The C_3-structure of the tournaments

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Abstract

Let $T = (V, E)$ be a tournament. The C_3-structure of T is the family $C_3(T)$ of the subsets $\{x, y, z\}$ of V such that the subtournament $T(\{x, y, z\})$ is a cycle on 3 vertices. In another respect, a subset X of V is an interval of T provided that for $a, b \in X$ and $x \in V - X$, $(a, x) \in E$ if and only if $(b, x) \in E$. For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are intervals of T, called trivial intervals. A tournament is indecomposable if all its intervals are trivial. Lastly, with each tournament $T = (V, E)$ is associated the dual tournament $T^\star = (V, E^\star)$ defined as: for $x, y \in V$, $(x, y) \in E^\star$ if $(y, x) \in E$. The following theorem is proved. Given tournaments $T = (V, E)$ and $T = (V, E')$ such that $C_3(T) = C_3(T')$, if T is indecomposable, then $T' = T$ or $T' = T^\star$. In order to treat the nonindecomposable case, the interval inversion is introduced. The paper concludes with an extension of this result to the digraphs which do not admit as subdigraphs $(\{0, 1, 2\}, \{(0, 1), (1, 0), (1, 2)\})$ and $(\{0, 1, 2\}, \{(0, 1), (1, 0), (2, 1)\})$, and with a brief consideration of the infinite case.

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1 Introduction

A *digraph* D consists of a finite set V of *vertices* together with a prescribed collection E of ordered pairs of distinct vertices called the set of *edges* of D. Such a digraph is denoted by (V, E). For example, given a set V, $(V, \emptyset)$ (resp. $(V, (V \times V) - \{(x, x); x \in V\})$ is the *empty* (resp. *complete*) digraph on V. Given a digraph $D = (V, E)$, with each subset X of V is associated the *subdigraph* $D(X) = (X, (X \times X) \cap E)$ of D *induced* by X. The subdigraph $D(V - X)$, where $X \subseteq V$, (resp. $D(V - \{x\})$, where $x \in V$,) is also denoted by $D - X$ (resp. $D - x$). A *tournament* is a digraph (V, E) provided that for $x \neq y \in V$, $(x, y) \in E$ if and only if $(y, x) \notin E$. For example, $(\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$ is a tournament called 3-*cycle*. Given a digraph $D = (V, E)$, the C_3-*structure* of D is the family $C_3(D)$ of the subsets $\{x, y, z\}$ of V such that $D(\{x, y, z\})$ is a 3-cycle. A *poset* is a digraph $P = (V, E)$ such that for $x, y \in V$, if $(x, y) \in E$, then $(y, x) \notin E$ and such that for $x, y, z \in V$, if $(x, y), (y, z) \in E$, then $(x, z) \in E$. In the case of a poset $P = (V, E)$, for $x \neq y \in V$, $x < y$ means $(x, y) \in E$. A *total order* is a tournament T such that $C_3(T) = \emptyset$. Given a poset P, the *comparability structure* of P is the family $C_2(P)$ of the pairs $\{x, y\}$ such that $x < y$ or $y < x$. In another respect, with each digraph $D = (V, E)$ is associated the *dual* digraph $D^\star = (V, E^\star)$ of D and the *complement* digraph $\overline{D} = (V, \overline{E})$ of D defined in the following manner. For $x \neq y \in V$, $(x, y) \in E^\star$ if $(y, x) \in E$ and $(x, y) \in \overline{E}$ if $(x, y) \notin E$.

Given a digraph $D = (V, E)$, a subset X of V is an *interval* [9, 10] (or an *autonomous* subset [5, 11] or a *clan* [4] or an *homogeneous* subset [2, 7] or a *module* [13]) of D provided that for any $a, b \in X$ and $x \in V - X$, $(a, x) \in E$ (resp. $(x, a) \in E$) if and only if $(b, x) \in E$ (resp. $(x, b) \in E$). For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are intervals of D, called *trivial* intervals. A digraph is then said to be *indecomposable* (or *prime* [2] or *primitive* [4]) if all its intervals are trivial. Otherwise, it is said to be *decomposable*. Given a digraph $D = (V, E)$, for each interval X of D is defined the digraph $\mathcal{I}(D, X) = (V, \mathcal{I}(E, X))$ obtained from D by the *interval inversion* as follows. For every $x \neq y \in V$, $(x, y) \in \mathcal{I}(E, X)$ if either $\{x, y\} - X \neq \emptyset$ and $(x, y) \in E$ or $\{x, y\} \subseteq X$ and $(y, x) \in E$. The transitive closure of the interval inversion is denoted by $\mathcal{I}$. More precisely, given two digraphs D and D' with the same set of vertices V, $D\mathcal{I}D'$ signifies that there are digraphs $D_0 = D, \ldots, D_n = D'$ such that for $0 \leq i \leq n - 1$, $D_{i+1} = \mathcal{I}(D_i, X_i)$, where X_i is an interval of D_i.

The next theorem follows from Gallai's decomposition theorem ([5], see also Theorem 1.2 of [11]).

Theorem 1 (Gallai [5]) *Given posets P and Q with the same set of vertices, $C_2(P) = C_2(Q)$ if and only if $P\mathcal{I}Q$.*

By considering the C_3-structure instead of the comparability structure, we establish the following.

Theorem 2 *Given tournaments $T = (V, E)$ and $T' = (V, E')$, $C_3(T) = C_3(T')$ if and only if $T\mathcal{I}T'$.*

In order to generalize these theorems, we have to forbid as subdigraphs the following two digraphs called *flags* : $(\{0, 1, 2\}, \{(0, 1), (1, 0), (1, 2)\})$ and $(\{0, 1, 2\}, \{(0, 1), (1, 0), (2, 1)\})$. In another vein, the digraphs $D = (V, E)$ and $D' = (V, E')$ are said to be *hemimorphic* if for every $X \subseteq V$ such that $| X |= 2$ or 3, $D'(X)$ is isomorphic to $D(X)$ or to $D^\star(X)$.

Theorem 3 *Given digraphs D and D' without flags, D and D' are hemimorphic if and only if $D\mathcal{I}D'$.*

Theorem 3 was announced in [1] without a proof.

2 Strongly connected components, difference relation and equality relation

Some notations are needed. Given a digraph $D = (V, E)$, for $x \neq y \in V$, $x \longrightarrow y$ means $(x, y) \in E$ and $(y, x) \notin E$, $x \longleftrightarrow y$ means $(x, y), (y, x) \in E$ and $x \cdots y$ means $(x, y), (y, x) \notin E$. For $x \in V$ and $Y \subseteq V$, $x \longrightarrow Y$ signifies $x \longrightarrow y$ for every $y \in Y$ and for $X, Y \subseteq V$, $X \longrightarrow Y$ signifies $x \longrightarrow Y$ for any $x \in X$. For $x \in V$ and for $X, Y \subseteq V$, $Y \longrightarrow x$, $x \longleftrightarrow Y$, $x \cdots Y$, $X \longleftrightarrow Y$ and $X \cdots Y$ are defined in the same way. Using these notations, for a digraph $D = (V, E)$, a subset X of V is an interval of D if for any $x \in V - X$, either $x \longrightarrow X$ or $X \longrightarrow x$ or $x \longleftrightarrow X$ or $x \cdots X$. This generalizes the classic notion of an interval of a total order. Moreover, it is clear in this form that D, $D^\star$ and $\overline{D}$ have the same intervals, for every digraph D. As shown by the following proposition, the intervals of a digraph and the usual intervals of a total order share the same properties.

Proposition 1 *Let $D = (V, E)$ be a digraph.*

1. *Given a subset W of V, if X is an interval of D, then $X \cap W$ is an interval of $D(W)$.*
2. *If X and Y are intervals of D, then $X \cap Y$ is an interval of D.*
3. *If X and Y are intervals of D such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is an interval of D.*
4. *If X and Y are intervals of D such that $X - Y \neq \emptyset$, then $Y - X$ is an interval of D.*
5. *If X and Y are intervals of D such that $X \cap Y = \emptyset$, then either $X \longrightarrow Y$ or $Y \longrightarrow X$ or $X \longleftrightarrow Y$ or $X \cdots Y$.*

The last assertion of the above proposition allows for the definition of the quotient of a digraph. Given a digraph $D = (V, E)$, a partition P of V is an *interval partition* of D if for any $X \in P$, X is an interval of D. With each interval partition P of D is associated the *quotient* $D/P = (P, E/P)$ of D by P defined in the following manner. For every $X \neq Y \in P$, $(X, Y) \in E/P$ if for $x \in X$ and $y \in Y$, $(x, y) \in E$.

To continue, the strongly connected components of a tournament are introduced. To each tournament $T = (V, E)$ is associated the equivalence relation $\mathcal{S}$ defined on V as follows. For any $x \neq y \in V$, $x\mathcal{S}y$ if there are $x_0 = x, \ldots, x_m = y \in V$ and $y_0 = y, \ldots, y_n = x \in V$ fulfilling : for $0 \leq i \leq m-1$, $x_i \longrightarrow x_{i+1}$ and for $0 \leq j \leq n-1$, $y_j \longrightarrow y_{j+1}$. The equivalence classes of $\mathcal{S}$ are called the *strongly connected components* ot T. The tournament T is *strongly connected* if it admits a single strongly connected component. The family of the strongly connected components of T is denoted by $\mathcal{S}(T)$. After examining the properties of the strongly connected components, we will give a simple proof of Gallai's decomposition theorem for the tournaments.

Lemma 1 *Given a tournament $T = (V, E)$, $\mathcal{S}(T)$ is an interval partition of T. Moreover, T is not strongly connected if and only if $T/\mathcal{S}(T)$ is a total order with $\mid \mathcal{S}(T) \mid \geq 2$.*

Proof. Let S be a strongly connected component of T. To show that S is an interval of T, it suffices to verify that for $x, y \in S$ and $z \in V - S$, if $x \longrightarrow z \longrightarrow y$, then $z \in S$. Indeed, since there are $y_0 = y, \ldots, y_n = x \in V$ such that $y_j \longrightarrow y_{j+1}$ for $0 \leq j \leq n-1$, x, y and z are equivalent modulo $\mathcal{S}$.

Let S, S' and S'' be distinct strongly connected components of T. Given $x \in S$, $x' \in S'$ and $x'' \in S''$, as x, x' and x'' are not equivalent modulo $\mathcal{S}$, $T(\{x, y, z\})$ is not a 3-cycle. It follows that $[T/\mathcal{S}(T)](\{S, S', S''\})$ is not a 3-cycle and, thus, $C_3[T/\mathcal{S}(T)] = \emptyset$. □

Lemma 2 *Given tournaments T and T' with the same set of vertices V, if $C_3(T) = C_3(T')$, then $\mathcal{S}(T) = \mathcal{S}(T')$.*

Proof. Given $x \neq y \in V$ such that $x\mathcal{S}y$ in T. For example, if $x \longrightarrow y$ in T, then consider the smallest integer n such that there are $y_0 = y, \ldots, y_n = x \in V$ satisfying : for $0 \leq i \leq n-1$, $y_i \longrightarrow y_{i+1}$ in T. By the minimality of n, for $0 \leq i \leq n-2$ and for $i+2 \leq j \leq n$, $y_j \longrightarrow y_i$ in T. Consequently, for $0 \leq i \leq n-2$, $T(\{y_i, y_{i+1}, y_{i+2}\})$ is a 3-cycle. Since $C_3(T) = C_3(T')$, $T'(\{y_i, y_{i+1}, y_{i+2}\})$ is a 3-cycle and, hence, y_i, y_{i+1} and y_{i+2} are equivalent modulo $\mathcal{S}$ in T'. By transitivity, $y_0 = y\mathcal{S}y_n = x$ in T'. □

By Lemma 1, if a tournament $T = (V, E)$ is not strongly connected, then $T/\mathcal{S}(T)$ is a total order. Clearly, if S is the minimum element of $T/\mathcal{S}(T)$, then $V - S$ is an interval of T. Such a subset S of V is called a cut of T. More generally, given a digraph $D = (V, E)$, a subset X of V is a *cut* of D if X and

$V - X$ are intervals of D. For example, $\emptyset$ and V are cuts of D, called *trivial* cuts.

¿From the last assertion of Proposition 1, it ensues that for every tournament $T = (V, E)$, if X is a cut of T, then $X \longrightarrow (V - X)$ or $(V - X) \longrightarrow X$. The next lemma follows immediately.

Lemma 3 *Given a tournament T, T is strongly connected if and only if all its cuts are trivial.*

Consequently, if S is a strongly connected component of a tournament $T = (V, E)$, then all the cuts of $T(S)$ are trivial. Let X be an interval of T such that $S \cap X \neq \emptyset$ and $X - S \neq \emptyset$. ¿From Proposition 1, since $X - S \neq \emptyset$, $S - X$ is an interval of T. Thus, $S \cap X$ is a cut of $T(S)$ and $S \subseteq X$. Such a subset S of V is called a strong interval of T. More generally, given a digraph $D = (V, E)$, a subset X of V is a *strong interval* of D [5, 11] provided that X is an interval of D and for every interval Y of D, if $X \cap Y \neq \emptyset$, then $X \subseteq Y$ or $Y \subseteq X$. It is easily verified that all the strong intervals of a total order are trivial. The following result characterizes such tournaments.

Proposition 2 *Given a tournament T, all the strong intervals of T are trivial if and only if T is a total order or T is indecomposable.*

Proof. It suffices to show that if T is decomposable and if all its strong intervals are trivial, then T is a total order. Among the nontrivial intervals of T, consider a maximal one X with respect to the inclusion. As X is not a strong interval of T, there is an interval Y of T such that $X \cap Y$, $X - Y$ and $Y - X$ are nonempty. By Proposition 1, since $X \cap Y \neq \emptyset$, $X \cup Y$ is an interval of T. As $X \subset X \cap Y$, it follows from the maximality of X that $X \cup Y = V$. Since $X - Y \neq \emptyset$, $V - X = Y - X$ is an interval of T. Consequently, X is a nontrivial cut of T. By Lemmas 1 and 2, $\mid \mathcal{S}(T) \mid \geq 2$ and $T/\mathcal{S}(T)$ is a total order. For every $S \in \mathcal{S}(T)$, since $S \neq V$ and since S is a strong interval of T, $\mid S \mid = 1$. Consequently, $\mathcal{S}(T) = \{\{x\}; x \in V\}$ and T is a total order. $\square$

The previous proposition leads us to attribute to any tournament a quotient, all the strong intervals of which are trivial. Given a digraph $D = (V, E)$, $P(D)$ denotes the family of maximal strong intervals of D under the inclusion which are distinct from V. The following three lemmas review some properties of $P(D)$.

Lemma 4 *For every digraph $D = (V, E)$, $P(D)$ is an interval partition.*

Proof. Given any $x \in V$, since $\{x\}$ is a strong interval of D, there is $X \in P(D)$ such that $\{x\} \subseteq X$. Let X and Y be elements of $P(D)$ such that $X \cap Y \neq \emptyset$. As X and Y are strong intervals of D, $X \subseteq Y$ or $Y \subseteq X$. It follows from the maximality of the elements of $P(D)$ that $X = Y$. $\square$

Lemma 5 *Given a tournament $T = (V, E)$, if T is not strongly connected, then $P(T) = \mathcal{S}(T)$.*

Proof. By Lemma 1, the elements of $\mathcal{S}(T)$ may be denoted by $X_0, \ldots, X_n$ in such a way that $T/\mathcal{S}(T)$ is the total order $X_0 < \cdots < X_n$. It suffices to prove that for any interval $X \neq V$ of T, if there is $i \in \{0, \ldots, n\}$ such that $X_i \subset X$, then X is not a strong interval. As X is an interval of T, the family $I = \{j \in \{0, \ldots, n\} : X_j \cap X \neq \emptyset\}$ is an interval of the usual total order on $\{0, \ldots, n\}$. Since $X_i \subset X \subset V$, there are $k < l \in \{0, \ldots, n\}$ such that $I = \{k, \ldots, l\}$. Since the strongly connected components of T are strong intervals of T, $X = X_k \cup \ldots \cup X_l$ and, thus, $(k, l) \neq (0, n)$. For example, if $k \neq 0$, then $X_0 \cup \ldots \cup X_k$ is an interval of T such that $X \cap (X_0 \cup \ldots \cup X_k)$, $X - (X_0 \cup \ldots \cup X_k)$ and $(X_0 \cup \ldots \cup X_k) - X$ are nonempty. □

Lemma 6 *For any digraph $D = (V, E)$, all the strong intervals of $D/P(D)$ are trivial.*

Proof. It is sufficient to verify that any nontrivial interval Q of $D/P(D)$ is not a strong interval of $D/P(D)$. The union $\bigcup Q$ of the element of Q is an interval of D such that for $X \in Q$, $X \subset \bigcup Q \subset V$. By the maximality of the elements of $P(D)$, $\bigcup Q$ is not a strong interval of D and, hence, there is an interval Y of D such that $Y \cap (\bigcup Q)$, $Y - (\bigcup Q)$ and $(\bigcup Q) - Y$ are nonempty. The family $Y/P(D) = \{Z \in P(D) : Y \cap Z \neq \emptyset\}$ is an interval of $D/P(D)$. Since $Y \cap (\bigcup Q) \neq \emptyset$ and since $Y - (\bigcup Q) \neq \emptyset$, $|Y/P(D)| \geq 2$. As the elements of $P(D)$ are strong intervals of D, $Y = \bigcup(Y/P(D))$ and, thus, $(Y/P(D)) \cap Q$, $(Y/P(D)) - Q$ and $Q - (Y/P(D))$ are nonempty. □

Gallai's decomposition theorem for tournaments is then stated as follows.

Theorem 4 *Let T be a tournament.*

1. *T is not strongly connected if and only if $T/P(T)$ is a total order.*
2. *T is strongly connected if and only if $|P(T)| \geq 3$ and $T/P(T)$ is indecomposable.*

Proof. By Proposition 2 and Lemma 6, it suffices to establish the first equivalence. If T is not connected, then, by Lemma 1, $T/\mathcal{S}(T)$ is a total order and, by Lemma 5, $P(T) = \mathcal{S}(T)$. Conversely, if $T/P(T)$ is a total order, then its minimum element is a nontrivial cut of T and Lemma 3 allows for the conclusion. □

To establish Theorem 2, the difference relation and the equality relation are introduced. Let $D = (V, E)$ and $D' = (V, E')$ be hemimorphic digraphs. A pair $\{x, y\}$ of elements of V is a *difference pair* (resp. an *equality pair*) if $D(\{x, y\})$ is a tournament such that $D(\{x, y\}) \neq D'(\{x, y\})$ (resp. $D(\{x, y\}) =$

$D'(\{x,y\}))$. The *difference relation* [12] (resp. the *equality relation*) is the equivalence relation $\mathcal{D}$ (resp. $\mathcal{E}$) defined on V by : for $x \neq y \in V$, $x\mathcal{D}y$ (resp. $x\mathcal{E}y$) if there are $x_0 = x, \ldots, x_n = y \in V$ such that for $0 \leq i \leq n-1$, $\{x_i, x_{i+1}\}$ is a difference pair (resp. an equality pair). The family of the equivalence classes of $\mathcal{D}$ (resp. $\mathcal{E}$) is denoted by $\mathcal{D}(D, D')$ (resp. $\mathcal{E}(D, D')$).

Lemma 7 (Hagendorf and Lopez [8]) *Given hemimorphic digraphs D and D', if D is without flags, then $\mathcal{D}(D, D')$ and $\mathcal{E}(D, D')$ are interval partitions of D and of D'.*

In order to demonstrate the next proposition, the following well known result is recalled.

Lemma 8 *Given a tournament $T = (V, E)$ with $\mid V \mid \geq 4$, T is strongly connected if and only if there are $x \neq y \in V$ such that $T - x$ and $T - y$ are strongly connected.*

Proposition 3 *Given tournaments $T = (V, E)$ and $T' = (V, E')$ such that $C_3(T) = C_3(T')$, if T is strongly connected, then $V \notin \mathcal{D}(T, T') \cap \mathcal{E}(T, T')$.*

Proof. Proceed by induction on $\mid V \mid$. Firstly, if $\mid V \mid = 3$, then, as T is strongly connected, T is a 3-cycle. Since $C_3(T) = C_3(T')$, T' is a 3-cycle and either $T' = T$ or $T' = T^\star$. In the first case, $V \notin \mathcal{D}(T, T')$ and, in the second one, $V \notin \mathcal{E}(T, T')$. Secondly, if $\mid V \mid > 3$, then, by the previous lemma, there is $x \in V$ such that $T - x$ is strongly connected. By the induction hypothesis, it may be assumed by considering $T^\star$ in place of T that $V - \{x\} \notin \mathcal{D}(T - x, T' - x)$. It is then sufficient to show that if $V \in \mathcal{E}(T, T')$, then $V \notin \mathcal{D}(T, T')$. Indeed, if $V \in \mathcal{E}(T, T')$, then there is $y \in V - \{x\}$ such that $\{x, y\}$ is an equality pair. For example, suppose that $x \longrightarrow y$. The element of $\mathcal{D}(T - x, T' - x)$ which contains y is denoted by Y. As $T - x$ is strongly connected, there is $Z \in \mathcal{D}(T - x, T' - x) - \{Y\}$ such that $Y \longrightarrow Z$. Since $x \longrightarrow y \longrightarrow Z$ and since $C_3(T) = C_3(T')$, for every $z \in Z$, $\{x, z\}$ is an equality pair. Furthermore, as Z is an equivalence class of the difference relation $\mathcal{D}(T - x, T' - x)$, for any $z \in Z$ and for any $z' \in (V - \{x\}) - Z$, $\{z, z'\}$ is an equality pair. It ensues that there do not exist $x_0, \ldots, x_n \in V$ such that $x_0 = x$, $x_n \in Z$ and for $0 \leq i \leq n-1$, $\{x_i, x_{i+1}\}$ is a difference pair. □

Theorem 2 for the indecomposable tournaments follows easily.

Corollary 1 *Given tournaments $T = (V, E)$ and $T' = (V, E')$ such that $C_3(T) = C_3(T')$, if T is indecomposable, then $T' = T$ or $T' = T^\star$.*

Proof. By the former proposition, it may be supposed by interchanging T and $T^\star$ that $V \notin \mathcal{D}(T, T')$. By Lemma 7, $\mathcal{D}(T, T')$ is an interval partition of T. Since T is indecomposable, $\mathcal{D}(T, T') = \{\{x\}; x \in V\}$ or, in other words, $T = T'$. □

The proof of the second corollary needs the following result.

Lemma 9 *Given a tournament $T = (V, E)$, if P is an interval partition of T such that $\mid P \mid \geq 3$ and T/P is indecomposable, then $P = P(T)$. As a consequence, T is strongly connected if and only if there is an interval partition P of T such that $\mid P \mid \geq 3$ and T/P is indecomposable.*

Proof. As stated, the second assertion follows from the first one by applying Theorem 4.2 . Consequently, it suffices to prove that for any interval partition R of T such that $\mid R \mid \geq 3$ and T/R is indecomposable, if X is an interval of T with $X \neq V$, then there is an element of R which contains X. Indeed, for every interval Y of T, $Y/R = \{Z \in R : Y \cap Z \neq \emptyset\}$ is an interval of T/R. Since T/R is indecomposable, either $\mid Y/R \mid \leq 1$ or $Y/R = R$. In the last instance, it must be shown that $Y = V$. Otherwise, there is $Z \in R$ such that $Z - Y \neq \emptyset$. By Proposition 1, $Y - Z$ is an interval of T and $(Y - Z)/R$ would be a nontrivial interval of T/R. □

Corollary 2 *Given tournaments $T = (V, E)$ and $T' = (V, E')$, if $C_3(T) = C_3(T')$, then $P(T) = P'(T)$.*

Proof. By Lemma 2, $\mathcal{S}(T) = \mathcal{S}(T')$, which means, T and T' are both strongly connected or not. If T and T' are not strongly connected, then, by Lemma 5, $P(T) = \mathcal{S}(T)$ and $P(T') = \mathcal{S}(T')$. If T and T' are strongly connected, then the following notion is utilized.

Given a partition P of a set S, a subset C of S is a *cross set* of P if for each $X \in P$, $\mid X \cap C \mid = 1$. The family of the cross sets of P is denoted by $C(P)$. Clearly, for every partitions P and Q of the same set S, $P = Q$ if and only if $C(P) = C(Q)$.

Since T is strongly connected, $T/P(T)$ is indecomposable and, hence, for every $A \in C[P(T)]$, $T(A)$ is indecomposable. By Corollary 1, $T'(A)$ is indecomposable and, thus, either there is $X \in P(T')$ such that $A \subseteq X$ or for every $X \in P(T')$, $\mid X \cap A \mid \leq 1$. In the first case, for any $x \in V$, there is $a \in A$ such that x and a belong to the same element of $P(T)$. Consequently, $(A - \{a\}) \cup \{x\} \in C[P(T)]$ and since $\mid [(A - \{a\}) \cup \{x\}] \cap X \mid \geq 2$, $(A - \{a\}) \cup \{x\} \subseteq X$. It ensues that for each $A \in C[P(T)]$ and for each $X \in P(T')$, $\mid X \cap A \mid \leq 1$. In particular, $\mid P(T) \mid \leq \mid P(T') \mid$. By interchanging T and T' in what preceeds, it is obtained that $\mid P(T) \mid = \mid P(T') \mid$ and, hence, $C[P(T)] = C[P(T')]$. □

Corollaries 1 and 2 allow for the following demonstration of Theorem 2.

Proof of Theorem 2. Let X be an interval of T. For any $C \subseteq V$ such that $T(C)$ is a 3-cycle, as $T(C)$ is indecomposable, $\mid C \cap X \mid = 0, 1$ or 3. It ensues that $\mathcal{I}(T, X)(C)$ is a 3-cycle as well and, thus, $C_3(T) \subseteq C_3[\mathcal{I}(T, X)]$. In the same way, since X is an interval of $\mathcal{I}(T, X)$ and since $\mathcal{I}[\mathcal{I}(T, X), X] = T$, $C_3[\mathcal{I}(T, X)] \subseteq C_3(T)$. As a consequence, $T\mathcal{I}T'$ implies $C_3(T) = C_3(T')$.

Conversely, it is proved by induction on $|V|$ that if $C_3(T) = C_3(T')$, then $T\mathcal{I}T'$. This assertion is easily verified if $|V| \leq 3$. Consequently, it is supposed that $|V| > 3$. Before continuing, a remark is brought into play.

Assume that T and T' admit a common nontrivial interval X. Given $x \in X$, $(V - X) \cup \{x\}$ is denoted by U. By supposing the induction hypothesis, as $T(U)\mathcal{I}T'(U)$, there are digraphs $S_0 = T(U), \ldots, S_m = T'(U)$ such that for $0 \leq i \leq m-1$, $S_{i+1} = \mathcal{I}(S_i, Y_i)$, where Y_i is an interval of S_i. For each $i \in \{0, \ldots, m-1\}$, the subset $\widetilde{Y_i}$ of V is defined from Y_i as : $\widetilde{Y_i} = Y_i$ if $x \notin Y_i$ and $\widetilde{Y_i} = Y_i \cup X$ if $x \in Y_i$. Now, the sequence $(\widetilde{S_i})_{0 \leq i \leq m}$ is defined by : $\widetilde{S_0} = T$ and for $i \in \{0, \ldots, m-1\}$, $\widetilde{S_{i+1}} = \mathcal{I}(\widetilde{S_i}, \widetilde{Y_i})$. Clearly, $\widetilde{S_m}(U) = T'(U)$, $\widetilde{S_m}(X) = T(X)$ or $T^\star(X)$ and since $C_3[T(X)] = C_3[T'(X)]$, $C_3[\widetilde{S_m}(X)] = C_3[T'(X)]$. By supposing again the induction hypothesis, there are digraphs $R_0 = \widetilde{S_m}(X), \ldots, R_n = T'(X)$ such that for $0 \leq i \leq n-1$, $R_{i+1} = \mathcal{I}(R_i, Z_i)$, where Z_i is an interval of R_i. By considering $\widetilde{R_0} = T''$ and for $0 \leq i \leq n-1$, $\widetilde{R_{i+1}} = \mathcal{I}(\widetilde{R_i}, Z_i)$, it is obtained that $\widetilde{R_n} = T'$. It ensues that the induction hypothesis allows for the conclusion provided that T and T' share a common nontrivial interval.

By Corollary 2, $P(T) = P(T')$ and, hence, it may be assumed that $P(T) = \{\{x\}; x \in V\}$. By Lemmas 2 and 5, and by Theorem 4, T and T' are total orders or T and T' are indecomposable. In the second instance, by Corollary 1, either $T' = T$ and $T' = \mathcal{I}(T, \emptyset)$ or $T' = T^\star$ and $T' = \mathcal{I}(T, V)$. In the first one, if X denotes the smallest interval of T which contains the minimum element of T and the minimum element of T', then $\mathcal{I}(T, X)$ and T' have the same minimum element m. Since $V - \{m\}$ is a non trivial interval of T and of T', it is sufficient to apply the previous remark. $\square$

By Theorems 1 and 2, the comparability structure of the posets and the C_3-structure of the tournaments appear to play a similar role. The characterization of the comparability structure attributed to Ghouila-Hari [6] leads us to state the following problem.

Problem 1 *Given a set V, find a necessary and sufficient condition for a family of subsets of cardinality 3 of V to be the C_3-structure of a tournament defined on V.*

3 The digraphs without flags

The purpose of the first part of the section is to extend Theorem 4 to any digraphs. We begin with the definition of diconnected digraphs, of connected digraphs and of coconnected digraphs. Since the notions of strong connectivity and of diconnectivity coincide in the case of the tournaments, the equivalence relation induced by the diconnectivity is still denoted by $\mathcal{S}$. Given a digraph $D = (V, E)$, the equivalence relations $\mathcal{S}$, $\mathcal{C}$ and $\overline{\mathcal{C}}$ are defined on V as below.

- For $x \neq y \in V$, $x\mathcal{S}y$ if there are $x_0 = x, \ldots, x_m = y \in V$ and $y_0 = y, \ldots, y_n = x \in V$ satisfying : for $0 \leq i \leq m-1$, either $x_i \longrightarrow x_{i+1}$ or $x_i \longleftrightarrow x_{i+1}$ or $x_i \cdots x_{i+1}$, and for $0 \leq j \leq n-1$, either $y_j \longrightarrow y_{j+1}$ or $y_j \longleftrightarrow y_{j+1}$ or $y_j \cdots y_{j+1}$.
- For $x \neq y \in V$, $x\mathcal{C}y$ if there are $x_0 = x, \ldots, x_m = y \in V$ such that for $0 \leq i \leq m-1$, either $x_i \longrightarrow x_{i+1}$ or $x_{i+1} \longrightarrow x_i$ or $x_i \longleftrightarrow x_{i+1}$.
- For $x \neq y \in V$, $x\overline{\mathcal{C}}y$ if there are $x_0 = x, \ldots, x_m = y \in V$ such that for $0 \leq i \leq m-1$, either $x_i \longrightarrow x_{i+1}$ or $x_{i+1} \longrightarrow x_i$ or $x_i \cdots x_{i+1}$.

The equivalence classes of $\mathcal{S}$, of $\mathcal{C}$ and of $\overline{\mathcal{C}}$ are respectively denoted by $\mathcal{S}(D)$, by $\mathcal{C}(D)$ and by $\overline{\mathcal{C}}(D)$. The digraph D is *diconnected* if $|\mathcal{S}(D)| = 1$, it is *connected* if $|\mathcal{C}(D)| = 1$ and it is *coconnected* if $|\overline{\mathcal{C}}(D)| = 1$.

Lemmas 1, 2, 3 and 5, Proposition 2 and Theorem 4 are extended to any digraphs in the following manner.

Lemma 10 *Let D be a digraph.*

1. *$\mathcal{S}(D)$, $\mathcal{C}(D)$ and $\overline{\mathcal{C}}(D)$ are interval partitions of D.*
2. *D is not diconnected if and only if $D/\mathcal{S}(D)$ is a total order with $|\mathcal{S}(D)| \geq 2$.*
3. *D is not connected if and only if $D/\mathcal{C}(D)$ is empty with $|\mathcal{C}(D)| \geq 2$.*
4. *D is not coconnected if and only if $D/\overline{\mathcal{C}}(D)$ is complete with $|\overline{\mathcal{C}}(D)| \geq 2$.*

Lemma 11 *Given digraphs D and D' defined on the same set of vertices V, if D and D' are hemimorphic, then $\mathcal{S}(D) = \mathcal{S}(D')$, $\mathcal{C}(D) = \mathcal{C}(D')$ and $\overline{\mathcal{C}}(D) = \overline{\mathcal{C}}(D')$.*

Proof. Since D and D' are hemimorphic, for each $X \subseteq V$ with $|X| = 2$, $D'(X)$ is isomorphic to $D(X)$ or to $D^\star(X)$. It follows from the definition of $\mathcal{C}$ and of $\overline{\mathcal{C}}$ that $\mathcal{C}(D) = \mathcal{C}(D')$ and $\overline{\mathcal{C}}(D) = \overline{\mathcal{C}}(D')$. Now, given distinct elements x and y of V which belong to the same element of $\mathcal{S}(D)$. If $x \cdots y$ or $x \longleftrightarrow y$, then $x\mathcal{S}y$ in D'. Otherwise, it is supposed that $x \longrightarrow y$ in D. Consequently, it may be considered the smallest integer n such that there are $y_0 = y, \ldots, y_n = x \in V$ satisfying : for $0 \leq i \leq n-1$, in D, either $y_i \longrightarrow y_{i+1}$ or $y_i \longleftrightarrow y_{i+1}$ or $y_i \cdots y_{i+1}$. As in the proof of Lemma 2, it is verified that for $i \in \{0, \ldots, n-2\}$, y_i, y_{i+1} and y_{i+2} are equivalent modulo $\mathcal{S}$ in D'. □

Lemma 12 *Given a digraph $D = (V, E)$, D is diconnected, connected and coconnected if and only if all its cuts are trivial. More specifically :*

1. *D is not diconnected if and only if there is a nontrivial cut X of D such that $X \longrightarrow (V - X)$.*

2. *D is not connected if and only if there is a nontrivial cut X of D such that $X \cdots (V - X)$.*

3. *D is not coconnected if and only if there is a nontrivial cut X of D such that $X \longleftrightarrow (V - X)$.*

The next result is useful to extend Proposition 2 and Lemma 5 to any digraphs.

Corollary 3 *Given a digraph D, every element of $\mathcal{S}(D) \cup \mathcal{C}(D) \cup \overline{\mathcal{C}}(D)$ is a strong interval of D.*

Proof. By contradiction, suppose that there are an element S of $\mathcal{S}(D)$ and an interval X of D such that $S \cap X$, $S - X$ and $X - S$ are nonempty. It ensues that $S \cap X$ is a non trivial cut of $D(S)$. Since $D(S)$ is diconnected, by Lemma 12, either $(S \cap X) \cdots (S - X)$ or $(S \cap X) \longleftrightarrow (S - X)$. By considering $\overline{D}$ in place of D, it may be supposed that $(S \cap X) \cdots (S - X)$. As X is an interval of D, $(S - X) \cdots (X - S)$ and, hence, $D(S \cup X)$ would be diconnected. Now, suppose, on the contrary, that there are an element C of $\mathcal{C}(D)$ and an interval X of D such that $C \cap X$, $C - X$ and $X - C$ are nonempty. It follows that $C \cap X$ is a nontrivial cut of $D(C)$. Since $D(C)$ is connected, by Lemma 12, either $(C \cap X) \longrightarrow (C - X)$ or $(C - X) \longrightarrow (C \cap X)$ or $(C \cap X) \longleftrightarrow (C - X)$. As X is an interval of D, either $(X - C) \longrightarrow (C - X)$ or $(C - X) \longrightarrow (X - C)$ or $(C - X) \longleftrightarrow (X - C)$. Consequently, $D(C \cup X)$ would be connected. To complete the proof, it suffices to note that $\overline{\mathcal{C}}(D) = \mathcal{C}(\overline{D})$. □

Proposition 4 *Given a digraph $D = (V, E)$, all the strong intervals of D are trivial if and only if D is complete or D is empty or D is a total order or D is indecomposable.*

Proof. As in the proof of Proposition 2, if D is decomposable and if all its strong intervals are trivial, then D admits a nontrivial cut X. By Lemma 12, D is not diconnected or D is not connected or D is not coconnected. The above three assertions, which follow from Lemma 10 and Corollary 3, allow for the conclusion.

- If D is not diconnected, then $\mathcal{S}(D) = \{\{x\}; x \in V\}$ and, thus, D is a total order.
- If D is not connected, then $\mathcal{C}(D) = \{\{x\}; x \in V\}$ and, thus, D is empty.
- If D is not coconnected, then $\overline{\mathcal{C}}(D) = \{\{x\}; x \in V\}$ and, thus, D is complete.

□

Lemma 13 *let $D = (V, E)$ be a digraph.*

1. *If D is not diconnected, then $P(D) = \mathcal{S}(D)$.*

2. *If D is not connected, then $P(D) = \mathcal{C}(D)$.*

3. *If D is not coconnected, then $P(D) = \overline{\mathcal{C}}(D)$.*

Proof. For the first assertion, the proof of Lemma 5 applies. In another respect, the third assertion follows from the second one by interchanging D and $\overline{D}$. Now, for the second implication, it is sufficient to prove that if X is an interval of D distinct from V such that there is $C \in \mathcal{C}(D)$ with $C \subset X$, then X is not a strong interval of D. Indeed, by Corollary 3, all the elements of $\mathcal{C}(D)$ are strong intervals of D and, hence, there is $Q \subset \mathcal{C}(D)$ such that $X = \bigcup Q$ and $|Q| > 1$. Given $Y \in Q$ and $Z \in \mathcal{C}(D) - Q$, $Y \cup Z$ is an interval of D such that $X \cap (Y \cup Z)$, $X - (Y \cup Z)$ and $(Y \cup Z) - X$ are nonempty. □

Gallai's decomposition theorem follows immediately.

Theorem 5 *Let D be a digraph.*

1. *D is not diconnected if and only if $D/P(D)$ is a total order.*

2. *D is not connected if and only if $D/P(D)$ is empty.*

3. *D is not coconnected if and only if $D/P(D)$ is complete.*

4. *D is strongly connected, connected and coconnected if and only if $D/P(D)$ is indecomposable with $|P(D)| \geq 3$.*

In [7], Habib adopts a different approach to establish this theorem. Given a digraph $D = (V, E)$, $\mathcal{P}(D)$ denotes the family of the interval partitions of D. The following poset is defined on $\mathcal{P}(D)$. Given $P \neq Q \in \mathcal{P}(D)$, $P \prec Q$ if for every $X \in P$, there is $Y \in Q$ such that $X \subseteq Y$. Clearly, $\{\{x\}; x \in V\}$ is the minimum element of $(\mathcal{P}(D), \prec)$ and $\{V\}$ is the maximum element of $(\mathcal{P}(D), \prec)$. Furthermore, given $P, Q \in \mathcal{P}(D)$, the *meet* of P and Q, denoted by $P \vee Q$, and the *join* of P and Q, denoted by $P \wedge Q$, are defined as follows.

- Given $x \neq y \in V$, x and y belong to the same set of $P \vee Q$ if there are $X_0, \ldots, X_n \in P \cup Q$ satisfying : $x \in X_0$, $y \in X_n$ and for $0 \leq i \leq n-1$, $X_i \cap X_{i+1} \neq \emptyset$.

- Given $x \neq y \in V$, x and y belong to the same set of $P \wedge Q$ if there are $X \in P$ and $Y \in Q$ such that $x, y \in X \cap Y$.

Consequently, $(\mathcal{P}(D), \prec)$ is a lattice. The maximal elements of $(\mathcal{P}(D) - \{V\}, \prec)$ are called the *coatoms* of $(\mathcal{P}(D), \prec)$. It is then verified that $P(D)$ is the join of all the coatoms of $(\mathcal{P}(D), \prec)$.

To extend Lemma 8 to any digraphs, the below three lemmas are required.

Lemma 14 (Ehrenfeucht - Rozenberg [4]) *Given an indecomposable digraph* $D = (V, E)$*, if* X *is a subset of* V *such that* $2 <\mid X \mid<\mid V \mid$ *and* $D(X)$ *is indecomposable, then there are* $u, v \in V - X$ *such that* $D - \{u, v\}$ *is indecomposable.*

Lemma 15 (Cournier - Ille [3]) *Let* $D = (V, E)$ *be an indecomposable digraph with* $\mid V \mid\geq 3$*.*

1. *There exists a subset* X *of* V *such that* $\mid X \mid= 3$ *or* 4 *and* $D(X)$ *is indecomposable.*

2. *For each* $x \in V$*, there exists a subset* X *of* V *such that* $x \in X$*,* $3 \leq\mid X \mid\leq 5$ *and* $D(X)$ *is indecomposable.*

The proof of the next lemma is the same as that of Lemma 9.

Lemma 16 *Given a digraph* D*, if* P *is an interval partition of* D *such that* $\mid P \mid\geq 3$ *and* P/D *is indecomposable, then* $P = P(D)$*. Consequently,* D *is diconnected, connected and coconnected if and only if there is an interval partition* P *of* D *such that* $\mid P \mid\geq 3$ *and* P/D *is indecomposable.*

The extension of Lemma 9 to the digraphs is then stated as below.

Lemma 17 *Let* $D = (V, E)$ *be a digraph.*

1. *If* $\mid V \mid\geq 5$ *and if* D *is diconnected, connected and coconnected, then there is* $x \in V$ *such that* $D - x$ *is diconnected, connected and coconnected.*

2. *Assume that* $\mid V \mid\geq 6$*. The digraph* D *is diconnected, connected and coconnected if and only if there exist distinct elements* x *and* y *of* V *such that* $D - x$ *and* $D - y$ *are diconnected, connected and coconnected.*

Proof. Suppose that D is diconnected, connected and coconnected. Firstly, if there is $X \in P(D)$ such that $\mid X \mid\geq 2$, then, by Lemma 16, for every $x \in X$, $D - x$ is diconnected, connected and coconnected. Secondly, assume that D is indecomposable with $\mid V \mid\geq 3$. The following remark is useful.

If $D-\{u, v\}$ is indecomposable, where $u, v \in V$, then there is $x \in \{u, v\}$ such that $D - x$ is diconnected, connected and coconnected. The remark is obvious if there is $x \in \{u, v\}$ such that $D - x$ is indecomposable. Otherwise, for $w = u$ or v, $D - w$ admits a nontrivial interval X. By Proposition 1, $X \cap (V - \{u, v\})$ is an interval of $D - \{u, v\}$ and, hence, $\mid X \cap (V - \{u, v\}) \mid= 1$ or $X = V - \{u, v\}$. However, if $V - \{u, v\}$ is an interval of $D - u$ and of $D - v$, then $V - \{u, v\}$ is an interval of D as well. It ensues that, for example, $V - \{u, v\}$ is not an interval of $D - u$. By what preceeds, there is $v' \in V - \{u, v\}$ such that $\{v, v'\}$ is an interval of $D - u$. As $\{\{v, v'\}\} \cup \{\{a\}; a \in V - \{u, v, v'\}\}$ is an interval partition of $D - u$ such that $(D - u)/(\{\{v, v'\}\} \cup \{\{a\}; a \in V - \{u, v, v'\}\})$ is indecomposable, it suffices to apply Lemma 15.

It results directly from Lemmas 14 and 15.1, and from the remark that there is $x \in V$ such that $D - x$ is diconnected, connected and coconnected. Furthermore, by Lemma 15.2, there is $X \subseteq V$ such that $x \in X$, $3 \leq |X| \leq 5$ and $D(X)$ is indecomposable. Consequently, if $|V| \geq 6$, then, by Lemma 14 and by the previous remark, there exists $y \in V - X$ such that $D - y$ is diconnected, connected and coconnected. $\Box$

As Lemma 8 is important to prove Proposition 3, the preceeding Lemma allows for the demonstration of the following result.

Proposition 5 *Given hemimorphic digraphs $D = (V, E)$ and $D' = (V, E')$, if D is diconnected, connected and coconnected, then $V \notin \mathcal{D}(D, D') \cap \mathcal{E}(D, D')$.*

Proof. We proceed by induction on $|V| \geq 3$. It follows from the definition of $\mathcal{D}$ and of $\mathcal{E}$ that if $V \in \mathcal{D}(D, D')$ (resp. $V \in \mathcal{E}(D, D')$), then the number of difference (resp. equality) pairs is at least $|V| - 1$. Consequently, if $|V| = 3$, then $V \notin \mathcal{D}(D, D') \cap \mathcal{E}(D, D')$. In the same manner, if $|V| = 4$ and if there are $x \neq y \in V$ such that $x \longleftrightarrow y$ or $x \cdots y$, then $V \notin \mathcal{D}(D, D') \cap \mathcal{E}(D, D')$. Furthermore, if $|V| = 4$ and if D is a tournament, then it is sufficient to apply Proposition 3. Now, assume that $|V| > 4$. By Lemma 17.1, there is $x \in V$ such that $D - x$ is diconnected, connected and coconnected. By the induction hypothesis, it may be supposed by considering $D^\star$ in place of D that $V - \{x\} \notin \mathcal{D}(D - x, D' - x)$. As in the proof of Proposition 3, we suppose that $V \in \mathcal{E}(D, D')$ and we consider $Y \in \mathcal{D}(D - x, D' - x)$ which contains an element y such that $x \longrightarrow y$ in D and in D'. The conclusion is then the same by considering $Z \in \mathcal{D}(D - x, D' - x) - \{Y\}$ such that $Y \longrightarrow Z$ or $Y \longleftrightarrow Z$ or $Y \cdots Z$. Such an element Z exists since $D - x$ is diconnected. $\Box$

As for the tournaments, the next two results follow easily.

Corollary 4 *Given hemimorphic digraphs D and D', if D is indecomposable and without flags, then $D' = D$ or $D' = D^\star$.*

This corollary does not hold if D contains some flags. Indeed, it suffices to consider the digraphs (see Figure 1) $(\{0, \ldots, n\}, \{(i, i+1), (i+1, i)\}_{1 \leq i \leq n-2} \cup \{(1, 0), (n-1, n)\})$ and $(\{0, \ldots, n\}, \{(i, i+1), (i+1, i)\}_{1 \leq i \leq n-2} \cup \{(1, 0), (n, n-1)\})$, where $n \geq 3$. In fact, Lemma 7 does not hold for these digraphs.

Corollary 5 *Given hemimorphic digraphs D and D', if D is without flags, then $P(D) = P(D')$.*

Proof. If D is not diconnected or if D is not connected or if D is not coconnected, then it suffices to apply Lemmas 11 and 13. If D is diconnected, connected and coconnected, then we conclude as in the proof of Corollary 2 by utilizing the cross sets of $P(D)$ and of $P(D')$. $\Box$

Corollaries 4 and 5 allow for the following demonstration of Theorem 3.

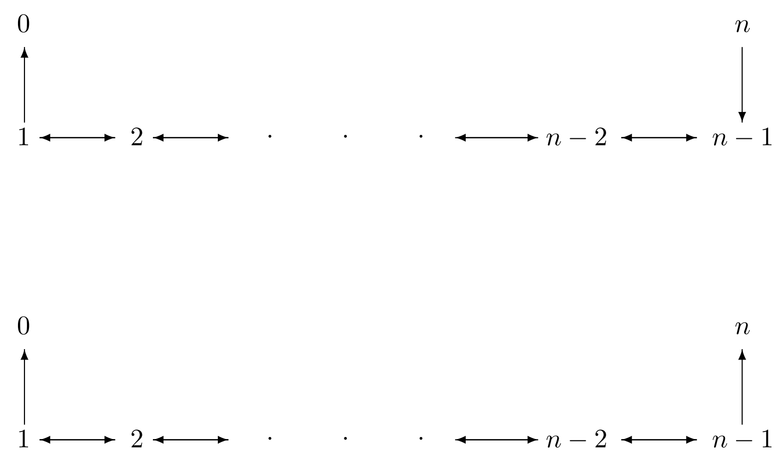

Figure 1:

Proof of Theorem 3. Let X be an interval of D. For any $A \subseteq V$ such that $\mid A \mid = 2$ or 3, if $\mid X \cap A \mid = 0$ or 1 or if $A \subseteq X$ or if $\mid X \cap A \mid = 2$ with $a \longleftrightarrow b$ or $a \cdots b$, where $X \cap A = \{a, b\}$, then $\mathcal{I}(D, X)(A) = D(A)$ or $D^\star(A)$. Thus, it is supposed that $\mid A \mid = 3$ and $X \cap A = \{a, b\}$, where $A = \{a, b, c\}$ and $a \longrightarrow b$ or $b \longrightarrow a$. Since X is an interval of D, either $c \longrightarrow X$ or $X \longrightarrow c$ or $c \longleftrightarrow X$ or $c \cdots X$. If $c \longleftrightarrow X$ or if $c \cdots X$, then $\mathcal{I}(D, X)(A) = D^\star(A)$. If $c \longrightarrow X$ or $X \longrightarrow c$, then $\mathcal{I}(D, X)(A)$ and $D(A)$ are total orders and, hence, they are isomorphic. It ensues that D and $\mathcal{I}(D, X)$ are hemimorphic. Consequently, if $D\mathcal{I}D'$, then D and D' are hemimorphic. The converse is proved by induction on $\mid V \mid$. Since the cases $\mid V \mid = 2$ or 3 are obvious, it is supposed that $\mid V \mid \geq 4$. As in the proof of Theorem 2, the induction hypothesis allows for the conclusion if D and D' share a nontrivial interval. By Corollary 5, $P(D) = P(D')$ and, hence, it may be assumed that $P(D) = \{\{x\}; x \in V\}$. It then results from Lemmas 10 and 11, and from Theorem 5 that D and D' are indecomposable or D and D' are total orders or D and D' are empty or D and D' are complete. If D and D' are indecomposable, then it suffices to apply Corollary 4. If D and D' are total orders, then it is concluded as in the proof of Theorem 2. If D and D' are empty or if D and D' are complete, then for any $X \subseteq V$ such that $1 < \mid X \mid < \mid V \mid$, X is a nontrivial interval of D and of D'. $\square$

Finally, it is mentionned that Corollary 4 may be extented to infinite digraphs by utilizing the following characterization of infinite indecomposable digraphs.

Theorem 6 (Ille [9]) *Given an infinite digraph $D = (V, E)$, D is indecomposable if and only if for every finite subset F of V, there is a finite subset G of V such that $F \subseteq G$ and $D(G)$ is indecomposable.*

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