Countable α-Extendable Graphs

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Abstract

Let us consider a countable graph G with vertex set $V(G)$. C.St.J.A. Nash-Williams introduced the notion of an n-path: a *0-path* is a finite path and for any $n \in \mathbb{N}$, an *(n+1)-path* is a path P such that, for every finite subset F of $V(G)$, P can be extended to an n-path containing F. This notion extends in a natural way to the concept of an α*-path*, where α is an ordinal. N. Polat proved that a countable graph which contains an ω_1-path has a hamiltonian path. The aim of this paper is to show that one cannot improve this theorem to an ordinal strictly less than ω_1: for any countable ordinal α, we exhibit a countable non-hamiltonian graph which contains an α-path. These graphs have maximal degree 4.

Can hamiltonicity of countable graphs be expressed in terms of some properties of finite paths? A necessary condition for a countable graph G to be a one-way hamiltonian graph is clearly that any finite subset of vertices of G can be covered by a finite path. Of course this condition is not sufficient: just consider the union of two countably infinite complete graphs linked to each other by finitely many edges. However, we can consider stronger conditions of this nature. Let us define a *0-path* in G to be a finite path in G and, inductively, define an *(n+1)-path* to be a finite path P such that, for every finite subset F of $V(G)$, P can be extended to an n-path containing F. In [2], C.St.J.A. Nash-Williams asked whether there exists a countable non-hamiltonian graph which contains an n-path for every positive integer n. He later found such a graph with vertices of infinite degree [3], and raised the following two questions: is there a locally finite countable graph with a 6-path but with no hamiltonian path? Is there a non-hamiltonian countable graph with an 8-path which has finitely many vertices of infinite degree? C. Thomassen provided an example with a 7-path for the second question. The purpose of this paper is to answer a more general version of both questions. Specifically, we shall define the concept of an α-path, where α is an ordinal number. We shall say that a graph G is α*-extendable* if the empty path is an α-path in G. We exhibit, for any countable ordinal α, a non-hamiltonian countable α-extendable graph, all of whose vertices have degree at most 4. Our construction needs some tools similar to those used in the proof of NP-completeness of hamiltonicity [1]. The case of narrow graphs (graphs with thin ends) is treated by N. Polat in [4]. He proved that there exists an integer function f such that whenever a narrow graph G has at most n disjoint rays, then G is a hamiltonian graph if and only if G has an $f(n)$-path.

1 The Notion of α-Path.

Definition 1 In this paper, graphs are understood to be simple (i.e. without loops or multiple edges). The sets of vertices and edges of a graph X will be denoted by $V(X)$ and $E(X)$ respectively. The symbol G will always denote a graph and $V(G)$ will be abbreviated to V, $\mathbb{N}$ denotes the set of positive integers , ω is the set of non-negative integers and ω_1 is the smallest uncountable ordinal.

A *path* in a graph is a sequence of distinct vertices of the form $(v_i)_{0\leq i\leq n}$ or $(v_i)_{i\in\omega}$ such that every two successive vertices v_i, v_{i+1} in the sequence are joined by an edge. (Thus, for the purposes of this paper, an infinite path is understood to be, in the language of some other authors, "one-way infinite". In particular, by a "hamiltonian path" of an infinite graph, we shall always mean a one-way infinite hamiltonian path, and an infinite graph will only be considered to be "hamiltonian" if it has a one-way infinite hamiltonian path.) An edge joining two successive vertices of a path P will be called an *edge of* P and $E(P)$ will denote the set of edges of P. The path $(v_i)_{0\leq i\leq p}$ is an *initial section* of $(v_i)_{0\leq i\leq n}$ if $p \leq n$ and is an *initial section* of $(v_i)_{i\in\omega}$ if $p\in\omega$. If a path P' is an initial section of a path P, we say that P' *extends to* P (or equivalently that P *extends* P'). We shall regard the empty sequence of vertices as a path; it will be understood to be an initial section of every path. A set $S\subseteq V$ is *covered* by a path P in G if every element of S is a term of P. If a finite graph G has one and only one hamiltonian path P with first term u and last term v, then $E(G;u,v)$ will denote the set $E(P)$.

For any ordinal α, we define by induction the notion of α-*path* of G : i) Any finite path is a 0-path. ii) Let P be a finite path of G. If, for every finite subset $F\subseteq V$, P extends to an α-path which covers F, then P is an $(\alpha+1)$-path. iii) If α is a limit ordinal and P is a β-path for every $\beta<\alpha$ then P is an α-path. We say that G is α-*extendable* if the empty path is an α-path of G. Note that a hamiltonian graph is α-extendable for every α.

Example 1 The graph in Fig. 1 is 3-extendable but has no hamiltonian path (Polat [4]) :

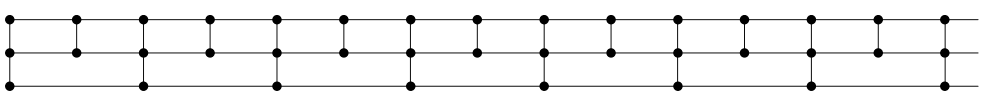

Fig. 1

The diagram in Fig. 2 shows how to cover three successive finite sets of vertices of this graph. The path P_1 extends to P_2 which in turn extends to P_3, and each of these paths can be arbitrarily long.

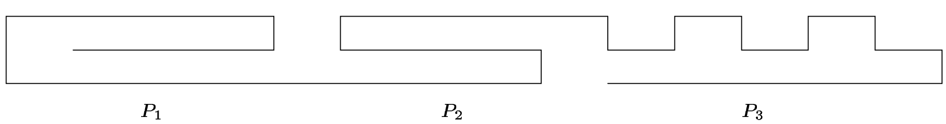

Fig. 2

Remark 1 *If P is an α-path in a graph G then*
i) P is a β-path for every $\beta<\alpha$.
ii) Every initial section of P is an α-path.

Theorem 1 (Polat [4]) *If G is a countable ω_1-extendable graph, G is a hamiltonian graph.*

Proof. Our definitions imply trivially that a finite graph is hamiltonian if it is ω_1-extendable (or even 1-extendable), and so we may assume that G is countably infinite. Let $(v_i)_{i\in\omega}$ be an enumeration of the vertices of G. Suppose that P is an ω_1-path in G and $k\in\omega$. Then, for each $\alpha<\omega_1$, P is an $(\alpha+1)$-path and so extends to an α-path P_α of G which covers $\{v_k\}$. There must be a path which is equal to P_α for uncountably many ordinals $\alpha<\omega_1$, and this path is an ω_1-path which covers $\{v_k\}$. This shows that for every ω_1-path P in G and every vertex v_k of G, P extends to an ω_1-path $\pi_k(P)$ which covers $\{v_k\}$. Now letting Q be the empty path, which is an ω_1-path, we can obtain a sequence of paths $\pi_0(Q), \pi_1(\pi_0(Q)), \pi_2(\pi_1(\pi_0(Q))), \ldots$ whose union is a hamiltonian path of G. $\square$

2 Construction of n-Extendable Graphs.

In this section, we construct for any integer $n \geq 2$, a hamiltonian graph G_n. We then modify G_n so as to make it non-hamiltonian but still n-extendable.

Example 2 The graph G_3 is depicted in Fig. 3.

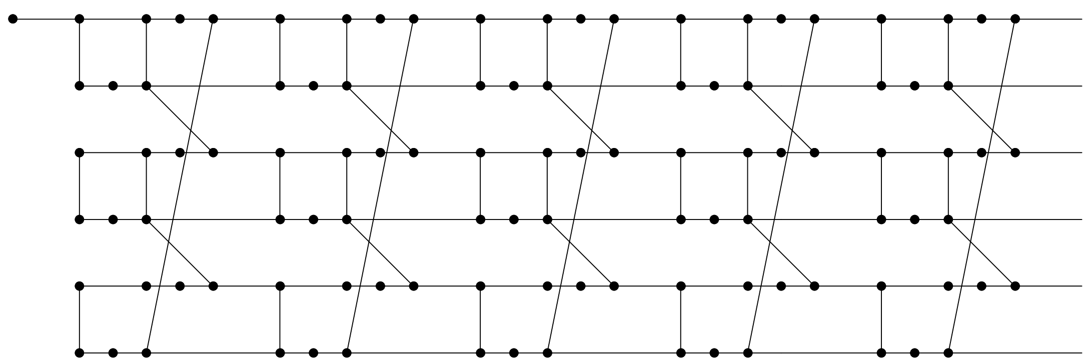

Fig. 3

Any initial part of G_3 is coverable by three successive paths, as shown in Fig. 4. Note that each path, from s to t, from t to u and from u can cover an arbitrarily long initial part of G_3. The vertices s, t and u have decreasing levels in G_3 and from u, the path cannot extend more than once.

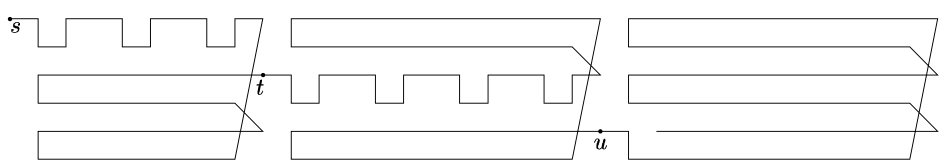

Fig. 4

The problem is that the graph G_3 has a hamiltonian path, as we can check in Fig. 5.

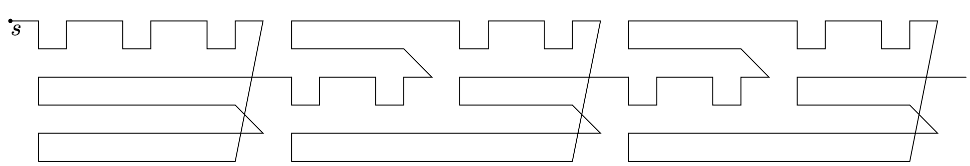

Fig. 5

We define now the graphs G_n for any integer $n \geq 2$.

Definition 2 A *block* is the finite graph depicted in Fig. 6. A *row* is the countable graph R depicted in Fig. 7. This graph is constructed on the set of blocks $\{B_j\}_{j \in \mathbb{N}}$.

The graph G_n $(n \geq 2)$ is constructed starting with the disjoint union of n rows, $R_1, R_2, \ldots, R_n$, where the subscripts $1, 2, \ldots, n$ belong to the set $\mathbb{Z}_n$ of residues modulo n. For each $i \in \mathbb{Z}_n$, $j \in \mathbb{N}$, the block $B_{i,j}$

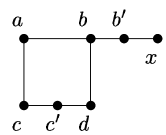

Fig. 6

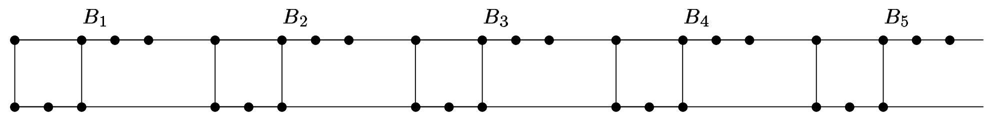

Fig. 7

has vertices $a_{i,j}$, $b_{i,j}$, $b'_{i,j}$, $c_{i,j}$, $c'_{i,j}$, $d_{i,j}$, $x_{i,j}$ and is the j^{th} block of the i^{th} row (the bottom row is R_1). The j^{th} *column* of G_n is the graph $C_j = \bigcup\{B_{i,j} : i \in \mathbb{Z}_n\}$. The edge $a_{i,j}c_{i,j}$ of G_n is denoted by $f_{i,j}$. The edge $b_{i,j}b'_{i,j}$ of G_n is denoted by $g_{i,j}$. The graph G_n is obtained as follows (the picture of G_3 in Fig. 3 is helpful in understanding the construction of G_n):

i) We add a single vertex s which is linked only to the vertex $a_{n,1}$. This vertex is now the origin of any possible hamiltonian path of G_n.

ii) We add the edges $e_{i,j} = d_{i,j}x_{i-1,j}$ $(i \in \mathbb{Z}_n,\ j \in \mathbb{N})$.

iii) We delete all the edges $b_{1,j}d_{1,j}$ $(j \in \mathbb{N})$; because of this, when the bottom level is reached, a path constructed on the lines of Fig. 4 can only be extended once.

Remark 2 Suppose that m, n are integers such that $0 \le m \le n$, and J is the subgraph of R (the "row" described above) induced by $V(B_m) \cup V(B_{m+1}) \cup \ldots \cup V(B_n)$. Then it is easily seen that there is a unique hamiltonian path of J from a_m to x_n and from d_n to x_n: they must be of one of the kinds indicated by Fig. 8 (drawn for the illustrative case $n - m = 4$). In particular, this implies that the notations $E(J; a_m, x_n)$ and $E(J; d_n, x_n)$ make sense.

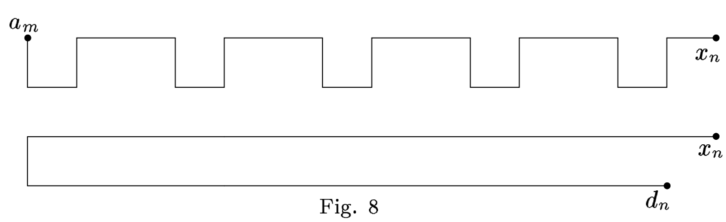

Fig. 8

Definition 3 An *edge forcing condition* of G is a formula $e_1 \Rightarrow e_2$ where e_1 and e_2 are edges of G. A path P of G *satisfies* this condition if either both e_1 and e_2 belong to $E(P)$ or else $e_1 \notin E(P)$. Let $\mathcal{F}$ be a set of edge forcing conditions of G. We say that $(G, \mathcal{F})$ is *hamiltonian* if there exists a hamiltonian path of G which satisfies all the edge forcing conditions of $\mathcal{F}$. In the graph G_n, we let $F_{i,j}$ denote the edge forcing condition $e_{i,j} \Rightarrow f_{i+1,j+1}$ for every $i \in \mathbb{Z}_n$, $j \in \mathbb{N}$ (see Fig. 9). The set $\{F_{i,j} : i \in \mathbb{Z}_n ,\ j \in \mathbb{N}\}$ is denoted by $\mathcal{F}_n$. A path P in G_n is *r-proper* if, for some $m \in \mathbb{N}$,

i) P starts at s and ends at $a_{r,m}$;

ii) $V(P) = (\bigcup\{V(C_j) : 1 \le j < m\}) \cup \{s, a_{r,m}\}$;

iii) P satisfies $F_{i,j}$ for all $i \in \mathbb{Z}_n$ and all $1 \le j < m - 1$.

An *r-proper section* of a path P of G_n is an initial section of P which is an r-proper path of G_n.

Theorem 2 *$(G_n, \mathcal{F}_n)$ is not hamiltonian.*

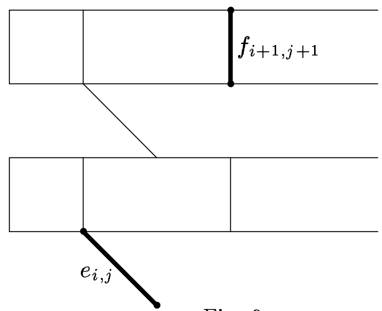

Fig. 9

Proof. We suppose, by way of contradiction, that G_n has a hamiltonian path P which satisfies the set of edge forcing conditions $\mathcal{F}_n$. Then the path $s, a_{n,1}$ is an n-proper section of P. Moreover, if P had a 1-proper section, say ending at $a_{1,m}$, then P could not cover both $b_{1,m}$ and $c_{1,m}$ and so could not be hamiltonian. Therefore there exists $l \in \mathbb{Z}_n \setminus \{1\}$ such that P has an l-proper section P' and has no $(l-1)$-proper section. We suppose that P' ends at $a_{l,m}$. Since the $e_{i,j}$ are the only edges that go between rows, infinitely many of them must be in $E(P)$. We can therefore choose $p \in \mathbb{Z}_n$ and an integer $q \geq m$ such that $e_{p,q} \in E(P)$ and, subject to these requirements, q is as small as possible. Then, by Remark 2, P contains the edges of $\bigcup\{C_k : m \leq k \leq q\}$ indicated in Fig. 10. Consequently, $b_{i,q}d_{i,q} \notin E(P)$ when $i \in \mathbb{Z}_n \setminus \{l\}$. Note also that $p \neq l$ since $d_{l,q}$ cannot be incident with three edges of P. We now make the following observations:

i) If $i \in \mathbb{Z}_n \setminus \{l\}$ and $f_{i,q+1} \in E(P)$ then $e_{i,q} \in E(P)$ (since $b_{i,q}d_{i,q} \notin E(P)$ and P must cover both $d_{i,q}$ and $c'_{i,q+1}$).

ii) If $i \in \mathbb{Z}_n$ and $e_{i,q} \in E(P)$ then $f_{i-1,q+1} \in E(P)$ (since P must cover both $a_{i-1,q+1}$ and $b'_{i-1,q}$).

iii) If $i \in \mathbb{Z}_n \setminus \{l+1\}$ and $e_{i,q} \in E(P)$ then $f_{i-1,q+1} \in E(P)$ by ii) and thus $e_{i-1,q} \in E(P)$ by i).

iv) If $i \in \mathbb{Z}_n \setminus \{l-1\}$ and $e_{i,q} \in E(P)$ then $f_{i+1,q+1} \in E(P)$ by the edge forcing conditions and consequently $e_{i+1,q} \in E(P)$ by i).

Since $e_{p,q} \in E(P)$, it follows from iii) and iv) that $e_{i,q} \in E(P)$ for every $i \in \mathbb{Z}_n \setminus \{l\}$. Thus P contains the edges of $\bigcup\{C_k : m \leq k \leq q\}$ indicated in Fig. 11 and so P has an $(l-1)$-proper section, a contradiction. □

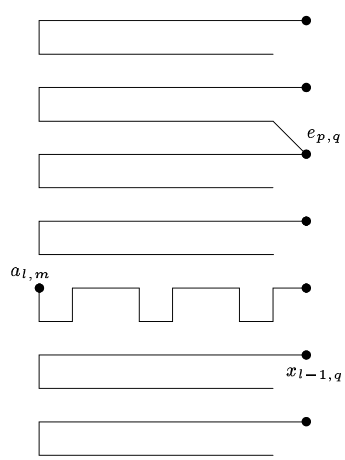

Fig. 10

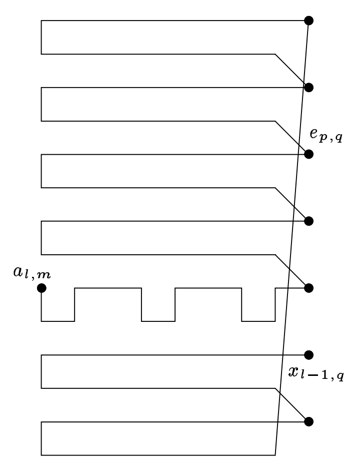

Fig. 11

3 Realization of Edge Forcing.

In this section, we construct, for any $n \geq 2$, an n-extendable graph which has no hamiltonian path. The construction is essentially based on the structure $(G_n, \mathcal{F}_n)$ defined in the previous section.

Definition 4 A *bound edge* of a graph G is an edge xy of $E(G)$ such that the degree of x or the degree of y is less than or equal to 2. Note that if G is an infinite hamiltonian graph with a degree one vertex, then every bound edge of G belongs to every hamiltonian path of G. An $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$-*bridge* is a graph isomorphic to the one depicted in Fig. 12. This kind of graph is widely used in the proofs of NP-completeness of hamiltonicity; see for example [1].

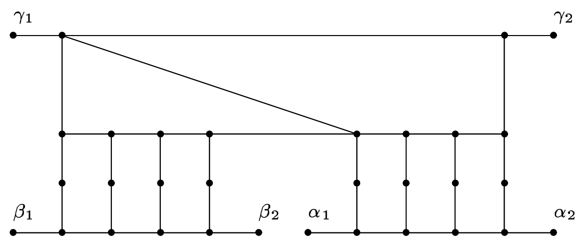

Fig. 12

Let G be an infinite graph. A triple (e, f, g) of pairwise nonadjacent edges of G such that g is a bound edge is a *forcing triple* of G. Let (x_1x_2, y_1y_2, u_1u_2) be a forcing triple of G, and denote by D an $(x_1, x_2, y_1, y_2, u_1, u_2)$-bridge such that $V(G) \cap V(D) = \{x_1, x_2, y_1, y_2, u_1, u_2\}$. The graph, with vertex set $V(G) \cup V(D)$ and edge set $(E(G) \setminus \{x_1x_2, y_1y_2, u_1u_2\}) \cup E(D)$, is called the *realization* of $(G, (x_1x_2, y_1y_2, u_1u_2))$; we denote it by $R(G, (x_1x_2, y_1y_2, u_1u_2))$. The graph D is the *bridge associated with* the forcing triple (x_1x_2, y_1y_2, u_1u_2).

Let (e, f, g) be a forcing triple of a graph G. A path P of G is *compatible* with (e, f, g) if $g \in E(P)$ or $E(P) \cap \{e, f, g\} = \emptyset$. Let P be a path of G compatible with (e, f, g) and D the bridge associated with (e, f, g). Set $A = E(P) \cap \{e, f, g\}$. We denote by $\Psi(P, (e, f, g))$ the path of $R(G, (e, f, g))$ with edge set $(E(P) \setminus \{e, f, g\}) \cup E_A$ where E_A is a subset of edges of D defined as follows: the set $E_\emptyset$ is empty, and the other sets of edges E_A are depicted by bold edges in Fig. 13.

Let G be an infinite graph and $T = \{(e_i, f_i, g_i) : i \in \mathbb{N}\}$ be a set of forcing triples of G such that every edge of G appears in at most one triple of T. Let $G^1 = G$, and inductively $G^{k+1} = R(G^k, (e_k, f_k, g_k))$ for every integer $k \geq 1$. We obtain a sequence of graphs G^1, G^2, $G^3, \ldots$ whose limit is the *realization* of (G, T). We denote this graph by $R(G, T)$. A *compatible path* of (G, T) is a path of G compatible with every forcing triple of T. Let P be a compatible path of (G, T), let P_0 be the path P in G^0, and inductively, P_{k+1} be the path $\Psi(P_k, (e_k, f_k, g_k))$ of $R(G^k, (e_k, f_k, g_k))$ for any $k \in \mathbb{N}$. The limit of the sequence P_0, P_1, ... is a path of the graph $R(G, T)$; we denote it by $\Psi(P, T)$.

Lemma 1 *Let G be an infinite graph with a degree one vertex. Let $T = \{(e_i, f_i, g_i) : i \in \mathbb{N}\}$ be a set of forcing triples of G such that every edge of G appears in at most one triple of T. The graph $R(G, T)$ has a hamiltonian path if and only if $(G, \{(e_i \Rightarrow f_i) : i \in \mathbb{N}\})$ is hamiltonian.*

Proof. Let P be a hamiltonian path of G which satisfies $e_i \Rightarrow f_i$ for every $i \in \mathbb{N}$. Since P is a hamiltonian path and G has a degree one vertex, every bound edge of G is an edge of P. Thus, P is a compatible path of (G, T) and by construction $\Psi(P, T)$ is a hamiltonian path of $R(G, T)$. Conversely, from every hamiltonian path Q of $R(G, T)$, it is routine to construct a hamiltonian path P of G which satisfies $\{(e_i \Rightarrow f_i) : i \in \mathbb{N}\}$ and such that $\Psi(P, T) = Q$. Indeed, for every bridge D associated with a forcing triple of T, the set of edges $E(Q) \cap E(D)$ is the set of bold edges in one of the figures $E_{\{g\}}$, $E_{\{g,f\}}$ or $E_{\{g,e,f\}}$ depicted in Fig. 13. □

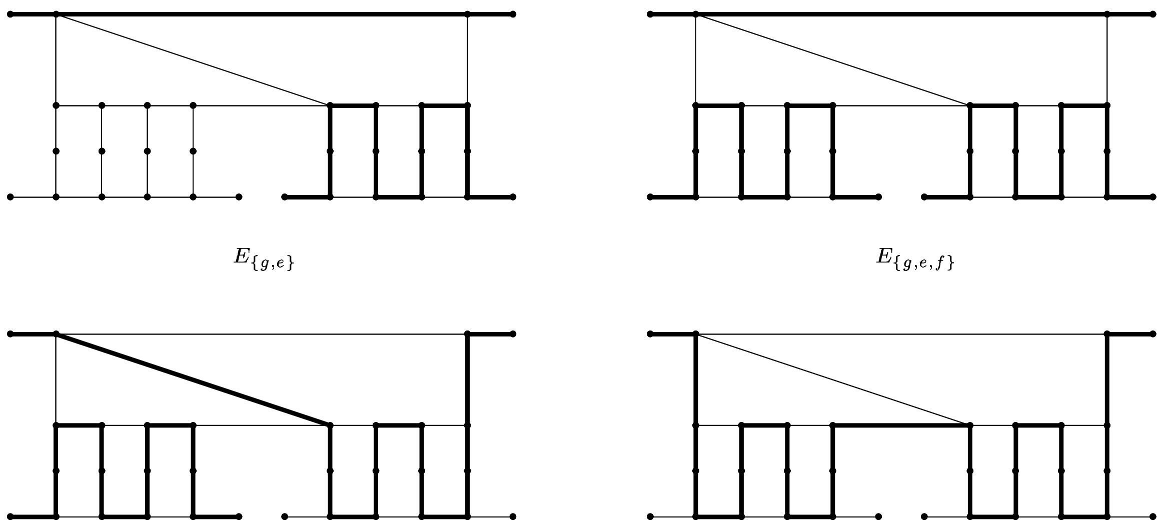

Fig. 13

Consider the graph G_n together with the set of forcing triples $T_n = \{(e_{i,j}, f_{i+1,j+1}, g_{i-1,j}) \ : \ i \in \mathbb{Z}_n,\ j \in \mathbb{N}\}$. Let $H_n = R(G_n, T_n)$ and let $D_{i,j}$ denote the bridge associated with $(e_{i,j}, f_{i+1,j+1}, g_{i-1,j})$ for all $i \in \mathbb{Z}_n$, $j \in \mathbb{N}$.

Lemma 2 *The graph H_n is n-extendable.*

Proof. If an r-proper path R of G_n ends at $a_{r,m}$ $(m \geq 2)$ then $E(R)$ must contain the bound edge $g_{r,m-1}$ and so the last four terms of R must be $b_{r,m-1}, b'_{r,m-1}, x_{r,m-1}, a_{r,m}$: we let R^* denote the initial section of R which ends at $b_{r,m-1}$. Let $l \in \mathbb{Z}_n \setminus \{1\}$, $p \in \mathbb{N}$ and P_l be an l-proper path of G_n which ends at $a_{l,p+1}$. Since P_l covers the set $\{x_{i,p} \ : \ i \in \mathbb{Z}_n\}$ and $a_{i,p+1} \notin V(P_l)$ when $i \in \mathbb{Z}_n \setminus \{l\}$, all the edges $e_{i,p}$ belong to $E(P_l)$ when $i \in \mathbb{Z}_n \setminus \{l+1\}$. Moreover, $E(P_l)$ contains the bound edge $g_{i,p}$ for all $i \in \mathbb{Z}_n$ and so $E(P_l^*)$ contains $g_{i-1,p}$ for all $i \in \mathbb{Z}_n \setminus \{l+1\}$. Now let R be any path which extends P_l such that $V(C_{p+1}) \setminus \{b'_{l-1,p+1}, x_{l-1,p+1}\} \subseteq V(R)$. (For example, this condition will be satisfied if P_{l-1} is any $(l-1)$-proper path which extends P_l and $R = P_{l-1}^*$.) Our goal is to show that $\Psi(R, T_n)$ extends $\Psi(P_l^*, T_n)$. Indeed, we just have to check that $E(\Psi(P_l^*, T_n)) \subset E(R, T_n))$. The critical point is to check this inclusion for all $D_{i,p}$, $i \in \mathbb{Z}_n \setminus \{l+1\}$. Note that R contains all the edges $f_{i,p+1}$ for all $i \in \mathbb{Z}_n$. Thus $\Psi(R, T_n)$ covers the vertices of $D_{i,p}$ for all $i \in \mathbb{Z}_n \setminus \{l+1\}$, as indicated over $E_{\{f,g,e\}}$ in Fig. 13. Since the edges of $E_{\{f,g,e\}}$ contain the edges of $E_{\{g,e\}}$, we conclude that $\Psi(R, T_n)$ extends $\Psi(P_l^*, T_n)$.

Now we are ready to prove that H_n is n-extendable. We define an integer valued function rk on $V(H_n)$ by letting $rk(v) = j$ if $v \in V(C_j)$ or $v \in V(D_{i,j})$ for some $i \in \mathbb{Z}_n$ and $rk(s) = 0$. Furthermore, we define $rk(F) = max(\{rk(v) \ : \ v \in F\})$ if F is a nonempty finite subset of $V(H_n)$ and $rk(\emptyset) = 0$. We prove now by induction that any path $\Psi(P^*, T_n)$ of H_n, where P is an l-proper path of G_n, is an l-path of H_n. To verify this for $l = 1$, observe that if P_1 is a 1-proper path of G_n ending at $a_{1,p}$ and F is a finite subset of $V(H_n)$ then, as illustrated in Fig. 4, there exists a finite path R of G_n which extends P_1 and covers $V(C_j)$ for all $j \leq max(rk(F), p) + 1$. Then, by the argument in the preceding paragraph, $\Psi(R, T_n)$ extends $\Psi(P_l^*, T_n)$. Moreover $\Psi(R, T_n)$ covers F. Since F was arbitrary , this proves that P_1 is a 1-path. Now let P be an $(l+1)$-proper path of G_n which ends at $a_{l+1,q}$ and F be a finite subset of $V(H_n)$. Let $r = max(rk(F), q) + 2$ and Q be an l-proper path of G_n which extends P and ends at $a_{l,r}$. Then, by the induction hypothesis, $\Psi(Q^*, T_n)$ is an l-path of H_n. Moreover $\Psi(Q^*, T_n)$ extends $\Psi(P^*, T_n)$ by the argument in the preceding paragraph, and clearly $\Psi(P^*, T_n)$ covers F. Since F is arbitrary, the path $\Psi(P^*, T_n)$ is an $(l+1)$-path of H_n. Consequently, the path $s, a_{n,1}$ is an n-path of H_n; in particular H_n is n-extendable. □

We claim that H_n is the graph we are looking for: by Theorem 2 and Lemma 1, it has no hamiltonian path; and by Lemma 2, it is n-extendable.

4 The case of α-extendable graphs.

We generalize the construction of our graphs to G_α, where α is a countable ordinal. We need first a mapping Φ from $\{1, \ldots, \alpha\}$ into $\mathbb{N}$ such that $\Phi(1) = \Phi(\alpha) = 1$ and $\Phi^{-1}(n)$ is finite for every $n \in \mathbb{N}$. For $j \in \mathbb{N}$ let Γ_j denote the finite set $\Phi^{-1}(\{1, \ldots, j\})$. If $\Gamma_j = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ where $1 = \gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_t = \alpha$, we shall say that γ_{i+1} is the Γ_j*-successor* of γ_i for $i = 1, \ldots, t-1$ and that $1\ (= \gamma_1)$ is the Γ_j*-successor* of $\alpha\ (= \gamma_t)$. For $\gamma \in \Gamma_j$, we denote the Γ_j-successor of γ by $\sigma_j(\gamma)$. We shall say that β is the Γ_j*-predecessor* of γ if $\gamma = \sigma_j(\beta)$, we denote the Γ_j-predecessor of γ by $\pi_j(\gamma)$. The graph G_α is constructed on the set of blocks $\{B_{\beta,j}\ :\ 1 \leq \beta \leq \alpha\ ,\ j \geq \Phi(\beta)\}$ where β is an ordinal and $j \in \mathbb{N}$. The set of vertices of the block $B_{\beta,j}$ is $\{a_{\beta,j}, b_{\beta,j}, b'_{\beta,j}, c_{\beta,j}, c'_{\beta,j}, d_{\beta,j}, x_{\beta,j}\}$; we denote also by $f_{\beta,j}$ the edge $a_{\beta,j}c_{\beta,j}$ and by $g_{\beta,j}$ the edge $b_{\beta,j}b'_{\beta,j}$. Each row R_β of G_α is constructed as in section 2 on the set of blocks $\{B_{\beta,j}\ :\ j \geq \Phi(\beta)\}$. We connect distinct rows by adding an edge $e_{\gamma,j} = x_{\beta,j}d_{\gamma,j}$ for each triple j, γ, β such that $j \in \mathbb{N}$, $\gamma \in \Gamma_j$ and β is the Γ_j-predecessor of γ. We add a vertex s joined just to $a_{\alpha,1}$. Finally, we delete the edges $b_{1,j}d_{1,j}$ for every $j \in \mathbb{N}$. This is our graph G_α.

When $j \in \mathbb{N}$, $\gamma \in \Gamma_j$ and $\delta = \sigma_j(\gamma)$, we let $F_{\gamma,j}$ denote the edge forcing condition $e_{\gamma,j} \Rightarrow f_{\delta,j+1}$. We let $\mathcal{F}_\alpha$ denote the set of edge forcing conditions $\{F_{\gamma,j}\ :\ j \in \mathbb{N}, \gamma \in \Gamma_j\}$ and T_α denote the set of forcing triples $\{(e_{\gamma,j}, f_{\delta,j+1}, g_{\beta,j})\ :\ j \in \mathbb{N}, \beta \in \Gamma_j, \gamma = \sigma_j(\beta), \delta = \sigma_j(\gamma)\}$. We define H_α to be $R(G_\alpha, T_\alpha)$, and $D_{\gamma,j}$ will denote the bridge associated with the triple $(e_{\gamma,j}, f_{\delta,j+1}, g_{\beta,j}) \in T_\alpha$.

We illustrate our construction by an example. In Fig. 14 are drawn the seven first columns of the graph $G_{\omega 2}$, the construction is based on the mapping Φ from $\{1, \ldots, \omega 2\}$ into $\mathbb{N}$ such that $\Phi(i) = i$ for every $i \in \omega \setminus \{0\}$, $\Phi(\omega + i) = i + 1$ for every $i \in \omega$, and $\Phi(\omega 2) = 1$. The bold path is, for instance, an $(\omega + 2)$-proper path of $G_{\omega 2}$. The graph $H_{\omega 2}$ is the realization of $(G_{\omega 2}, T_{\omega 2})$.

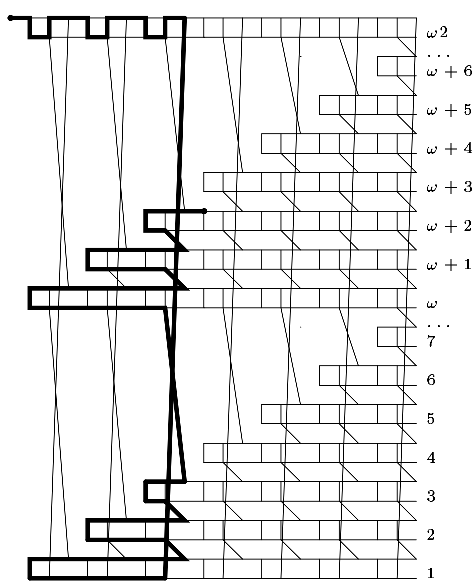

Fig. 14

If $n, m \in \mathbb{N}$ and $m \leq n$ and $\gamma \in \Gamma_n$, then $J(\gamma, m, n)$ will denote the induced subgraph of G_α whose set of vertices is $\bigcup(V(B_{\gamma,i}) : \max(m, \Phi(\gamma)) \leq i \leq n)$. We see from Remark 2 that $E(J(\gamma, m, n), d_{\gamma,n}, x_{\gamma,n})$ is well defined in these circumstances and that $E(J(\gamma, m, n), a_{\gamma,m}, x_{\gamma,n})$ is well defined when $m, n \in \mathbb{N}$ and

$m \leq n$ and $\gamma \in \Gamma_m$. Let us extend now our definition of r-proper section to the transfinite case. Let α and ρ be countable ordinals such that $1 \leq \rho \leq \alpha$. A path P in G_α is *ρ-proper* if, for some $m \geq \Phi(\rho)$,

i) P starts at s and ends at $a_{\rho,m}$;

ii) $V(P) = (\bigcup\{V(B_{\gamma,j} : \Phi(\gamma) \leq j < m\ ,\ 1 \leq \gamma \leq \alpha\}) \cup \{s, a_{\rho,m}\}$;

iii) P satisfies $F_{\gamma,j}$ for all pairs γ, j such that $1 \leq j < m-1$ and $\gamma \in \Gamma_j$.

A *ρ-proper section* of a path P of G_α is an initial section of P which is a ρ-proper path of G_α. If $1 \leq \lambda \leq \alpha$ and P is a λ-proper path of G_α ending at $a_{\lambda,n+1}$ (where $n \geq 1$) then P covers $b'_{\lambda,n}$ and so its last three terms must be $b'_{\lambda,n}, x_{\lambda,n}, a_{\lambda,n+1}$. We let P^* denote the initial section of P obtained by omitting these three terms, i.e. the initial section of P ending at $b_{\lambda,n}$. We again define an integer valued function rk on $V(H_\alpha)$ by letting $rk(v) = j$ if $v \in V(B_{\gamma,j})$ or $v \in V(D_{\gamma,j})$ for some $\gamma \in \Gamma_j$ and $rk(s) = 0$. Furthermore, we define $rk(F) = max(\{rk(v)\ :\ v \in F\})$ if F is a nonempty finite subset of $V(H_\alpha)$, and $rk(\emptyset) = 0$.

Lemma 3 *Let α, λ, ρ be countable ordinals such that $1 \leq \lambda < \rho \leq \alpha$ and P be a ρ-proper path of G_α and F be a finite subset of $V(H_\alpha)$. Then, for some π such that $\lambda \leq \pi < \rho$, there exists a π-proper extension Q of P in G_α such that $\Psi(Q^*, T_\alpha)$ covers F in H_α.*

Proof. The path P ends at a vertex $a_{\rho,m}$, where $m \geq \Phi(\rho)$. Choose $n \in \mathbb{N}$ such that $n \geq \max(\Phi(\lambda), m, rk(F)) + 2$. Since $n > \Phi(\lambda)$ and $n > m \geq \Phi(\rho)$, it follows that $\lambda, \rho \in \Gamma_n$ and so $\lambda \leq \pi < \rho$, where π is the Γ_n-predecessor of ρ. The set

$$E(P) \cup E(J(\rho, m, n), a_{\rho,m}, x_{\rho,n}) \cup \bigcup(E(J(\gamma, m, n), d_{\gamma,n}, x_{\gamma,n}) \cup \{e_{\gamma,n}\}\ :\ \gamma \in \Gamma_n \setminus \{\rho\}) \cup \{x_{\pi,n}a_{\pi,n+1}\}$$

is the set of edges of a π-proper extension Q of P. (In the example of Fig. 15, the edges in $E(Q) \setminus E(P)$ form the bold path.) Moreover $\Psi(Q^*, T_\alpha)$ covers F since $n \geq rk(F) + 2$. □

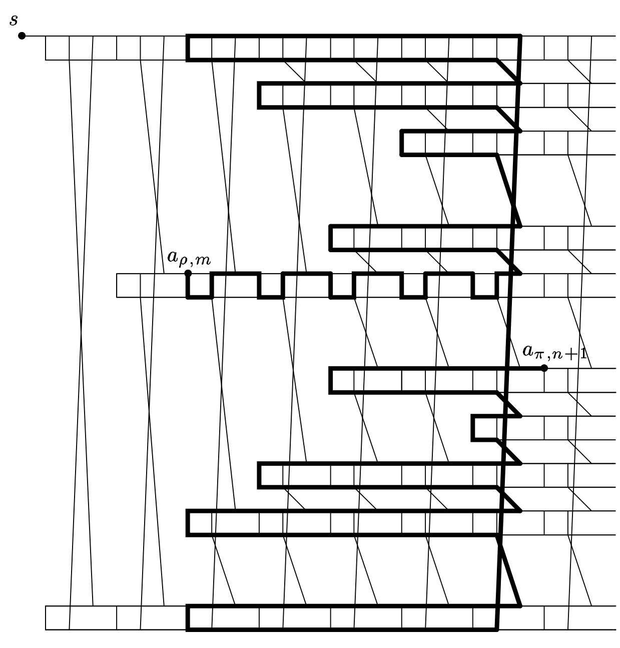

Fig. 15

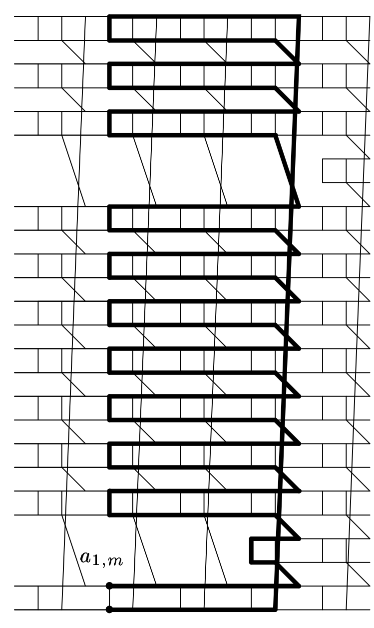

Fig. 16

Theorem 3 $(G_\alpha, \mathcal{F}_\alpha)$ *is not hamiltonian.*

Proof. We suppose, by way of contradiction, that G_α has a hamiltonian path P which satisfies the set of edge forcing conditions $\mathcal{F}_\alpha$. Then the path $s, a_{\alpha,1}$ is an α-proper section of P. Therefore we can define ρ

to be the least ordinal such that P has a ρ-proper section P'. Suppose that P' ends at $a_{\rho,m}$. If $\rho = 1$ then P could not cover both $b_{1,m}$ and $c_{1,m}$ and so could not be hamiltonian. Therefore $1 < \rho \leq \alpha$. Since the $e_{\gamma,j}$ are the only edges that connect the rows, infinitely many of them must be in $E(P)$. We can therefore choose an ordinal β and an integer $q \geq m$ such that $e_{\beta,q} \in E(P)$ and, subject to these requirements, q is as small as possible. Then, by Remark 2, P contains the set of edges

$$E(J(\rho, m, q), a_{\rho,m}, x_{\rho,q}) \cup \bigcup (E(J(\gamma, m, q), d_{\gamma,q}, x_{\gamma,q}) \ : \ \gamma \in \Gamma_q \setminus \{\rho\})$$

and the edge $e_{\beta,q}$. Consequently, $b_{\gamma,q}d_{\gamma,q} \notin E(P)$ when $\gamma \in \Gamma_q \setminus \{\rho\}$. Note also that $\rho \neq \beta$ since $d_{\rho,q}$ cannot be incident with three edges of P. We now make the following observations:

i) If $\gamma \in \Gamma_q \setminus \{\rho\}$ and $f_{\gamma,q+1} \in E(P)$ then $e_{\gamma,q} \in E(P)$ (since $b_{\gamma,q}d_{\gamma,q} \notin E(P)$ and P must cover both $d_{\gamma,q}$ and $c'_{\gamma,q+1}$).

ii) If $\gamma \in \Gamma_q$ and $e_{\gamma,q} \in E(P)$ then $f_{\pi_q(\gamma),q+1} \in E(P)$ (since P must cover both $a_{\pi_q(\gamma),q+1}$ and $b'_{\pi_q(\gamma),q}$).

iii) If $\gamma \in \Gamma_q \setminus \{\sigma_q(\rho)\}$ and $e_{\gamma,q} \in E(P)$ then $f_{\pi_q(\gamma),q+1} \in E(P)$ by ii) and thus $e_{\pi_q(\gamma),q} \in E(P)$ by i).

iv) If $\gamma \in \Gamma_q \setminus \{\pi_q(\rho)\}$ and $e_{\gamma,q} \in E(P)$ then $f_{\sigma_q(\gamma),q+1} \in E(P)$ by the edge forcing conditions and consequently $e_{\sigma_q(\gamma),q} \in E(P)$ by i).

Since $e_{\beta,q} \in E(P)$, it follows from iii) and iv) that $e_{\gamma,q} \in E(P)$ for every $\gamma \in \Gamma_q \setminus \{\rho\}$. Thus P contains the set of edges

$$E(J(\rho, m, q), a_{\rho,m}, x_{\rho,q}) \cup \bigcup (E(J(\gamma, m, q), d_{\gamma,q}, x_{\gamma,q}) \cup \{e_{\gamma,q}\} \ : \ \gamma \in \Gamma_q \setminus \{\rho\}) \cup \{x_{\pi_q(\rho),q}a_{\pi_q(\rho),q+1}\}$$

and so P has a $\pi_q(\rho)$-proper section, a contradiction. □

Lemma 4 *The graph H_α is α-extendable.*

Proof. We prove by induction that $\Psi(P^*, T_\alpha)$ is a ρ-path of H_α if P is a ρ-proper path of G_α. Let us prove it for $\rho = 1$. Let P be a 1-proper path of G_α (ending at $a_{1,m}$ say). If F is any finite subset of $V(H_\alpha)$, we choose n such that $n \geq m+2$ and $n \geq rk(F)+2$ and then F will be covered by the extension $\Psi(Q^*, T_\alpha)$ of $\Psi(P^*, T_\alpha)$ where Q is the path of G_α whose set of edges is

$$(E(P) \cup \bigcup (E(J(\gamma, m, n), d_{\gamma,n}, x_{\gamma,n}) \cup \{e_{\gamma,n}\} \ : \ \gamma \in \Gamma_n)) \setminus \{f_{1,m}\}.$$

The set of edges $E(Q) \setminus E(P)$ is illustrated in the example of Fig. 16. When $\rho = \pi + 1$ ($\pi \geq 1$) and F is a finite subset of $V(H_\alpha)$, there exists, by Lemma 3, a π-proper extension Q of P such that $\Psi(Q^*, T_\alpha)$ covers F. Then $\Psi(Q^*, T_\alpha)$ is, by induction, a π-path of H_α. Moreover $\Psi(Q^*, T_\alpha)$ extends $\Psi(P^*, T_\alpha)$, as may be seen by adapting the proof of the corresponding statement in the proof of Lemma 2. Therefore $\Psi(P^*, T_\alpha)$ is a ρ-path of H_α. When ρ is a limit ordinal, let us prove that if $1 \leq \lambda < \rho$ then $\Psi(P^*, T_\alpha)$ is a λ-path of H_α. By Lemma 3, there exist an ordinal π and a path Q such that $\lambda \leq \pi < \rho$ and Q is a π-proper extension of P, and therefore, from the inductive hypothesis, $\Psi(Q^*, T_\alpha)$ is a π-path of H_α. Thus $\Psi(P^*, T_\alpha)$ is a λ-path of H_α since $\lambda \leq \pi$ and $\Psi(Q^*, T_\alpha)$ extends $\Psi(P^*, T_\alpha)$. □

Consequently, by Lemma 1 and Theorem 3, the graph H_α is a non-hamiltonian graph and contains an α-path. Moreover, every vertex of H_α has degree at most four.

Problem 1 *Is it possible to find planar 3-regular examples of such graphs?*

Acknowledgements. We gratefully thank J.A. Bondy for his help during the preparation of this paper and the anonymous referees for their helpful comments.

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