

Paths with two blocks in n -chromatic digraphs

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September 27, 2006

Abstract

We show that every oriented path of order $n \geq 4$ with two blocks is contained in every n -chromatic digraph.

1 Introduction

Gallai-Roy's celebrated theorem [11, 12] states that every n -chromatic digraph contains a directed path of length $n - 1$. More generally, one can ask which digraphs are contained in every n -chromatic digraph. Such digraphs are called n -universal. Since there exist n -chromatic graphs with arbitrarily large girth [7], n -universal digraphs must be oriented trees. Burr [3] proved that every oriented tree of order n is $(n - 1)^2$ -universal and he conjectured that every oriented tree of order n is $(2n - 2)$ -universal. This is a generalization of Sumner's conjecture which states that every oriented tree of order n is contained in every tournament (orientation of a complete graph) of order $2n - 2$. The first linear bound for tournaments was given by Häggkvist and Thomason [8]. The best bound so far, $3n - 3$, was obtained by El Sahili [5], refining an idea of [10].

Regarding oriented paths in general, there is no better upper bound than the one given by Burr for oriented trees. However in tournaments, Havet and Thomassé [9] proved that except for three particular cases, every tournament of order n contains every oriented path of order n .

A *path with two blocks* is an oriented path of order $k + l + 1$ starting with k forward arcs and followed by l backward arcs for some $k \geq 1$ and $l \geq 1$. We denote such a path by $P(k, l)$. El-Sahili conjectured [4] that every path of order $n \geq 4$ with two blocks is n -universal, and Bondy and El-Sahili [4] proved it if one of the two blocks has length one. The condition $n \geq 4$ is necessary because of odd circuits. Recently, El-Sahili and Kouider [6] showed that every path of order n with two blocks is $(n + 1)$ -universal.

In this paper, we show that every path of order $n \geq 4$ with two blocks is n -universal, proving El-Sahili's conjecture.

A natural question is to ask for cycles with two blocks instead of paths. In this context, Benhocine and Wojda [1] proved that every tournament on $n \geq 4$ vertices contains every cycle of order n with two blocks. As pointed out by Gyárfás and Thomassen, this does not extend to n -chromatic digraphs. Consider for this the following inductive construction: Let D_1 be the singleton digraph. Then, D_{i+1} is constructed starting with i disjoint copies C_1, \dots, C_i of D_i and adding, for every set X of i vertices, one

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in each C_i , a vertex dominated exactly by X . By construction, the chromatic number of D_i is exactly i and there are no cycle with two blocks.

However the digraphs D_i are not strongly connected and it is easy to see that every strongly connected digraph which is not a directed cycle contains two vertices x and y linked by two independent paths (i. e. having only x and y in common). We do not know if the strong connectivity condition ensures the existence of two vertices linked by two “long” independent paths.

Problem 1 Let D be an n -chromatic strongly connected digraph ($n \geq 4$) and k, l be positive integers such that $k + l = n$.

Does there exist two vertices of D which are linked by two independent paths P_1 and P_2 of length at least k and l respectively?

In other words, does there exists an oriented cycle with two blocks such that one block has length at least k and the other one length at least l ?

This problem maybe seen as an extension of Bondy’s theorem (Theorem 1) which proves this statement for directed cycles ($l = 0$).

2 Final spanning out-forests.

An *out-arborescence* T is an oriented tree having exactly one vertex r with in-degree zero. The vertex r is the *root* of T . An *out-forest* is a disjoint union of out-arborescences. Let F be an out-forest and x a vertex of F . The *level* of x is the number of vertices of a longest directed path of F ending at x . For instance, the level 1 vertices are the roots of the out-arborescences of F . We denote by F_i the set of vertices with level i in F . A vertex y is a *descendant* of x in F if there is a directed path from x to y in F .

Let F be a spanning out-forest of D . If there is an arc xy in D from F_i to F_j , with $i \geq j$, and x is not a descendant of y , then the out-forest F' obtained by adding xy and removing the arc of F with head y (if such exists that is if $j > 1$) is called an *elementary improvement* of F . An out-forest F' is an *improvement* of F if it can be obtained from an out-forest F by a sequence of elementary improvements. The key-observation is that if F' is an improvement of F then the level of every vertex in F' is at least its level in F . Moreover, at least one vertex of F has its level in F' strictly greater than its level in F . Thus, one cannot perform infinitely many improvements. A spanning out-forest F is *final* if there is no elementary improvement of F .

We say that x *dominates* y if xy is an arc of D . The following proposition follows immediately from the definition of final spanning out-forest:

Proposition 1 (El-Sahili and Kouider [6]) *Let D be a digraph and F a final spanning out-forest of D . If a vertex $x \in F_i$ dominates in D a vertex $y \in F_j$ for $j \leq i$ then x is a descendant of y in F . In particular, every level of F is a stable set in D .*

The notion of final forests is useful in the context of universal digraphs. As shown by El-Sahili and Kouider [6], it gives an easy proof of Gallai-Roy’s theorem. Indeed, consider a final spanning out-forest of an n -chromatic digraph D . Since every level is a stable set by Proposition 1, there are at least n levels. Hence D contains a directed path of length at least $n - 1$. Final forests are also useful for finding paths with two blocks, as illustrated by the following proof due to El-Sahili and Kouider [6].

Lemma 1 (El-Sahili and Kouider [6]) *Let F be a final spanning out-forest of a digraph D . We assume that there is an arc vw from F_i to F_j . Then*

- (i) *If $k \leq i < j - l$, then D contains a $P(k, l)$.*

(ii) If $k < j \leq i - l$, then D contains a $P(k, l)$.

Proof. (i) Let P_l be the directed path of F which starts at F_{j-l} and ends at w and P_{k-1} be the directed path in F starting at $F_{i-(k-1)}$ and ending at v . Then $P_{k-1} \cup vw \cup P_l$ is a $P(k, l)$.

(ii) Let P_{l-1} be the directed path in F which starts at F_{i-l+1} and ends at v . Let P_k be the directed path in F starting at F_{j-k} and ending at w . Then $P_k \cup P_{l-1} \cup vw$ is a $P(k, l)$. ■

Corollary 1 (El-Sahili and Kouider [6]) *Every digraph with chromatic number at least $k + l + 2$ contains a $P(k, l)$.*

Proof. Let F be a final spanning out-forest of D . Color the levels F_1, \dots, F_k of F with colors $1, \dots, k$. Then color the level F_i , where $i > k$, with color $j \in \{k + 1, \dots, k + l + 1\}$ such that $j \equiv i \pmod{l + 1}$. Since this is not a proper coloring, there exists an arc which satisfies the hypothesis of Lemma 1. ■

Our goal is now to extend this proof to the case of $(k + l + 1)$ -chromatic digraphs.

3 Good circuits; the strongly connected case.

Let us recall the following extension of Gallai-Roy's theorem to strongly connected digraphs:

Theorem 1 (Bondy [2]) *Every strongly connected digraph D has a circuit of length at least $\chi(D)$.*

Let $S \subset V(D)$ be a set of vertices. We denote by $D[S]$ the subdigraph induced by the vertices of S . Let k be a positive integer and D be a digraph. A directed circuit C of D is k -good if $|C| \geq k$ and $\chi(D[V(C)]) \leq k$. Note that Theorem 1 states that every strongly connected digraph D has a $\chi(D)$ -good circuit.

Lemma 2 *Let D be a strongly connected digraph and k be in $\{3, \dots, \chi(D)\}$. Then D has a k -good circuit.*

Proof. By Bondy's theorem, there exists a circuit with length at least $\chi(D)$, implying the claim for the value $k = \chi(D)$. Suppose $3 \leq k < \chi(D)$, in particular $\chi(D) > 3$. Let us now consider a shortest circuit C with length at least k . We claim that C is k -good. Suppose for contradiction that $\chi(D[V(C)]) \geq k + 1$. We may assume by induction on the number of vertices that $D = D[V(C)]$. Furthermore, if D contains a circuit of length 2, we can remove one of its arcs, in such a way that $\chi(D)$ and the circuit C are unchanged. Thus, we can assume that D has no circuit of length two, has a hamiltonian circuit C of length at least k , has chromatic number greater than k , and that every circuit of length at least k is hamiltonian. Our goal is to reach a contradiction.

We claim that every vertex u has in-degree at most $k - 2$ in D . Indeed, if v_1, \dots, v_{k-1} were in-neighbors of u , listed in such a way that v_1, \dots, v_{k-1}, u appear in this order along C , the circuit obtained by shortcutting C through the arc $v_{k-2}u$ would have length at least k since the out-neighbor of u in C is not an in-neighbor of u . This contradicts the minimality of C . The same argument gives that every vertex has out-degree at most $k - 2$ in D .

A *handle decomposition* of D is a sequence H_1, \dots, H_r such that:

- i) H_1 is a circuit of D .
- ii) For every $i = 2, \dots, r$, H_i is a handle, that is, a directed path of D (with possibly the same endvertices) starting and ending in $V(H_1 \cup \dots \cup H_{i-1})$ but with no inner vertex in this set.
- iii) $D = H_1 \cup \dots \cup H_r$.

An H_i which is an arc is a *trivial handle*. It is well-known that r is invariant for all handle decompositions of D (indeed, r is the number of arcs minus the number of vertices plus one). However the number of nontrivial handles is not invariant. Let us then consider H_1, \dots, H_r , a handle decomposition of D with minimum number of trivial handles. Free to enumerate first the nontrivial handles, we can

assume that H_1, \dots, H_p are not trivial and H_{p+1}, \dots, H_r are arcs. Let $D' := H_1 \cup \dots \cup H_p$. Clearly D' is a strongly connected spanning subgraph of D . Observe that since $\chi(D) > 3$, D is not an induced circuit, so in particular $p > 1$.

We denote by x_1, \dots, x_q the handle H_p minus its endvertices.

If $q = 1$, the digraph $D' \setminus x_1$ is strongly connected, and therefore $D \setminus x_1$ is also strongly connected. Moreover its chromatic number is at least k . Thus by Bondy's theorem, there exists a circuit of length at least k in $D \setminus x_1$. This circuit is not hamiltonian in D , a contradiction.

If $q = 2$, note that x_2 is the unique out-neighbor of x_1 in D , otherwise we would make two non trivial handles out of H_p , contradicting the maximality of the number of non trivial handles. Similarly, x_1 is the unique in-neighbor of x_2 . Since the outdegree and the indegree of every vertex is at most $k - 2$, both x_1 and x_2 have degree at most $k - 1$ in the underlying graph of D . Since $\chi(D) > k$, it follows that $\chi(D \setminus \{x_1, x_2\}) > k$. Since $D \setminus \{x_1, x_2\}$ is strongly connected, it contains, by Bondy's theorem, a circuit with length at least k , contradicting the minimality of C .

Hence, we may assume $q > 2$. For every $i = 1, \dots, q - 1$, by the maximality of p , the unique arc in D leaving $\{x_1, \dots, x_i\}$ is $x_i x_{i+1}$ (otherwise we would make two nontrivial handles out of H_p). Similarly, for every $j = 2, \dots, q$, the unique arc in D entering $\{x_j, \dots, x_q\}$ is $x_{j-1} x_j$. In particular, as for $q = 2$, x_1 has out-degree 1 in D and x_q has in-degree 1 in D .

Another consequence is that the underlying graph of $D \setminus \{x_1, x_q\}$ has two connected components $D_1 := D \setminus \{x_1, x_2, \dots, x_q\}$ and $D_2 := \{x_2, \dots, x_{q-1}\}$. Since the degrees of x_1 and x_q in the underlying graph of D are at most $k - 1$ and D is at least $(k + 1)$ -chromatic, it follows that $\chi(D_1)$ or $\chi(D_2)$ is at least $(k + 1)$ -chromatic. Each vertex has in-degree at most $k - 2$ in D and $d_{D_2}^+(x_i) \leq 1$ for $2 \leq i \leq q - 1$, so $\Delta(D_2) \leq k - 1$ and $\chi(D_2) \leq k$. Hence D_1 is at least $(k + 1)$ -chromatic and strongly connected. Thus by Bondy's theorem, D_1 contains a circuit of length at least k but shorter than C . This is a contradiction. ■

The existence of good circuits directly gives our main theorem in the case of strongly connected digraphs. However, we will not need this result for the proof of the general case.

Lemma 3 *Let $k + l = n - 1$ and D be a strongly connected n -chromatic digraph. If D contains an $(l + 1)$ -good circuit then D contains a $P(k, l)$.*

Proof. Suppose C is an $(l + 1)$ -good circuit. Since $\chi(D[V(C)]) \leq l + 1$, the chromatic number of the (strongly connected) contracted digraph D/C is at least $k + 1$. Thus by Bondy's theorem, D/C has a circuit of length at least $k + 1$, and in particular the vertex C is the end of a path P of length k in D/C . Finally $P \cup C$ contains a $P(k, l)$. ■

Corollary 2 *Let $k + l = n - 1 \geq 3$ and D be an n -chromatic strongly connected digraph. Then D contains a $P(k, l)$.*

Proof. Since $P(k, l)$ and $P(l, k)$ are isomorphic, we may assume that $l \geq 2$. By Lemma 2, D has an $(l + 1)$ -good circuit, and thus contains a $P(k, l)$ according to Lemma 3. ■

4 The general case.

We now turn to the proof of the main result.

Theorem 2 *Let $k + l = n - 1 \geq 3$ and D be an n -chromatic digraph. Then D contains a $P(k, l)$.*

Proof. We again assume that $l \geq k$, and therefore $l \geq 2$. Suppose for contradiction that D does not contain $P(k, l)$. Let F be a final spanning out-forest of D .

We first prove that D contains an $(l+1)$ -good circuit C which is disjoint from $F_1 \cup \dots \cup F_{k-1}$. For this, we consider the following coloring of D (called *canonical*): for $1 \leq i \leq k-1$, the vertices of F_i are colored i , and for $i \geq k$, the vertices of F_i are colored j , where $j \in \{k, \dots, k+l\}$ and $j \equiv i \pmod{l+1}$. Since we colored D with less than n colors, this coloring is improper. In particular, there exists an arc vw from F_i to F_j where $i, j \geq k$ and $j \equiv i \pmod{l+1}$. By Lemma 1 (i), we reach a contradiction if $i < j$. Thus $j < i$, and by Lemma 1 (ii), we necessarily have $j = k$ and $i \geq k+l+1$. By Proposition 1, v is a descendant of w in F . In particular $F \cup vw$ has a circuit C of length at least $l+1$. If the induced digraph on C has chromatic number at most $l+1$, $C^0 := C$ is $(l+1)$ -good. If not, by Lemma 2, it contains an $(l+1)$ -good circuit C^0 .

We inductively define couples (D^i, F^i) as follows: Set $D^0 := D$, $F^0 := F$. Then, if there exists an $(l+1)$ -good circuit C^i of $D^i \setminus (F_1^i \cup \dots \cup F_{k-1}^i)$, define $D^{i+1} := D^i \setminus V(C^i)$ and let F^{i+1} be any final improvement of $F^i \setminus V(C^i)$.

With the previous definitions, we have $D^1 = D \setminus V(C^0)$. This inductive definition certainly stops on some (D^p, F^p) where D^p admits a canonical coloring as a proper coloring.

Let us be a little more precise: at each inductive step, the circuit C^i must contain a vertex v^i of F_k^i , otherwise the union of C^i and a path of F^i starting at F_1^i and ending at C^i would certainly contain a $P(k, l)$, since C^i has length at least $l+1$. We denote by u^i the unique in-neighbor of v^i in F_{k-1}^i . Observe that the level of u^i in F^j , where $j > i$, always increases since we apply improvements. Observe also that u^i cannot reach a level greater than $k-1$, otherwise u^i would be the end of a path P of length $k-1$ in $D \setminus C^i$ and thus $C^i \cup P \cup u^i v^i$ would contain a $P(k, l)$. Thus every circuit C^i , $i = 0, \dots, p-1$, has an in-neighbor u^i in F_{k-1}^p .

Let us now reach a contradiction, by properly coloring D with $n-1$ colors. We first color the levels F_1^p, \dots, F_{k-1}^p with colors $1, \dots, k-1$. We will now color the remaining induced graph $D' := D \setminus (F_1^p \cup \dots \cup F_{k-1}^p)$ with colors $k, \dots, k+l$. To this end, we first establish some claims. The proof of some of them follows easily from the fact that D has no $P(k, l)$ and $l \geq k$ and is left to the reader.

Claim 1 *There is no arc between two distinct C^i 's.*

Claim 2 *No vertex of C^i has a neighbor, in- or out-, in any level F_j^p for any $j > k$. Moreover, no vertex of C^i has an in-neighbor in F_k^p .*

Let us call *dangerous vertices* the out-neighbors of the C^i 's in F_k^p and *safe vertices* the non-dangerous vertices in F_k^p .

Claim 3 *A dangerous vertex b has in-neighbors in a unique C^i .*

Claim 4 *A dangerous vertex b has at most l in-neighbors in C^i .*

Proof. Suppose for contradiction that w_1, \dots, w_{l+1} are in-neighbors of b in C^i , enumerated with respect to the cyclic order of C^i and so that w_1 is the first vertex w_j along C^i which follows v^i (in other words $C^i[v^i, w_1] \cap \{w_1, \dots, w_{l+1}\} = \{w_1\}$). Let P be the path of F^p starting at F_1^p and ending at u^i . Now $P \cup u^i v^i \cup C[v^i, w_1] \cup w_1 b \cup C[w_2, w_{l+1}] \cup w_{l+1} b$ contains a $P(k, l)$, a contradiction. ■

If b is a dangerous vertex, we denote by S_b be the set of descendants of b in F^p , i. e. the set of vertices x such that there is a path from b to x in F^p , including b itself.

Claim 5 *If b is dangerous, every arc xy entering S_b in $D' = D \setminus (F_1^p \cup \dots \cup F_{k-1}^p)$ is such that $y = b$ and $x \in C^i$.*

Proof. Let xy be an arc of D' with $y \in S_b$ and $x \notin S_b$. If $y \neq b$, y would be a strict descendant of b in F^p . By Claim 2, x is not in some C^j . Thus $x \in F^p$, and is not a descendant of b by hypothesis. In

particular $F^p \cup xy$ contains two (F_k^p, y) -directed paths P_1, P_2 such that $P_1 \cap P_2 = y$ and one of them, say P_1 , starts at b . Extending P_1 via C^i and P_2 via $F_1^p \cup \dots \cup F_{k-1}^p$ would give a $P(k, l)$.

So $y = b$. By Proposition 1 and the fact that $x \notin S_b$, x is not a vertex of F^p . So x belongs to some C^j , and by Claim 3, x belongs to C^i . ■

Claim 6 *If b is dangerous, there is no arc leaving S_b in D' .*

Proof. Assume for contradiction that xy is an arc of D' such that $x \in S_b$ and $y \notin S_b$. If $y \in F^p$, there exists two paths ending at y , one starting from b and the other starting from another vertex of F_k^p , which is impossible. Thus y belongs to some C^j , but this is again impossible because of Claim 2. ■

Let us now color the vertices of D' .

Every C^i is $(l+1)$ -good and thus $(l+1)$ -colorable. Moreover, by Claim 1, we can properly color the union of the C_i 's with the colors $k, \dots, k+l$.

By Claim 2 and the definition of safe vertices, there is no arc between the C^i 's and the descendants of safe vertices in F^p . Hence we can properly extend our coloring to the safe vertices and their descendants in a canonical way. Now we have to properly extend the coloring to S_b for every dangerous vertex b . Observe that between S_b and $D' \setminus S_b$, by Claim 5 and 6, there are only arcs starting at some given C^i and ending at b . By Claim 4, there are at most l of these arcs. Thus, there is one color c amongst $k, \dots, k+l$ which is not used by one in-neighbor of b in C^i . Color b with color c . Then extend to a proper coloring to S_b in a periodical way: a vertex in $F_i \cap S_b$ is assigned $j \in \{k, \dots, k+l\}$ if $j \equiv i + c_b \pmod{l+1}$. Doing this for every dangerous vertex yields a proper $(n-1)$ -coloring of D and thus a contradiction. ■

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