

# Spanning a strong digraph by $\alpha$ circuits: A proof of Gallai's conjecture.

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## Abstract

In 1963, Tibor Gallai [9] asked whether every strongly connected directed graph  $D$  is spanned by  $\alpha$  directed circuits, where  $\alpha$  is the stability of  $D$ . We give a proof of this conjecture.

## 1 Coherent cyclic orders.

In this paper, circuits of length two are allowed. Since loops and multiple arcs play no role in this topic, we will simply assume that our digraphs are loopless and simple. A directed graph (digraph) is *strongly connected*, or simply *strong*, if for all vertices  $x, y$ , there exists a directed path from  $x$  to  $y$ . A *stable set* of a directed graph  $D$  is a subset of vertices which are not pairwise joined by arcs. The *stability* of  $D$ , denoted by  $\alpha(D)$ , is the number of vertices of a maximum stable set of  $D$ . It is well-known, by the Gallai-Milgram theorem [10] (see also [1] p. 234 and [3] p. 44), that  $D$  admits a vertex-partition into  $\alpha(D)$  disjoint paths. We shall use in our proof a particular case of this result, known as Dilworth's theorem [8]: A partial order  $P$  admits a vertex-partition into  $\alpha(P)$  chains (linear orders). Here  $\alpha(P)$  is the size of a maximal antichain. In [9], Gallai raised the problem, when  $D$  is strongly connected, of spanning  $D$  by a union of circuits. Precisely, he made the following conjecture (also formulated in [1] p. 330, [2] and [3] p. 45):

**Conjecture 1** *Every strong digraph with stability  $\alpha$  is spanned by the union of  $\alpha$  circuits.*

The case  $\alpha = 1$  is Camion's theorem [6]: Every strong tournament has a hamilton circuit. The case  $\alpha = 2$  is a corollary of a result of Chen and Manalastas [7] (see also Bondy [4]): Every strong digraph with stability two is spanned by two circuits intersecting each other on a (possibly empty) path. In [11] was proved the case  $\alpha = 3$ . In the next section of this paper, we will give a proof of Gallai's conjecture for every  $\alpha$ .

Let  $D$  be a strong digraph on vertex set  $V$ . An enumeration  $E = v_1, \dots, v_n$  of  $V$  is *elementary equivalent* to  $E'$  if one the following holds:  $E' = v_n, v_1, \dots, v_{n-1}$ , or  $E' = v_2, v_1, v_3, \dots, v_n$  if neither  $v_1v_2$  nor  $v_2v_1$  is an arc of  $D$ . Two enumerations  $E, E'$  of  $V$  are *equivalent* if there is a sequence  $E = E_1, \dots, E_k = E'$  such that  $E_i$  and  $E_{i+1}$  are elementary equivalent, for  $i = 1, \dots, k - 1$ . The classes of this equivalence relation are called the *cyclic orders* of  $D$ . Roughly speaking, a cyclic order is a class of enumerations of the vertices on the integers modulo  $n$ , where one stay in the class while switching consecutive vertices which are not joined by an arc. We fix an enumeration  $E = v_1, \dots, v_n$  of  $V$ , the following definitions are understood with respect to  $E$ . An arc  $v_i v_j$  of  $D$  is a *forward arc* if  $i < j$ , otherwise it is a *backward arc*. A directed path of  $D$  is a *forward path* if it only contains forward arcs. The *index* of a directed circuit  $C$  of  $D$  is the number of backward arcs of  $C$ , we denote it by  $i_E(C)$ .

This correspond to the winding number of the circuit. Observe that  $i_E(C) = i_{E'}(C)$  if  $E'$  is elementary equivalent to  $E$ . Consequently, the index of a circuit is invariant in a given cyclic order  $\mathcal{C}$ , we denote it by  $i_{\mathcal{C}}(C)$ . By extension, the index  $i(\mathcal{S})$  of a set of circuits  $\mathcal{S}$  is the sum of the indices of the circuits of  $\mathcal{S}$ . A circuit is *simple* if it has index one. A cyclic order  $\mathcal{C}$  is *coherent* if every arc of  $D$  is contained in a simple circuit, or, equivalently, if for every enumeration  $E$  of  $\mathcal{C}$  and every backward arc  $v_j v_i$  of  $E$ , there exists a forward path from  $v_i$  to  $v_j$ . We denote by  $\text{cir}(D)$  the set of all directed circuits of  $D$ .

**Lemma 1** *Every strong digraph has a coherent cyclic order.*

**Proof.** Let us consider a cyclic order  $\mathcal{C}$  which is minimum with respect to  $i_{\mathcal{C}}(\text{cir}(D))$ . We suppose for contradiction that  $\mathcal{C}$  is not coherent. There exists an enumeration  $E = v_1, \dots, v_n$  and a backward arc  $a = v_j v_i$  which is not in a simple circuit. Assume moreover that  $E$  and  $a$  are chosen in order to minimize  $j - i$ . Let  $k$  be the largest integer  $i \leq k < j$  such that there exists a forward path from  $v_i$  to  $v_k$ . Observe that  $v_k$  has no out-neighbour in  $]v_k, v_j]$ . If  $k \neq i$ , by the minimality of  $j - i$ ,  $v_k$  has no in-neighbour in  $]v_k, v_j]$ . In particular the enumeration  $E' = v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_j, v_k, v_{j+1}, \dots, v_n$  is equivalent to  $E$ , and contradicts the minimality of  $j - i$ . Thus  $k = i$ , and by the minimality of  $j - i$ , there is no in-neighbour of  $v_i$  in  $]v_i, v_j]$ . In particular the enumeration  $E' = v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_i, v_j, \dots, v_n$  is equivalent to  $E$ . Observe now that in  $E'' = v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_j, v_i, \dots, v_n$ , every circuit  $C$  satisfies  $i_{E''}(C) \leq i_{E'}(C)$ , and the inequality is strict if the arc  $a$  belongs to  $C$ , a contradiction.  $\square$

A direct corollary of Lemma 1 is that every strong tournament has a hamilton circuit, just consider for this any coherent cyclic order.

## 2 Cyclic stability versus spanning circuits.

The *cyclic stability* of a coherent cyclic order  $\mathcal{C}$  is the maximum  $k$  for which there exists an enumeration  $v_1, \dots, v_n$  of  $\mathcal{C}$  such that  $\{v_1, \dots, v_k\}$  is a stable set of  $D$ . We denote it by  $\alpha(\mathcal{C})$ , observe that we clearly have  $\alpha(\mathcal{C}) \leq \alpha(D)$ .

**Lemma 2** *Let  $D$  be a strong digraph and  $v_1, \dots, v_n$  be an enumeration of a coherent cyclic order  $\mathcal{C}$  of  $D$ . Let  $X$  be a subset of vertices of  $D$  such that there is no forward path between two distinct vertices of  $X$ . Then  $|X| \leq \alpha(\mathcal{C})$ .*

**Proof.** We consider an enumeration  $E = v_1, \dots, v_n$  of  $\mathcal{C}$  such that there is no forward path between two distinct vertices of  $X$ , and chosen in such a way that  $j - i$  is minimum, where  $v_i$  is the first element of  $X$  in the enumeration, and  $v_j$  is the last element of  $X$  in the enumeration. Suppose for contradiction that  $X \neq \{v_i, \dots, v_j\}$ . There exists  $v_k \notin X$  for some  $i < k < j$ . There cannot exist both a forward path from  $X \cap \{v_i, \dots, v_{k-1}\}$  to  $v_k$  and a forward path from  $v_k$  to  $X \cap \{v_{k+1}, \dots, v_j\}$ . Without loss of generality, we assume that there is no forward path from  $X \cap \{v_i, \dots, v_{k-1}\}$  to  $v_k$ . Suppose moreover that  $v_k$  is chosen with minimum index  $k$ . Clearly,  $v_k$  has no in-neighbour in  $\{v_i, \dots, v_{k-1}\}$ , and since  $\mathcal{C}$  is coherent,  $v_k$  has no out-neighbour in  $\{v_i, \dots, v_{k-1}\}$ . Thus the enumeration  $v_1, \dots, v_{i-1}, v_k, v_i, \dots, v_{k-1}, v_{k+1}, \dots, v_n$  belongs to  $\mathcal{C}$ , contradicting the minimality of  $j - i$ . Consequently,  $X = \{v_i, \dots, v_j\}$ , and there is no forward arcs, and then no backward arcs, between the vertices of  $X$ . Considering now the enumeration  $v_i, \dots, v_n, v_1, \dots, v_{i-1}$ , we conclude that  $|X| \leq \alpha(\mathcal{C})$ .  $\square$

Let  $P = x_1, \dots, x_k$  be a directed path, we call  $x_1$  the *head* of  $P$  and  $x_k$  the *tail* of  $P$ . We denote the restriction of  $P$  to  $\{x_i, \dots, x_j\}$  by  $P[x_i, x_j]$ .

**Theorem 1** *Let  $D$  be a strong digraph with a coherent cyclic order  $\mathcal{C}$ . The minimal  $i_{\mathcal{C}}(\mathcal{S})$ , where  $\mathcal{S}$  is a spanning set of circuits of  $D$  is equal to  $\alpha(\mathcal{C})$ .*

**Proof.** We consider a coherent cyclic order  $\mathcal{C}$  of  $D$  with cyclic stability  $k := \alpha(\mathcal{C})$ . Let  $E = v_1, \dots, v_n$  be an enumeration of  $\mathcal{C}$  such that  $S = \{v_1, \dots, v_k\}$  is a stable set of  $D$ . Clearly, if a circuit  $C$  contains  $q$  vertices of  $S$ , the index of  $C$  is at least  $q$ . In particular the inequality  $i_{\mathcal{C}}(\mathcal{S}) \geq k$  is satisfied for every spanning set of circuits of  $D$ . To prove that equality holds, we consider an auxiliary acyclic digraph  $D'$  on vertex set  $V \cup \{v'_1, \dots, v'_k\}$  which arc set consists of every forward arc of  $E$  and every arc  $v_i v'_j$  for which  $v_i v_j$  is an arc of  $D$ . We call  $T'$  the transitive closure of  $D'$ . Let us prove that the size of a maximal antichain in the partial order  $T'$  is exactly  $k$ . Consider such an antichain  $A$ , and set  $A_1 := A \cap \{v_1, \dots, v_k\}$ ,  $A_2 := A \cap \{v_{k+1}, \dots, v_n\}$  and  $A_3 := A \cap \{v'_1, \dots, v'_k\}$ . Since one can arbitrarily permute the vertices of  $S$  in the enumeration  $E$  and still remain in  $\mathcal{C}$ , we may assume that  $A_3 = \{v'_1, \dots, v'_j\}$  for some  $0 \leq j \leq k$ . Since every vertex is in a simple circuit, there is a directed path in  $D'$  from  $v_i$  to  $v'_i$ , and consequently we cannot both have  $v_i \in A$  and  $v'_i \in A$ . Clearly, the enumeration  $E' = v_{j+1}, \dots, v_n, v_1, \dots, v_j$  belongs to  $\mathcal{C}$ . By the fact that  $A$  is an antichain of  $T'$ , there is no forward path joining two elements of  $(A \cap V) \cup \{v_1, \dots, v_j\}$  in  $E'$ , and thus, by Lemma 2,  $|A| = |(A \cap V) \cup \{v_1, \dots, v_j\}| \leq k$ . Observe also that  $\{v_1, \dots, v_k\}$  are the sources of  $T'$  and  $\{v'_1, \dots, v'_k\}$  are the sinks of  $T'$ , and both are maximal antichains of  $T'$ . We apply Dilworth's theorem in order to partition  $T'$  into  $k$  chains (thus starting in the set  $\{v_1, \dots, v_k\}$  and ending in the set  $\{v'_1, \dots, v'_k\}$ ), and by this, there exists a spanning set  $P_1, \dots, P_k$  of directed paths of  $D'$  with heads in  $\{v_1, \dots, v_k\}$  and tails in  $\{v'_1, \dots, v'_k\}$ . We can assume without loss of generality that the head of  $P_i$  is exactly  $v_i$ , for all  $i = 1, \dots, k$ . Let us now denote by  $\sigma$  the permutation of  $\{1, \dots, k\}$  such that  $v'_{\sigma(i)}$  is the tail of  $P_i$ , for all  $i$ . Assume that among all spanning sets of paths, we have chosen  $P_1, \dots, P_k$  (with respective heads  $v_1, \dots, v_k$ ) in such a way that the permutation  $\sigma$  has a maximum number of cycles. We claim that if  $(i_1, \dots, i_p)$  is a cycle of  $\sigma$  (meaning that  $\sigma(i_j) = i_{j+1}$  and  $\sigma(i_p) = i_1$ ), then the paths  $P_{i_1}, \dots, P_{i_p}$  are pairwise vertex-disjoint. If not, suppose that  $v$  is a common vertex of  $P_{i_l}$  and  $P_{i_m}$ , and replace  $P_{i_l}$  by  $P_{i_l}[v_{i_l}, v] \cup P_{i_m}[v, v_{\sigma(i_m)}]$  and  $P_{i_m}$  by  $P_{i_m}[v_{i_m}, v] \cup P_{i_l}[v, v_{\sigma(i_l)}]$ . This is a contradiction to the maximality of the number of cycles of  $\sigma$ . Now, in the set of paths  $P_1, \dots, P_k$ , contract all the pairs  $\{v_i, v'_i\}$ , for  $i = 1, \dots, k$ . This gives a spanning set  $\mathcal{S}$  of circuits of  $D$  which satisfies  $i_{\mathcal{C}}(\mathcal{S}) = k$ .  $\square$

**Corollary 1.1** *Every strong digraph  $D$  is spanned by  $\alpha(D)$  circuits.*

**Proof.** By Lemma 1,  $D$  has a coherent cyclic order  $\mathcal{C}$ . By Theorem 1,  $D$  is spanned by a set  $\mathcal{S}$  of circuits such that  $|\mathcal{S}| \leq i_{\mathcal{C}}(\mathcal{S}) = \alpha(\mathcal{C}) \leq \alpha(D)$ .  $\square$

We now establish the arc-cover analogue of Theorem 1. Again, a minimax result holds.

### 3 Cyclic feedback arc set versus arc cover.

Let  $\mathcal{C}$  be a cyclic order of a strong digraph  $D$ . We denote by  $\beta(\mathcal{C})$  the maximum  $k$  for which there exists an enumeration of  $\mathcal{C}$  with  $k$  backward arcs. We call  $k$  the *maximal feedback arc set* of  $\mathcal{C}$ . Since every vertex of  $D$  has indegree at least one, we clearly have  $\alpha(\mathcal{C}) \leq \beta(\mathcal{C})$ .

**Theorem 2** *Let  $D = (V, A)$  be a strong digraph with a coherent cyclic order  $\mathcal{C}$ . The minimal  $i_{\mathcal{C}}(\mathcal{S})$ , where  $\mathcal{S}$  is a set of circuits which covers the arc set of  $D$ , is equal to  $\beta(\mathcal{C})$ .*

**Proof.** If  $\mathcal{S}$  is a set of circuits which spans the arcs of  $D$ , every backward arc in any enumeration of  $\mathcal{C}$  must be in a circuit of  $\mathcal{S}$ . In particular the inequality  $i_{\mathcal{C}}(\mathcal{S}) \geq \beta(\mathcal{C})$  clearly holds. Let  $D'$  be the subdivision of  $D$ , i.e. the digraph with vertex set  $V \cup A$  and arc set  $\{(v, e) : v \text{ is the head of } e \text{ and } e \in A\} \cup \{(e, v) : v \text{ is the tail of } e \text{ and } e \in A\}$ . There is a one-to-one correspondence  $\phi$  between the circuits of  $D'$  and the circuits of  $D$ . Let  $E = v_1, \dots, v_n$  be an enumeration of  $\mathcal{C}$  with backward arc set  $\{e_1, \dots, e_k\}$ , where  $k = \beta(\mathcal{C})$ . Consider the enumeration  $E'$  of  $D'$  given by

$$E' := e_1, \dots, e_k, v_1, f_1^1, \dots, f_{n_1}^1, v_2, f_1^2, \dots, f_{n_2}^2, \dots, v_n$$

where  $\{f_1^i, \dots, f_{n_i}^i\}$  is the set of forward arcs in  $E$  with head  $v_i$ . Let  $\mathcal{C}'$  be the cyclic order of  $E'$ . The index in  $\mathcal{C}'$  of a circuit  $C$  of  $D'$  is equal to the index in  $\mathcal{C}$  of  $\phi(C)$ , thus  $\mathcal{C}'$  is coherent. Let  $F'$  be any enumeration of  $\mathcal{C}'$ . We denote by  $F$  the enumeration induced by  $F'$  on  $V$ . Since  $\mathcal{C}'$  is coherent, if  $e := xy$  is a forward arc of  $F$ ,  $xy$  is a forward path of  $F'$ . In particular,  $F$  belongs to  $\mathcal{C}$  (since one cannot switch  $x$  and  $y$  in  $F'$ ). Moreover, if  $e := xy$  is a backward arc of  $F$ , exactly one of  $xe$  or  $ey$  is a backward arc of  $F'$ . Thus, we have  $\beta(\mathcal{C}') \leq \beta(\mathcal{C})$ . By Theorem 1, the vertex set of  $D'$  is spanned by a set of circuits  $\mathcal{S}'$  with  $i_{\mathcal{C}'}(\mathcal{S}') \leq \alpha(\mathcal{C}') \leq \beta(\mathcal{C}') \leq \beta(\mathcal{C})$ . To conclude, observe that  $\mathcal{S} := \{\phi(C), C \in \mathcal{S}'\}$  is a set of circuits which covers the arc set of  $D$  and verify  $i_{\mathcal{C}}(\mathcal{S}) = i_{\mathcal{C}'}(\mathcal{S}') \leq \beta(\mathcal{C})$ .  $\square$

## 4 Longest circuit versus minimum cyclic coloration.

In this section, we present in terms of cyclic orders a proof of a theorem of Bondy [5]. Our proof is merely an adaptation of his argument, the main difference being that, in this context, the result is again a min-max theorem. The *cyclic chromatic number* of a strong digraph  $D$  together with a coherent cyclic order  $\mathcal{C}$  is the minimum  $k$  for which there exists an enumeration  $E = v_1, \dots, v_{i_1}, v_{i_1+1}, \dots, v_{i_2}, v_{i_2+1}, \dots, v_{i_k}$  of  $\mathcal{C}$  for which  $v_{i_j+1}, \dots, v_{i_{j+1}}$  is a stable set for all  $j = 0, \dots, k-1$  (with  $i_0 := 0$ ).

**Theorem 3** *Let  $D = (V, A)$  be a strong digraph and  $C$  be a longest circuit of  $D$ . If a coherent cyclic order  $\mathcal{C}$  of  $D$  is such that  $C$  is simple, the cyclic chromatic number of  $\mathcal{C}$  is exactly  $|C|$ .*

**Proof.** Clearly, the cyclic chromatic number of  $\mathcal{C}$  is at least  $|C|$ . Now, we consider an enumeration  $E = v_1, \dots, v_{i_1}, v_{i_1+1}, \dots, v_{i_2}, v_{i_2+1}, \dots, v_{i_k}$  of  $\mathcal{C}$  where  $C = v_{i_1}, v_{i_2}, \dots, v_{i_k}$ . We also write  $I_{j+1} := v_{i_j+1}, \dots, v_{i_{j+1}}$ , with  $j = 0, \dots, k-1$ . A forward arc is *bad* if its endvertices both belong to the same  $I_j$ . Let us suppose that  $E$  is chosen in order to minimize the number of bad arcs. If  $E$  has no bad arcs, since it is coherent, every  $I_j$  is a stable set, and we have our conclusion. If not, we consider a bad arc  $xy$ . Without loss of generality, we suppose that  $x, y \in I_1$ . Let  $D'$  be the subdigraph of  $D$  consisting of bad arcs and arcs  $uv$  such that  $u \in I_j$  and  $v \in I_{j+1}$ , or  $u \in I_k$  and  $v \in I_1$ . Observe that  $D'$  contains  $C$ . Moreover,  $x$  and  $y$  cannot be both in a strong component of  $D'$ : since a path  $P$  from  $y$  to  $x$  in  $D'$  has length at least  $k$ , the circuit  $P \cup xy$  would be longer than  $C$ . In particular there is no path in  $D'$  from  $C$  to  $x$ , or there is no path from  $y$  to  $C$ . By directional duality, we assume that this last case holds. Let  $Y$  be the set of vertices  $z$  such that there exists a path from  $y$  to  $z$  in  $D'$ . Set  $K_j := I_j \cap Y$  and  $L_j := I_j \setminus Y$ , both with the enumeration induced by  $E$ . By construction, the endvertex of  $L_j$  is  $v_{i_j}$  and there is no arc from  $K_i$  to  $L_{i+1}$  or from  $K_k$  to  $L_1$  (and since  $\mathcal{C}$  is coherent, there is no arc from  $L_{i+1}$  to  $K_i$  or from  $L_1$  to  $K_k$ ). In particular, the enumeration  $E' = K_k L_1 K_1 L_2 K_2 \dots K_{k-1} L_k$  is in  $\mathcal{C}$ . Moreover, every bad arc of  $E'$  is a bad arc of  $E$ . But the arc  $xy$  is no longer a bad arc in  $E'$  since  $x \notin Y$  (and thus  $x \in L_1$ ) and  $y \in K_1$ . A contradiction to the minimality of the number of bad arcs.  $\square$

**Corollary 3.1** (Bondy [5]) *Every strong digraph has a circuit with length at least  $\chi(D)$ .*

**Proof.** Given a circuit  $C$  of  $D$ , there exists a coherent cyclic order of  $D$  for which  $C$  is simple, the proof is basically the same as the one of Lemma 1. Then, one just has to apply Theorem 3.  $\square$

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