaves E_1 . If \mathcal{T} is not faithful to E_1 , we can modify \mathcal{T}_1 in \mathcal{T} to form a tighter branch-decomposition \mathcal{T}' of H which is faithful to E_1 .*

Proof. Fix the vertex $e \cap \mathcal{T}_1$ as a root of \mathcal{T}_1 . Our goal is to change the binary rooted tree \mathcal{T}_1 into another binary rooted tree \mathcal{T}'_1 . For every connected component C of E_1 , consider the subtree \mathcal{T}_C of \mathcal{T}_1 which contains the root of \mathcal{T}_1 and has set of leaves C . Observe that \mathcal{T}_C is not necessarily binary since \mathcal{T}_C may contain paths having internal vertices with only one descendant. We simply replace these paths by edges to obtain our rooted tree \mathcal{T}'_C . Now, consider any rooted binary tree BT with $c(E_1)$ leaves and identify these leaves to the roots of \mathcal{T}'_C , for all components C of E_1 . This rooted binary tree is our \mathcal{T}'_1 . We denote by \mathcal{T}' the branch-decomposition we obtain from \mathcal{T} by replacing \mathcal{T}_1 by \mathcal{T}'_1 . Roughly speaking, we merged all subtrees of \mathcal{T}_1 induced by the components of E_1 together with $\mathcal{T} \setminus \mathcal{T}_1$ to form \mathcal{T}' . Let us prove that \mathcal{T}' is tighter than \mathcal{T} . For this, consider an edge f' of \mathcal{T}' . If $f' \notin \mathcal{T}'_1$, the f' -separations of \mathcal{T} and \mathcal{T}' are the same. If $f' \in BT$, by Lemma 1, we have $w_m(f') \leq w_m(e)$ and $\delta(f') \subseteq \delta(e)$. So the only case we have to care of is when f' is an edge of some tree \mathcal{T}'_C , where C is a component of E_1 . Recall that f' corresponds to a path P of \mathcal{T}_C .

Let f be any edge of P . Let $(F, E \setminus F)$ be the f -separation of \mathcal{T} , where $F \subseteq E_1$. Therefore, the f' -separation of \mathcal{T}' is $(F \cap C, E \setminus (F \cap C))$. Since F is a subset of E_1 , the connected components of F are subsets of the connected components of E_1 . Thus $F \cap C$ is a union of connected components of F . By Lemma 1, we have $\delta(f') \subseteq \delta(f)$ and $w_m(f') \leq w_m(f)$.

We have proved that $w(\mathcal{T}') \leq w(\mathcal{T})$ and that $G_{\mathcal{T}'}$ is a subgraph of $G_{\mathcal{T}}$, thus \mathcal{T}' is tighter than \mathcal{T} . \blacksquare

3 Connected branch-decompositions.

Let $F \subseteq E$ be a component. The hypergraph on vertex set V and edge set $(E \setminus F) \cup \{V(F)\}$ is denoted by $H * F$. In other words, $H * F$ is obtained by merging the edges of F into one edge. A partition (E_1, E_2) of E is *connected* if $c(E_1) = c(E_2) = 1$. A branch-decomposition \mathcal{T} is *connected* if all its \mathcal{T} -separations are connected.

Lemma 3 *If \mathcal{T} is tight, every \mathcal{T} -separation (E_1, E_2) is such that E_1 or E_2 is connected.*

Proof. Suppose for contradiction that there exists a \mathcal{T} -separation (E_1, E_2) such that neither E_1 nor E_2 is connected. By Lemma 2, we can assume that \mathcal{T} is faithful to E_1 and to E_2 . Let \mathcal{C}_1 and \mathcal{C}_2 be respectively the sets of components of E_1 and E_2 . Consider the graph on set of vertices $\mathcal{C}_1 \cup \mathcal{C}_2$ where $C_1 C_2$ is an edge whenever $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ have nonempty intersection. This graph is connected and is not a star. Thus, it has a vertex-partition into two connected subgraphs, each having at least two vertices. This vertex-partition corresponds to a partition $(\mathcal{C}'_1, \mathcal{C}'_2)$ of $\mathcal{C}_1 \cup \mathcal{C}_2$.

Consider any rooted binary tree BT with $|\mathcal{C}'_1|$ leaves. Since every $C \in \mathcal{C}'_1$ is an element of $\mathcal{C}_1 \cup \mathcal{C}_2$, $(C, E \setminus C)$ is an e -separation of \mathcal{T} . We denote by \mathcal{T}_C the tree of $\mathcal{T} \setminus e$ with set of leaves C . Root \mathcal{T}_C with the vertex $e \cap \mathcal{T}_C$ in order to get a binary rooted tree. Now identify the leaves of BT with the roots of \mathcal{T}_C , for $C \in \mathcal{C}'_1$. This rooted tree is our \mathcal{T}'_1 . We construct similarly \mathcal{T}'_2 . Adding an edge between the roots of \mathcal{T}'_1 and \mathcal{T}'_2 gives the branch-decomposition \mathcal{T}' of H . By Lemma 1, $w(\mathcal{T}') \leq w(\mathcal{T})$ and $G_{\mathcal{T}'}$ is a subgraph of $G_{\mathcal{T}}$. Let us now show that $G_{\mathcal{T}'}$ is a strict subgraph of $G_{\mathcal{T}}$. Indeed, since \mathcal{C}'_1 is connected and has at least two elements, it contains $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2$ is nonempty. By construction, every vertex x of $C_1 \cap C_2$ is such that $x \notin \delta(\mathcal{C}'_1)$ and $x \in \delta(\mathcal{C}_1)$. Similarly, there is a vertex y spanned by \mathcal{C}'_2 such that $y \notin \delta(\mathcal{C}'_2)$ and $y \in \delta(\mathcal{C}_2)$. Thus xy is an edge of $G_{\mathcal{T}}$ but not of $G_{\mathcal{T}'}$, contradicting the fact that \mathcal{T} is tight. \blacksquare

Theorem 1 *For every branch-decomposition \mathcal{T} of a connected hypergraph H , there exists a tighter branch-decomposition \mathcal{T}' such that for every \mathcal{T}' -separation (E_1, E_2) with $c(E_1) > 1$, E_1 consists of components of $H \setminus e$, for some $e \in E_2$. In particular, if H is bridgeless, it has an optimal connected branch-decomposition.*

Proof. Let us prove the theorem by induction on $|V| + |E|$. The statement is obvious if $|E| \leq 3$, so we assume now that H has at least four edges. Call *achieved* a branch-decomposition satisfying the conclusion of Theorem 1. If \mathcal{T} is not tight, we

can replace it by a tight branch-decomposition tighter than \mathcal{T} . So we may assume that \mathcal{T} is tight.

If there is an edge $e \in E$ such that $H \setminus e$ is not connected, we can assume by Lemma 2 that \mathcal{T} is faithful to $E \setminus e$. Let E_1 be a connected component of $E \setminus e$. Let \mathcal{T}_1 be the branch-decomposition induced by \mathcal{T} on $E_1 \cup e$. Let also \mathcal{T}_2 be the branch-decomposition induced by \mathcal{T} on $E \setminus E_1$. Observe that both $E_1 \cup e$ and $E \setminus E_1$ are connected, so by the induction hypothesis, there exists two achieved branch-decompositions \mathcal{T}'_1 and \mathcal{T}'_2 , respectively tighter than \mathcal{T}_1 and \mathcal{T}_2 . Identify the leaf e of the trees \mathcal{T}'_1 and \mathcal{T}'_2 , and attach a leaf labelled by e to the identified vertex. Call \mathcal{T}' this branch-decomposition of H . Observe that it is tighter than \mathcal{T} and achieved.

So we assume now that H is bridgeless. We can also assume that all the vertices of H have degree at least two, since we can simply delete the vertices of H with degree 0 or 1, and apply induction. The key-observation is that if there is a connected \mathcal{T} -separation (E_1, E_2) with $|E_1| \geq 2$ and $|E_2| \geq 2$, we can apply the induction hypothesis on $H * E_1$ and $H * E_2$ and merge the two branch-decompositions to obtain an optimal connected branch-decomposition of H . Therefore, we assume that every \mathcal{T} -separation (E_1, E_2) with $|E_1| \geq 2$ and $|E_2| \geq 2$ is such that E_1 or E_2 is not connected. We now orient the edges of \mathcal{T} . If (E_1, E_2) is an e -separation such that E_2 is connected and $|E_2| > 1$, we orient e from E_1 to E_2 . Since H is bridgeless, every edge of \mathcal{T} incident to a leaf is oriented from the leaf. By Lemma 3, every edge get at least one orientation. And by the key-observation, every edge of \mathcal{T} has exactly one orientation.

This orientation of \mathcal{T} has no circuit, thus there is a vertex $t \in \mathcal{T}$ with outdegree zero. Since every leaf has outdegree one, t has indegree three. Let us denote by A, B, C the set of leaves of the three trees of $\mathcal{T} \setminus t$. Observe that by construction, $A \cup B$, $A \cup C$ and $B \cup C$ are connected. By Lemma 2, we can assume moreover that \mathcal{T} is faithful to A, B and C . We claim that A is a disjoint union of edges, i.e. the connected components of A are edges of H . To see this, assume for contradiction that a component C_A of A is not an edge of H . Since \mathcal{T} is faithful to A , $(C_A, E \setminus C_A)$ is a \mathcal{T} -separation. But this is simply impossible since $B \cup C$ being connected, $E \setminus C_A$ is also connected, against the fact that every edge of \mathcal{T} has a unique orientation. So the hypergraph H consists of three sets of disjoint edges A, B, C . Call this partition the *canonical partition* of H . Call $(A, E \setminus A)$, $(B, E \setminus B)$ and $(C, E \setminus C)$ the *main* \mathcal{T} -separations. Note that the width of every other \mathcal{T} -separation is at most $\text{bw}_m(H)$. Since every vertex of H belongs to two or three edges, it is spanned by at least two of the sets $\delta(A), \delta(B), \delta(C)$. In particular $G_{\mathcal{T}}$ is a complete graph, and thus every optimal branch-decomposition of H is tighter than \mathcal{T} . Therefore, every optimal branch-decomposition of H has a canonical partition, otherwise we can conclude by induction. Set $\delta_{AB} := |\delta(A) \cap \delta(B)|$, $\delta_{AC} := |\delta(A) \cap \delta(C)|$, $\delta_{BC} := |\delta(B) \cap \delta(C)|$ and $\delta_{ABC} := |\delta(A) \cap \delta(B) \cap \delta(C)|$. We now prove some properties of H .

1. Two of the sets A, B, C have at least two edges. Indeed, assume for contradiction that $A = \{a\}$ and $B = \{b\}$. Since $|E| \geq 4$, there are at least two edges in C . Let $c \in C$. Observe that c intersects both a and b since $A \cup C$ and $B \cup C$ are connected. Assume without loss of generality that $|a \cap c| \geq |b \cap c|$. Now form a new branch-decomposition \mathcal{T}' by *moving* c to A , i.e. \mathcal{T}' has a

separation $(A \cup c, B \cup (C \setminus c))$ and then four branches $A, c, B, (C \setminus c)$. We have

$$w_m(A \cup c, B \cup (C \setminus c)) \leq |\delta(A)| + |b \cap c| - |a \cap c| \leq |\delta(A)| \leq w_m(A)$$

since both parts are connected. In particular \mathcal{T}' is tighter than \mathcal{T} , and since the \mathcal{T}' -separation $(A \cup c, B \cup (C \setminus c))$ is connected and both of its branches have at least two vertices, we can apply induction to conclude.

2. Every set A, B, C have at least two edges. Indeed, assume for contradiction that A consists of a single edge a . Let b be an edge of B . If $|b \cap \delta(C)| \leq |b \cap a|$, we can as previously move b to A in order to form a tighter branch-decomposition \mathcal{T}' . If moreover $(B \cup C) \setminus b$ is connected, we are done since we have now a connected separation $(A \cup b, (B \cup C) \setminus b)$, on which we can apply induction. If $(B \cup C) \setminus b$ is not connected, the canonical partition of \mathcal{T}' must be $A, b, (B \cup C) \setminus b$ since all of these pair of branches are connected. But this is impossible since $(B \cup C) \setminus b$ does not consist of disjoint edges, which should be the case in a canonical partition. Call $|b \cap \delta(C)| - |b \cap a|$ the *excess* of b . Similarly, call $|c \cap \delta(B)| - |c \cap a|$ the *excess* of an edge $c \in C$. Let s be the minimum excess of an edge e_s of $B \cup C$. Observe that $s \geq 1$ and that every $b \in B$ satisfies $|b \cap \delta(C)| \geq |b \cap a| + s$. Thus, summing for all edges of B , we obtain $\delta_{BC} \geq \delta_{AB} + s|B|$. Similarly, $\delta_{BC} \geq \delta_{AC} + s|C|$. Note also that $\text{bw}_m(H) \geq w_m(C) = \delta_{BC} + \delta_{AC} - \delta_{ABC} - |C| + 1$ and $\text{bw}_m(H) \geq w_m(B) = \delta_{BC} + \delta_{AB} - \delta_{ABC} - |B| + 1$. In all

$$2 \text{bw}_m(H) \geq 2\delta_{BC} - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2.$$

Then $2 \text{bw}_m(H) \geq \delta_{AB} + s|B| + \delta_{AC} + s|C| - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2$. Finally, $\text{bw}_m(H) \geq \delta_{AC} + \delta_{AB} - \delta_{ABC} + 1 + ((s-1)|C| + (s-1)|B|)/2$. Since $|\delta(A)| = \delta_{AC} + \delta_{AB} - \delta_{ABC}$, we have $\text{bw}_m(H) \geq |\delta(A)| + s$. But then we can move e_s to A to conclude.

3. We have $\text{bw}_m(H) = w_m(A)$. If not, pick two edges a, a' of A and merge them together. The hypergraph we obtain is still connected and bridgeless, and the branch-decomposition still has the same width. Apply induction to get an achieved branch-decomposition. Then replace the merged edge by the two original edges. This branch-decomposition \mathcal{T}' is optimal with canonical partition $\{a\}, \{a'\}, E \setminus \{a, a'\}$. Thus by property 1, we can apply induction. Similarly, $\text{bw}_m(H) = w_m(B) = w_m(C)$.
4. We have $\text{bw}_m(H) \geq \beta + 1$, where β is the size of a maximum edge of H . Observe that H has no edge of size one. Indeed if such an edge e belongs to, say, A , it is also included in another edge, say in B . But then moving e to B would give our conclusion. So the size of an edge of H is at least two. Assume for instance that $e \in A$ has size β . Since A has at least one component of size β and one of size 2, we have $\text{bw}_m(H) = w_m(A) \geq |\delta(A)| - c(B \cup C) - c(A) + 2 = |\delta(A)| - |A| + 1 \geq \beta + 1$.
5. We have $\delta_{ABC} = 0$. Indeed, suppose for contradiction that there exists a vertex z in $\delta(A) \cap \delta(B) \cap \delta(C)$. Consider the hypergraph H_z obtained from H by

removing the vertex z from all its edges. Observe that H_z is connected since z is incident to three edges and H is bridgeless. The branch-decomposition \mathcal{T} induces a branch-decomposition \mathcal{T}_z of H_z having width at most $w_m(\mathcal{T}) - 1$. We apply induction on \mathcal{T}_z to obtain an achieved branch-decomposition \mathcal{T}'_z of H_z . Now add back the vertex z to the edges of H_z and call \mathcal{T}' the branch-decomposition obtained from \mathcal{T}'_z . Let us show that \mathcal{T}' is optimal. Observe that if a \mathcal{T}'_z -separation (E_1, E_2) is connected, adding z will raise by at most one its width in \mathcal{T}' . Moreover if a \mathcal{T}'_z -separation (E_1, E_2) is not connected, say $c(E_2) > 1$, adding z can raise by at most two its width in \mathcal{T}' (either by merging three components of E_2 into one, or by merging two and increasing the border by one). Since \mathcal{T}'_z is achieved, E_2 is a set of components of $E \setminus e$ for some edge e of H_z . But then in \mathcal{T}'_z , we have $w_m(E_1, E_2) \leq |\delta(E_2)| - 3 + 2 \leq |e| - 1 \leq \beta - 1 \leq \text{bw}_m(H) - 2$, and thus $w_m(E_1, E_2) \leq \text{bw}_m(H)$ in \mathcal{T}' . Therefore \mathcal{T}' is optimal. Moreover every \mathcal{T}' -separation (E_1, E_2) is connected. Indeed, if (E_1, E_2) is connected in \mathcal{T}'_z , we are done. If E_1 is not connected in \mathcal{T}'_z , E_1 consists of components of $H_z \setminus e$, for some edge e of H_z . But since H is bridgeless, every component of E_1 in H must contain z , otherwise they would be components of $H \setminus e$. Consequently E_1 is connected in H .

6. Every edge of H is incident to at least four other edges. Indeed, assume for contradiction that an edge a of A is incident to only one edge b of B and at most two edges of C (the case where a is only incident to edges of C is obvious, we just move a to C). Moving a to B increases $w_m(B)$ by $|a \cap \delta(C)| - |a \cap b|$ and does not increase $w_m(A)$ and $w_m(C)$. Therefore, if $|a \cap \delta(C)| \leq |a \cap b|$, we can move a to B , and this new branch-decomposition \mathcal{T}' is strictly tighter than \mathcal{T} since the vertices of $a \cap b$ are no more joined to $(\delta(A) \setminus a) \cap \delta(C)$ in the graph $G_{\mathcal{T}'}$. Thus $|a \cap \delta(C)| \geq |a \cap b| + 1$. Moreover, moving a to C increases $w_m(C)$ by at most $|a \cap b| - |a \cap \delta(C)| + 1$, since at most two components of C can merge. So $|a \cap b| + 1 > |a \cap \delta(C)|$, a contradiction. This implies in particular that the size of any edge is at least four. In particular, $w_m(e) \leq \text{bw}_m(H) - 3$ whenever e is not one of the main \mathcal{T} -separations. Therefore $\beta \leq \text{bw}_m(H) - 3$.
7. The hypergraph H is triangle-free. Indeed, suppose that there exists three edges $a \in A$, $b \in B$ and $c \in C$ and three vertices $x \in a \cap b$, $y \in b \cap c$ and $z \in c \cap a$. Let H/xyz be the hypergraph obtained by contracting x, y, z to a single vertex v . The branch-decomposition \mathcal{T} induces a branch-decomposition \mathcal{T}/xyz of H/xyz . Note that H/xyz is still connected and bridgeless, and that $w_m(\mathcal{T}/xyz) = w_m(\mathcal{T}) - 1$ since we decrease by one the border of every main separation. By induction, we can find an achieved branch-decomposition \mathcal{T}' of H/xyz which is tighter than \mathcal{T}/xyz . We claim that \mathcal{T}' is also an achieved branch-decomposition of H . Consider for this a \mathcal{T}' -separation (E_1, E_2) of E . If a, b, c belong to the same part, say E_1 , the width of (E_1, E_2) is the same in H/xyz and in H . If a, b belong to one part and c to the other, the width of (E_1, E_2) is one less in H/xyz than in H . Thus $\text{bw}_m(H) \leq \text{bw}_m(H/xyz) + 1$, and in particular \mathcal{T}' is optimal. Finally, since (E_1, E_2) is connected in H/xyz , it is also connected in H . Thus, \mathcal{T}' is achieved.

Now we are ready to finish the proof. Note that $\text{bw}_m(H) = (\text{w}_m(A) + \text{w}_m(B) + \text{w}_m(C))/3 = (2|V| - |E|)/3 + 1$. Consider the line multigraph $L(H)$ of H , i.e. the multigraph on vertex set $A \cup B \cup C$ and edge set V such that $v \in V$ is the edge which joins the two edges e, f of H such that $v \in e$ and $v \in f$. The multigraph $L(H)$ satisfies the hypothesis of Lemma 4 (proved in the next section), thus it admits a vertex-partition of its vertices as in the conclusion of Lemma 4. This corresponds to a partition of $A \cup B \cup C$ into two subsets $E_1 := A_1 \cup B_1 \cup C_1$ and $E_2 := A_2 \cup B_2 \cup C_2$ such that $|\delta(E_1, E_2)| \leq (2|V| - |E|)/3 + 1$ and both E_1 and E_2 have at least $\lfloor |E|/2 \rfloor - 1$ internal vertices. In particular, the separation (E_1, E_2) has width at most $\text{bw}_m(H)$. Let us show that one of $\text{w}_m(A_1 \cup B_1)$, $\text{w}_m(B_1 \cup C_1)$, and $\text{w}_m(C_1 \cup A_1)$ is also at most $\text{bw}_m(H)$. For this, observe that the set $\delta(A_1 \cup B_1) \cup \delta(B_1 \cup C_1) \cup \delta(C_1 \cup A_1)$ covers twice every vertex of V which is not an internal vertex of E_2 . Thus

$$|\delta(A_1 \cup B_1)| + |\delta(B_1 \cup C_1)| + |\delta(C_1 \cup A_1)| \leq 2|V| - 2\lfloor |E|/2 \rfloor + 2 \leq 2|V| - |E| + 3.$$

Without loss of generality, we can assume that $\delta(A_1 \cup B_1) \leq (2|V| - |E|)/3 + 1 = \text{bw}_m(H)$, and thus we split E_1 into two branches $A_1 \cup B_1$ and C_1 . We similarly split E_2 to obtain an optimal branch-decomposition \mathcal{T}' of H . Observe that in the graph $G_{\mathcal{T}'}$, there is no edge between the internal vertices of E_1 and E_2 . This contradicts the fact that \mathcal{T} is tight. ■

4 The partition Lemma

Let G be a multigraph on vertex set V and X, Y two subsets V . We denote by $e(X, Y)$ the number of edges of G between X and Y . We also denote by $e(X)$ the number of edges in X . The *degree* of a vertex x in a subset Y of G is $d_Y(x) := e(x, Y)$. When $Y = V$, we simply note $d(x)$. The *underlying degree* of x in Y is the number of neighbors of x in Y , i.e. we forget the multiplicity of edges. A graph is *2-connected* if it is connected and the removal of any vertex leaves it connected.

Lemma 4 *Let G be a 2-connected triangle-free multigraph on $n \geq 5$ vertices and m edges. Assume that its minimum underlying degree is at least four and that its maximum degree is at most $(2m - n)/3 + 1$. There exists a partition (X, Y) of the vertex set of G such that $e(X) \geq \lfloor n/2 \rfloor - 1$, $e(Y) \geq \lfloor n/2 \rfloor - 1$ and $e(X, Y) \leq (2m - n)/3 + 1$.*

Proof. Call *good* a partition which satisfies the conclusion of Lemma 4. Assume first that there are vertices x, y such that $e(x, y) \geq \lfloor n/2 \rfloor - 1$. The minimum degree in $V \setminus \{x, y\}$ is at least two, so $e(V \setminus \{x, y\})$ is at least $n - 2$ and hence at least $\lfloor n/2 \rfloor - 1$. Thus, if the partition $(V \setminus \{x, y\}, \{x, y\})$ is not good, we necessarily have $d(x) + d(y) - 2e(x, y) > (2m - n)/3 + 1$. By the maximum degree hypothesis, both $d(x)$ and $d(y)$ are greater than $2e(x, y)$. Since G is triangle-free, there exists a partition (X, Y) where $(N(x) \cup x) \setminus y \subseteq X$ and $(N(y) \cup y) \setminus x \subseteq Y$. Observe that $e(X) \geq d(x) - e(x, y) > e(x, y) \geq \lfloor n/2 \rfloor - 1$. Similarly $e(Y) \geq \lfloor n/2 \rfloor - 1$. Moreover, since $m \geq 2n$ by the minimum degree four hypothesis, we have

$$e(X, Y) \leq m - (d(x) + d(y) - 2e(x, y)) < m - (2m - n)/3 - 1 \leq (m + n)/3 - 1.$$

Thus $e(X, Y) \leq (2m - n)/3 + 1$ and (X, Y) is a good partition. We assume from now on that the multiplicity of an edge is less than $\lfloor n/2 \rfloor - 1$.

Let $a + b = n$, where $a \leq b$. A partition (X, Y) of V is an *a-partition* if $|X| \leq a$, $e(X) \geq a - 1$, $e(Y) \geq b - 1$, $e(X, Y) \leq (2m - n)/3 + 1$, and the additional requirement that X contains a vertex of G with maximum degree.

Note that there exists a 1-partition, just consider for this $X := \{x\}$, where x has maximum degree in G (the minimum degree in Y is at least three, insuring that $e(Y) \geq n - 2$). We consider now an *a-partition* (X, Y) with maximum a . If $a \geq b - 1$, this partition is good and we are done. So we assume that $a < b - 1$. In particular $e(X) = a - 1$.

The key-observation is that there exists at most one vertex y of Y such that $e(Y \setminus y) < b - 2$. Indeed, if there is a vertex of Y with degree one in Y , we simply move it to X , and we obtain an $(a + 1)$ -partition ($e(X)$ increases, $e(Y)$ decreases by one, and $e(X, Y)$ decreases). Thus the minimum degree in Y is at least two, and hence $e(Y) \geq |Y|$. Moreover, if there is a vertex z of Y with degree two in Y , we can still move it to X : indeed $e(X)$ increases, $e(Y \setminus z) \geq |Y| - 2$ and $e(X, Y)$ does not increase. So the minimum degree in Y is at least three (but the minimum underlying degree may be one). This implies that $e(Y) \geq 3|Y|/2$. Let $Y := \{y_1, \dots, y_{|Y|}\}$ where the vertices are indexed in the increasing order according to their degree in Y . For every $i \neq |Y|$, we have $e(Y) \geq (3(|Y| - 2) + d_Y(y_i) + d_Y(y_{|Y|}))/2$. Furthermore,

$$e(Y \setminus y_i) \geq (3(|Y| - 2) + d_Y(y_i) + d_Y(y_{|Y|}))/2 - d_Y(y_i) \geq 3(|Y| - 2)/2 \geq |Y| - 2 \geq b - 2.$$

We now discuss the two different cases depending if there exists $y \in Y$ such that $e(Y \setminus y) < b - 2$ or not. In the following, the *excess* of a vertex $y \in Y$ is $exc(y) := d_Y(y) - d_X(y)$.

- Assume that $e(Y \setminus y) \geq b - 2$ for every $y \in Y$. We denote by Y' the (nonempty) set of vertices of Y with at least one neighbour in X . We let $Y'' := Y \setminus Y'$, by definition every vertex of Y'' has underlying degree at least four in Y . Note that we can move a vertex of Y' to X if it does not have positive excess. Denote by c the minimum excess of a vertex of Y' , we have $c > 0$. The sum of the degrees of the vertices of Y' is at least $2e(X, Y) + c|Y'|$. Now, summing the degrees of all the vertices of Y , we get $2e(Y) + e(X, Y) \geq 4|Y''| + 2e(X, Y) + c|Y'|$, and hence:

$$2e(Y) \geq e(X, Y) + 4|Y''| + c|Y'| \quad (4.1)$$

Let $y \in Y'$ such that $exc(y) = c$. Since the partition $(X \cup y, Y \setminus y)$ is not an $(a + 1)$ -partition, we have $e(X, Y) + c > (2m - n)/3 + 1$. Since $m = e(X, Y) + e(X) + e(Y)$, this implies

$$e(X, Y) + 3c > 2e(X) + 2e(Y) - n + 3 \quad (4.2)$$

Equations (4.1) and (4.2) give:

$$3c > 2e(X) + 4|Y''| + c|Y'| - n + 3 \quad (4.3)$$

Since $e(X) \geq a - 1 \geq n - |Y| - 1$, we get $3c > n - 2|Y| + 4|Y''| + c|Y'| + 1$. From $|Y| = |Y'| + |Y''|$, we get $3c > n + 2|Y| + (c - 4)|Y'| + 1$, and finally $n + 2|Y| < (c - 4)(3 - |Y'|) + 11$. If $c = 4$, we get $n + 2|Y| \leq 10$, which is impossible since $n \geq 5$ and $|Y| > n/2$. If $c = 3$, we get $n + 2|Y| - |Y'| \leq 7$, again impossible. If $c = 2$, we get $n + 2|Y| - 2|Y'| \leq 4$, again impossible. If $c = 1$, we get $n + 2|Y| - 3|Y'| \leq 1$, which can only hold if $|Y| = |Y'| = n - 1$. Thus, X consists of a single vertex, completely joined to Y , against the fact that G is triangle-free and has minimum underlying degree 4. Finally $c > 4$, and consequently $|Y'| < 3$. Observe that $|Y'| > 1$ since G is 2-connected. Thus $|Y'| = 2$. Let y_1, y_2 be the vertices of Y' , indexed in such a way that $e(y_1, X) + e(y_2, Y'') \geq e(y_2, X) + e(y_1, Y'')$. Let $X_1 := X \cup y_1$ and $Y_1 := Y \setminus y_1$. Since $y_1 \in Y'$, we have that $e(X_1) \geq a$. Moreover $e(Y_1) \geq b - 2$. We claim that $e(y_1, y_2) \leq e(Y \setminus \{y_1, y_2\})$: this is obvious if $e(y_1, y_2) = 0$, and if there is an edge between y_1 and y_2 , since G has minimum underlying degree four, the minimum degree in $Y \setminus \{y_1, y_2\}$ is at least two. So

$$e(Y'') = e(Y \setminus \{y_1, y_2\}) \geq |Y| - 2 \geq \lfloor n/2 \rfloor - 1 \geq e(y_1, y_2).$$

Thus

$$e(X_1, Y_1) = e(y_1, y_2) + e(y_1, Y'') + e(y_2, X) \leq e(Y'') + e(y_2, Y'') + e(y_1, X).$$

In particular, $e(X_1, Y_1) \leq e(X_1) + e(Y_1) \leq m/2$ and since $m \geq 2n$, we have $e(X_1, Y_1) \leq (2m - n)/3 + 1$. So the partition (X_1, Y_1) is good.

- Now assume that there exists a vertex $y \in Y$ such that $e(Y \setminus y) \leq b - 3 \leq |Y| - 3$. We denote by Y' the set of vertices of $Y \setminus y$ with at least one neighbour in X . Set $Y'' := Y \setminus (Y' \cup y)$. Observe that since every vertex of Y'' has underlying degree four in Y , we have $e(Y \setminus y) \geq 3|Y''|/2$. Thus, $|Y''| \leq (2|Y| - 6)/3$. Since $|Y| > 3$, we have $|Y''| < |Y| - 3$, and finally $|Y'| \geq 3$. Denote by c the minimum excess of a vertex of Y' , again $c > 0$. Summing the degrees of the vertices of Y gives $2e(Y) \geq e(X, Y) + 4|Y''| + c|Y'| + exc(y)$. Equation (4.2) still holds, so

$$exc(y) < 3c + n - 3 - 2e(X) - 4|Y''| - c|Y'| \leq 3c - 1 - e(X) - 3|Y''| - (c - 1)|Y'|$$

since $e(X) + |Y''| + |Y'| \geq n - 2$. Therefore $exc(y) < -e(X) - 3|Y''| - (c - 1)(|Y'| - 3) + 2$. Since $|Y'| \geq 3$ and $c \geq 1$, we have $exc(y) \leq 1 - e(X)$. Recall that the minimum degree in Y is at least three, hence summing the degrees in Y of the vertices of $Y \setminus y$ gives that $3(|Y| - 1) \leq 2e(Y \setminus y) + d_Y(y) \leq 2b - 6 + d_Y(y)$. Finally, $d_Y(y) \geq |Y| + 3$ and by the fact that $exc(y) \leq 1 - e(X)$, we have $d_X(y) \geq |Y| + e(X) + 2$. In all, we have $d(y) \geq 2|Y| + e(X) + 5$. Recall that X contains a vertex x with maximum degree in G . In particular both x and y have degree at least $2|Y| + e(X) + 5$. Observe that $d_X(x)$ is at most $|e(X)|$, and consequently $d_Y(x)$ is at least $2|Y| + 5$. Now the end of the proof is straightforward, it suffices to switch x and y to obtain the good partition $(X_1, Y_1) := ((X \cup y) \setminus x, (Y \cup x) \setminus y)$. The only fact to care of is $e(x, y)$. Indeed if $e(x, y)$ is at most $e(X)$, we have:

1. $e(Y_1) \geq d_{Y_1}(x) \geq 2|Y| + 5 - e(x, y) \geq 2|Y| - e(X) \geq |Y| \geq n/2$.
2. $e(X_1) \geq d_{X_1}(y) \geq |Y| + e(X) + 2 - e(x, y) \geq n/2$.
3. Finally, since the excess of y is at most $1 - e(X)$, we have $d_{X_1}(y) + e(x, y) = d_X(y) \geq d_Y(y) + e(X) - 1$, hence $d_{X_1}(y) \geq d_Y(y) - 1$. Moreover $d_{Y_1}(x) \geq 2|Y| + 5 - e(X) \geq e(X) + 5 \geq d_X(x) + 5$. Thus, $e(X_1, Y_1) = e(X, Y) + d_Y(y) - d_{X_1}(y) + d_X(x) - d_{Y_1}(x) \leq e(X, Y) - 4$. Therefore $e(X_1, Y_1) \leq (2m - n)/3 + 1$, since (X, Y) is an a -partition.

To conclude, we just have to show that $e(x, y)$ is at most $e(X)$. Assume for contradiction that $e(x, y) \geq a$. We consider the partition into $X_2 := \{x, y\}$ and $Y_2 := V \setminus \{x, y\}$. Observe that the minimum underlying degree in Y_2 is at least two. Thus $e(Y_2) \geq n - 2 \geq b - 2$. By the maximality of a , (X_2, Y_2) is not an $(a + 1)$ -partition, therefore $e(X_2, Y_2) > (2m - n)/3 + 1$, hence

$$d(x) + d(y) - 2e(x, y) > (2m - n)/3 + 1. \quad (4.4)$$

To conclude, we now claim that any partition (X_3, Y_3) such that $(x \cup N(x)) \setminus y \subseteq X_3$ and $(y \cup N(y)) \setminus x \subseteq Y_3$ is good. Indeed, we have $e(X_3) \geq d(x) - e(x, y) \geq 2|Y| + e(X) + 5 - n/2 \geq n/2$. Similarly $e(Y_3) \geq n/2$. So, if (X_3, Y_3) is not good, we must have $e(X_3, Y_3) > (2m - n)/3 + 1$. Therefore $m - (d(x) + d(y) - 2e(x, y)) > (2m - n)/3 + 1$, and by Equation (4.4), we have $m > 2(2m - n)/3 + 2$. Finally $m < 2n - 6$ which is impossible since the minimum degree in G is at least four. ■

An independent proof of the equality of branchwidth of cycle matroids and graphs was also given by Hicks and McMurray [2]. Their method is based on matroid tangles and is slightly more involved than ours.

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