Small degree out-branchings.*

Jørgen Bang-Jensen[†] Stéphan Thomassé[‡] Anders Yeo [§]

Abstract

Using a suitable orientation, we give a short proof of a result of Czumaj and Strothmann [3]: Every 2-edge-connected graph G contains a spanning tree T with the property that $d_T(v) \leq \frac{d_G(v)+3}{2}$ for every vertex v. Trying to find an analogue of this result in the directed case, we prove that every 2-arc-strong digraph D has an out-branching B such that $d_B^+(x) \leq \frac{d_D^+(x)}{2} + 1$. As a corollary, every k-arc-strong digraph D has an out-branching B such that $d_B^+(v) \leq \frac{d_D^+(v)}{2^r} + r$, where $r = \lfloor \log_2 k \rfloor$. We conjecture that in this case $d_B^+(x) \leq \frac{d_D^+(x)}{k} + 1$ would be the right (and best possible) answer. We prove that any degree requirement in out-branchings of acyclic digraphs can be polynomially checked.

1 Introduction

Finding a spanning tree with restrictions on the degrees (e.g. the maximum degree) in a graph is a well-known problem which has many practical applications e.g. in communication, design of reliable networks etc. Such problems have been studied extensively both in the mathematical and the computer science litterature. We refer to the references for a small sample of such papers. If we insist on finding a spanning tree where the maximum degree is at most some given bound, then the problem is NP-complete, but one can find, in polynomial time, a spanning tree whose maximum degree is at most one more than the optimum [6]. Czumaj and Strothmann [3] showed that if the input graph G is k-connected, then one can find, in polynomial time, a spanning tree T of G such that

$$\Delta(T) \leq \frac{\Delta(G) - 2}{k} + 2, \tag{1}$$

where $\Delta(T)$ and $\Delta(G)$ denote the maximum degrees in T and G respectively. They also showed that for the special case of 2-connected graphs one can obtain a stronger result: Every 2-connected graph G contains a spanning tree T with the property that

$$d_T(v) \leq \frac{d_G(v) + 3}{2} \text{ for every vertex } v. \tag{2}$$

In this paper we give another proof of this result, with the weaker hypothesis of 2-edge-connectivity. We also prove an analogue of the results in [3] for out-degrees of vertices in out-branchings in digraphs.

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[†]Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark (email jbj@imada.sdu.dk).

[‡]LaPCS, Université Claude Bernard, Lyon 1, France (email thomasse@jonas.univ-lyon1.fr)

[§]Department of Computer Science, Royal Holloway, University of London, Egham;Surrey TW20 0EX, UK (email:yeo@dcs.rhbnc.ac.uk)

Precisely, we prove that when D is a 2-arc-strong digraph then it has an out-branching for which the out-degree of every vertex is at most one plus half of its original out-degree. Using this result in a recursive way we obtain a result for k-arc-strong digraphs which resembles the result of [3] for k-connected graphs.

In Section 4 we show that for acyclic digraphs one can decide the existence of an out-branching using at most a specified number of out-neighbours at every vertex in polynomial time.

We generally follow the notation from [1]. An **out-arborescence** rooted at s is a (not necessarily spanning) tree T in the underlying graph of D for which every vertex of $V(T)$ except s has in-degree one. An **out-branching** is an out-arborescence which is spanning. We sometimes denote such an out-branching F_s^+ and we may also say that F_s^+ is an out-branching **from** s in D. An **in-arborescence** and an **in-branching** rooted at s are defined analogously and sometimes an in-branching rooted at s is denoted F_s^-.

2 Balanced Orientations

It turns out that the existence of a tree which satisfies (2) in a 2-edge-connected graph G follows directly from the following result (here a digraph D is **balanced** if $|d_D^+(x) - d_D^-(x)| \leq 1$ for all vertices x of D).

Theorem 2.1 *Every $2k$-edge-connected graph G has a k-arc-strong balanced orientation D.*

Proof: For the sake of self-containness, we give a proof of this result in the case $k = 1$. We say that an orientation D of a subgraph H of G is **good** if

i) D is balanced and strong on the set S of vertices it spans.

ii) there is at most one vertex a such that $d_D^+(v) = d_D^-(v) + 1$ and $d_H(v) < d_G(v)$.

iii) there is at most one vertex b such that $d_D^-(v) = d_D^+(v) + 1$ and $d_H(v) < d_G(v)$.

Clearly if $H = G$, we are done. Now suppose that H is a strict subgraph of G. If neither a nor b exist, take any path with both endpoints in S (this exists by 2-edge-connectedness) and add it as a directed path of D to form a good orientation D'. Now suppose that b exists in D, and thus $b \in S$. Grow a path P from b to another vertex c of S, direct P from b to c and add it to D. This new orientation is good. Starting with a directed circuit and continuing this process gives a balanced strong orientation of G. □

Corollary 2.2 *Every 2-edge-connected graph G contains a spanning tree T with the property that*

$$d_T(v) \leq \frac{d_G(v) + 3}{2} \text{ for every vertex } v.$$

Proof: Consider a strongly connected balanced orientation D of G and take any out-branching B of D. The underlying tree T of B is the tree we are looking for. □

Note that we only need edge-connectivity to conclude. A similar new approach of the undirected k-connected case would follow from the following conjecture:

Conjecture 2.3 *Let D be a k-arc-strong digraph. There exists a spanning out-branching B such that $d_B^+(x) \leq \frac{d_D^+(x)}{k} + 1$ for all vertices x of D.*

This would be best possible as there are k-arc-strong k-regular digraphs with no hamiltonian path. Now take G a $2k$-edge-connected graph, by Theorem 2.1, it has a k-arc-strong balanced orientation. Hence, Conjecture 2.3 would give a spanning out-branching B whose underlying tree T satisfies $d_T^+(x) \leq \frac{d_T(x) + 1}{2k} + 2$ for all vertices x. Our next section is devoted to Conjecture 2.3.

3 Small degree out-branchings

We shall use the following classical result by Edmonds characterizing when a digraph has k arc-disjoint out-branchings from a chosen vertex s.

Theorem 3.1 (Edmonds) *[5] A digraph $D = (V, A)$ contains k arc-disjoint out-branchings from a specified vertex $s \in V$ if and only if*

$$d^-(X) \geq k \text{ for all } X \subseteq V - s. \tag{3}$$

By Theorem 3.1, every 2-arc-strong digraph is in particular the union of two out-branchings rooted at a given point r. So the case $k = 2$ of Conjecture 2.3 follows from the following theorem.

Theorem 3.2 *Let D be a digraph which is the union of two out-branchings rooted at r. There exists an out-branching A rooted at r such that $d^+_A(x) \leq \frac{d^+_D(x)}{2} + 1$ for all vertices x of D.*

Proof: Let A' be an out-arborescence rooted at r, we denote by D' the subdigraph of D which is induced by the vertices of A'. We say that A' is *good* if for all vertices x of A':

$$d^+_{A'}(x) \leq \frac{d^+_{D'}(x)}{2} + 1 \text{ when } d^+_{D'}(x) = d^+_D(x) \text{ and } d^+_{A'}(x) \leq \frac{d^+_{D'}(x) + 1}{2} \text{ otherwise.}$$

Clearly the out-arborescence consisting of just the vertex r is good and if one can find a good spanning out-arborescence, the proof is achieved. It suffices then to prove that every non spanning good out-arborescence A' is strictly contained in a good out-arborescence.

Call a vertex x of A' an *out-vertex* if it has an out-neighbour in D which belongs to $V - V(A')$. Suppose one vertex x of A' has precisely one out-neighbour in D which belongs to $V - V(A')$ (i.e. $d^+_{D'}(x) = d^+_D(x) - 1$) then taking $A'' = A' \cup xy$ and letting D'' be the subdigraph induced by $V(A'')$ in D we have

$$\begin{aligned} d^+_{A''}(x) = d^+_{A'}(x) + 1 &\leq \frac{d^+_{D'}(x) + 1}{2} + 1 \\ &= \frac{d^+_D(x)}{2} + 1 \\ &= \frac{d^+_{D''}(x)}{2} + 1, \end{aligned}$$

implying that A'' is good. Hence we can assume that every out-vertex $x \in A'$ has at least two out-neighbours belonging to $V - V(A')$ in D.

Start now from any out-vertex x_1 and denote by y_1 one of its outneighbours in $V - V(A')$. As D contains two arc-disjoint (s, y_1)-paths there exists a path P_1 starting at some vertex x_2 of A' and ending at y_1, with all internal vertices outside of A' and which does not use the arc x_1y_1. Since x_2 has at least two out-neighbours in $V - V(A')$ it is the origin of an arc $e = x_2y_2$, where $y_2 \notin A'$ and e is not the first arc of P_1. Applying the same argument as above we see that y_2 is the end of a path P_2 starting at some vertex x_3 of A', with all internal vertices in $V - V(A')$, and which does not use the arc e. We continue this construction, and let k be the largest integer such that $V(P_1), \ldots, V(P_{k-1})$ are pairwise disjoint. We denote by a the first repeated vertex (which belongs to P_k. Here, by the first repeated vertex we mean the first vertex among $V(P_1) \cup \ldots V(P_{k-1})$ that we encounter by moving backwards on P_k starting in y_k). If $a = x_1$, we consider the out-arborescence $A' \cup P_1 \cup \ldots \cup P_k$, which is good: every new vertex that we add to A' has out-degree one and the out-degree of x_i, $i = 1, 2, \ldots, k$ is increased by one and we include at least two arcs out of x_i in the digraph induced by the new out-arborescence. If $a \neq x_1$, there exists a unique P_i, $i < k$ such that $a \in P_i$. Again it is easy to see that the out-arborescence $A' \cup P_i[x_{i+1}, a] \cup P_{i+1} \cup \ldots \cup P_{k-1} \cup P_k[a, y_k]$ is good. □

Theorem 3.3 *If the vertex s has k arc-disjoint paths to every other vertex in D, then D has an out-branching T rooted at s such that $d^+_T(v) \leq \frac{d^+_D(v)}{2^r} + r$, where $r = \lfloor \log_2 k \rfloor$.*

Proof: Let r be chosen such that $2^r \leq k < 2^{r+1}$ and let $p = 2^r$. By Theorem 3.1, D contains p arc-disjoint out-branchings $T_1, T_2, \ldots, T_p$ all of which are rooted at s. For each $1 \leq i < j \leq p$ we denote by $D_{i,j}$ the digraph formed by the union of the out-branchings $T_i, T_{i+1}, \ldots, T_j$ (that is the vertex set is V and the arc set is the union of the arcs from $T_i, T_{i+1}, \ldots, T_j$).

Now consider $D_{1,2}$ which is the union of two out-branchings. By Theorem 3.2, for each $i = 1, 2, \ldots, p/2$ the digraph $D_{2i-1,2i}$ contains an out-branching $T_{2i-1,2i}$ which satisfies for all $v \in V$ that

$$d^+_{T_{2i-1,2i}}(v) \leq \frac{d^+_{D_{2i-1,2i}}(v)}{2} + 1. \tag{4}$$

Next we form $p/4$ new digraphs $D'_{4i-3,4i}$ for $i = 1, 2, \ldots, p/4$ as follows. The digraph $D'_{4i-3,4i}$ has vertex set V and arc set the union of the arcs in the two out-branchings $T_{4i-3,4i-2}$ and $T_{4i-1,4i}$. Note that by (4), for all $v \in V$ we have

$$\begin{aligned} d^+_{D'_{4i-3,4i}}(v) &= d^+_{T_{4i-3,4i-2}}(v) + d^+_{T_{4i-1,4i}}(v) \\ &\leq (\frac{d^+_{D_{4i-3,4i-2}}(v)}{2} + 1) + (\frac{d^+_{D_{4i-1,4i}}(v)}{2} + 1) \\ &= \frac{d^+_{D_{4i-3,4i}}(v)}{2} + 2. \end{aligned}$$

Applying Theorem 3.2 to $D'_{4i-3,4i}$ for each $i = 1, 2, \ldots, p/4$, we see that $D'_{4i-3,4i}$ contains an out-branching $T_{4i-3,4i}$ rooted at s such that for every $v \in V$

$$d^+_{T_{4i-3,4i}}(v) \leq \frac{d^+_{D'_{4i-3,4i}}(v)}{2} + 1 \tag{5}$$

$$\leq \frac{\frac{d^+_{D_{4i-3,4i}}(v)}{2} + 2}{2} + 1 \tag{6}$$

$$= \frac{d^+_{D_{4i-3,4i}}(v)}{4} + 2. \tag{7}$$

Repeating the procedure above r times in total it follows that the digraph $D'_{1,p}$ contains an out-branching $T_{1,p}$ rooted at s such that for all $v \in V$

$$d^+_{T_{1,p}}(v) \leq \frac{d^+_{D_{1,p}}(v)}{p} + r \leq \frac{d^+_D(v)}{p} + r. \tag{8}$$

□

4 Acyclic digraphs

For a set of vertices X in a digraph D we denote by X^- the set of vertices with at least one arc to a vertex in X. Note that $X \cap X^-$ is non-empty whenever two vertices in X are connected by an arc.

Theorem 4.1 *Let $D = (V, A)$ be an acyclic digraph and let $f : V \rightarrow Z_+ \cup \{0\}$. Suppose that D has precisely one vertex s of in-degree zero. Then D has an out-branching T rooted at s satisfying*

$$d^+_T(v) \leq f(v) \text{ for all } v \in V \tag{9}$$

if and only if

$$\sum_{x \in X^-} f(x) \geq |X| \text{ for all } X \subset V - s. \tag{10}$$

Furthermore, there exists a polynomial algorithm to test whether a given acyclic digraph contains an out-branching satisfying (9) with respect to a given non-negative integer assignment to its vertices.

Proof: Given $D = (V, A)$ we form a flow network N as follows. The vertex set of N consists of two copies v', v'' of each vertex $V \in V - s$, one copy s'' of s and finally a new vertex z. The arc set of N is $A(N) = \{u'' \rightarrow v' : u \rightarrow v \in A\} \cup \{v' \rightarrow z : v \in V - s\} \cup \{z \rightarrow v'' : v \in V\}$. There are the following capacities and lower bounds on the arcs:

- all arcs of the type $v' \rightarrow z$ have capacity and lower bound equal to one.
- all arcs $u'' \rightarrow v'$ corresponding to arcs in D have lower bound zero and infinite capacity.
- all arcs of the type $z \rightarrow v''$ have lower bound zero and capacity equal to $f(v)$.

We claim that D has an out-branching T rooted at s satisfying (9) if and only if N has a feasible circulation. First assume that T is an out-branching rooted at s in D which satisfies (9). Since the in-degree of every vertex except s is precisely one in T it follows that the following x is a feasible circulation in N:

- $x(u'' \rightarrow v') = 1$ if $u \rightarrow v$ is an arc of T.
- $x(v' \rightarrow z) = 1$ for all vertices of the type v'.
- $x(z \rightarrow v'') = d^+_T(v)$ for all vertices of the type v''.

Conversely, if x is a feasible integer valued circulation in N, then let A' be the set of those arcs $u \rightarrow v$ in D for which $x(u'' \rightarrow v') = 1$. It is easy to see that these arcs form a spanning acyclic subdigraph T' of D with $n-1$ arcs and in which s is the only vertex of in-degree zero. Thus T' is exactly an out-branching rooted at s. Furthermore, by the capacity constraint on the arcs $z \rightarrow v''$ no vertex v is the tail of more than $f(v)$ arcs in T'.

Now we are ready to prove the first claim of the theorem. Since every vertex except s has in-degree one in an out-branching from s it is easy to see that (10) must hold if D has an out-branching satisfying (9). Suppose now that D and f satisfy (10). By the arguments above it suffices to prove that N has a feasible circulation.

Assume this is not the case. Then by Hoffman's circulation theorem (see e.g. [1, Theorem 3.8.2]) there exists some partition $S, \bar{S}$ of $V(N)$ such that the sum $l(S, \bar{S})$ of the lower bounds on the arcs from S to $\bar{S}$ is strictly larger than the sum $u(\bar{S}, S)$ of the capacities on the arcs from $\bar{S}$ to S. Since only arcs into z have a non-zero lower bound we have $z \in \bar{S}$. Let X' be the set of those v' that belong to S and let X be the corresponding set of vertices in D. By the choice of capacities in N we see that every vertex w'' which has an arc to a vertex in X' must belong to S. Since z has an arc to all such vertices w'' with capacity $f(w)$ and each such arc contributes to $u(\bar{S}, S)$ we have

$$l(S, \bar{S}) > u(\bar{S}, S) \geq \sum_{w \in X^-} f(w) \geq |X| = l(S, \bar{S}),$$

a contradiction. Hence N has a feasible circulation and the desired branching exists in D.

The second part of the theorem follows from the fact that our proof can be turned into an algorithm for checking the existence of the desired branching (and finding one if it exists) by using flow techniques to search for a feasible integer valued circulation in the corresponding network N.

□

Corollary 4.2 *Let D be an acyclic digraph such that there is exactly one vertex s of in-degree zero and exactly one vertex t of out degree zero in D. Then D contains arc-disjoint branchings F_s^+ and F_t^- where the first is an out-branching rooted at s and the second is an in-branching rooted at t if and only if we have*

$$\sum_{x \in X^-} (d^+(x) - 1) \geq |X| \text{ for all } X \subset V - s. \tag{11}$$

Furthermore it can be decided in polynomial time whether D has such branchings.

Proof: It is an easy fact that an acyclic digraph H has an in-branching rooted at a vertex z if and only if z is the unique vertex of out-degree zero in H. Now we see that D has the desired branchings if and only if D has an out branching rooted at s which satisfies (9) with respect to $f(v) = d^+(v) - 1$ for $v \neq t$ and $f(t) = 0$. By Theorem 4.1 this is equivalent to requirering that (11) must hold.

The complexity claim follows from the last part of Theorem 4.1. □

5 Open problems

Since the hamiltonian path problem is NP-complete for 2-regular digraphs we have the following.

Proposition 5.1 *It is NP-complete to decide whether a digraph D contains an out-branching T such that $D - A(T)$ has out-degree at least one at every vertex.*

We proved in this paper that if a digraph is 2-strong and every vertex has outdegree at least 3, such an out-branching can be found. But one can ask for a stronger property of $D - A(T)$, for instance to contain an in-branching. Whereas Corollary 4.2 completely solve the question in the case of acyclic digraphs, it is not known if there is a bound on the arc-strong-connectivity which insure the existence of an in-branching and an out-branching rooted at a given vertex.

If true Conjecture 2.3 would imply the following.

Conjecture 5.2 *Every k-arc-strong digraph with maximum out-degree less than $2k$ contains an out-branching rooted at s with maximum out-degree 2 for every choice of s.*

Sometimes one may not only be interested in finding an out-branching from s with small degree (relative to the degree in D) at every vertex, but one may also want to find such a branching that still leaves another out-branching at s when removed. It follows from our proof of Theorem 3.3 that if s has at least $2^r + t$ arc-disjoint paths to every other vertex then we can find an out-branching from s in which every vertex has out-degree at most r plus 2^{-r} times its original out-degree and such that removing this out-branching still leaves t arc-disjoint out-branchings from s. Furthermore, by dividing a set of 2^r arc-disjoint out-branchings into two groups and applying the proof of Theorem 3.3 to each we can find two arc-disjoint out-branchings such that each of these have out-degree at most $r - 1$ plus 2^{1-r} times its original out-degree.

Problem 5.3 *Is it true that every digraph which is the union of k out-branchings formed at root s always contains an out-branching T such that $D - A(T)$ contains an out-branching from s and*

$$d_T^+(x) \leq \frac{d_D^+(x)}{k} + c \text{ for some constant } c? \tag{12}$$

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