Hoàng-Reed conjecture holds for tournaments

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Abstract

Hoàng-Reed conjecture asserts that every digraph D has a collection $\mathcal{C}$ of circuits $C_1, \ldots, C_{\delta^+}$, where δ^+ is the minimum outdegree of D, such that the circuits of $\mathcal{C}$ have a forest-like structure. Formally, $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| \leq 1$, for all $i = 2, \ldots, \delta^+$. We verify this conjecture for the class of tournaments.

1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph D on n vertices and with minimum outdegree n/k has a circuit of length at most k. Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A *circuit-tree* is either a singleton or consists of a set of circuits $C_1, \ldots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| = 1$ for all $i = 2, \ldots, k$, where $V(C_j)$ is the set of vertices of C_j. A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique xy-directed path for every distinct vertices x and y. A vertex-disjoint union of circuit-trees is a *circuit-forest*. When all circuits have length three, we speak of a *triangle-tree*. For short, a k-circuit-forest is a circuit-forest consisting of k circuits.

Conjecture 1 (Hoàng and Reed [3]) *Every digraph has a δ^+-circuit-forest.*

This conjecture is not even known to be true for $\delta^+ = 3$. In the case $\delta^+ = 2$, C. Thomassen proved in [6] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament T, that is the 3-uniform hypergraph on vertex set V which edges are the 3-circuits of T.

Indeed, if a tournament T has a δ^+-circuit-forest, by the fact that every circuit contains a directed triangle, T also has a δ^+-triangle-forest. Observe that a δ^+-triangle-forest spans exactly $2\delta^+ + c$ vertices, where c is the number of components of the triangle-forest. When T is a regular tournament with outdegree δ^+, hence with $2\delta^+ + 1$ vertices, a δ^+-triangle-forest of T is necessarily a spanning δ^+-triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

Theorem 1 *Every tournament has a δ^+-triangle-tree.*

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2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

Lemma 1 *Let $k \geq 1$ and let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be two sequences of positive reals. Let $A = \sum_{i=1}^k a_i$ and $B = \sum_{j=1}^k b_j$. If $\sum_{i=1}^k a_i b_i = \frac{AB}{2} + q$, where $q \geq 0$, then there is an i such that $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$.*

Proof. If $k = 1$, then the lemma follows immediately as $q = \frac{AB}{2}$ and $A + B \geq \frac{A+B}{2} + \sqrt{AB}$. So assume that $k > 1$. Without loss of generality, we may assume that $(a_1, b_1) \geq (a_2, b_2) \geq \ldots \geq (a_k, b_k)$ in the lexicographical order. Let r be the minimum value such that $b_r \geq b_i$ for all $i = 1, 2, \ldots, k$. Note that $a_1 \geq |A|/2$, since otherwise $\sum_{i=1}^k a_i b_i < \sum_{i=1}^k A b_i/2 = AB/2$. Analogously $b_r \geq |B|/2$. Define a' and b' so that $a_1 = A/2 + a'$ and $b_r = B/2 + b'$.

If $r \neq 1$, then the following holds:

$$\begin{array}{rcl} \sum_{i=1}^k a_i b_i & \leq & a_1 b_1 + \sum_{i=2}^k a_i b_r \\ & \leq & a_1(B - b_r) + (A - a_1) b_r \\ & = & (\frac{A}{2} + a')(\frac{B}{2} - b') + (\frac{A}{2} - a')(\frac{B}{2} + b') \\ & = & \frac{AB}{2} - 2a'b' \\ & \leq & \frac{AB}{2} \end{array}$$

As $q \geq 0$, this implies we have equality everywhere above, which means that $b_1 = B - b_r$. As $B = b_1 + b_r$, we must have $k = 2$. As there was equality everywhere above we have $b' = 0$ or $a' = 0$ which implies that $a_1 = a_2 = A/2$ or $b_1 = b_2 = B/2$. In both cases we would have $r = 1$, a contradiction.

Suppose now that $r = 1$. Then

$$\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = (\frac{A}{2} + a')(\frac{B}{2} + b') + (\frac{A}{2} - a')(\frac{B}{2} - b')$$

This implies that $q \leq 2a'b'$. The minimum value of $a' + b'$ is obtained when $a' = b' = \sqrt{q/2}$. Therefore the minimum value of $a_1 + b_1$ is $A/2 + B/2 + 2\sqrt{q/2}$. This completes the proof of the lemma. ∎

Corollary 1 *Let G be a bipartite graph with partite sets A and B. If $|E(G)| = \frac{|A||B|}{2} + q$, where $q \geq 0$, then there is a component in G of size at least $|V(G)|/2 + \sqrt{2q}$.*

Proof. Let $Q_1, Q_2, \ldots, Q_k$ be the components of G. Let $a_i = |A \cap Q_i|$ and $b_i = |B \cap Q_i|$ for all $i = 1, 2, \ldots, k$. We note that $\sum_{i=1}^k a_i b_i \geq \frac{|A||B|}{2} + q$. By Lemma 1, we have $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$ for some i. This completes the proof. ∎

Lemma 2 *Let T be a triangle-tree in a digraph D, and let $X \subseteq V(T)$ and $Y \subseteq V(T)$ be such that $|X| + |Y| \geq |V(T)| + 2$. Then there exists a triangle C in T such that the three disjoint triangle-trees in $T - E(C)$ can be named T_1, T_2, T_3 such that Y intersects both T_1 and T_2 and X intersects both T_2 and T_3.*

Proof. We show this by induction. As $|X| + |Y| \geq |V(T)| + 2$, we note that T contains at least one triangle. If T only contains one triangle then the lemma holds as either X or Y equals $V(T)$, and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that T contains at least two triangles. Let $T = T_1 \cup C$, where C is a triangle and T_1 is a triangle-tree. If $|X \cap V(T_1)| + |Y \cap V(T_1)| \geq |V(T_1)| + 2$, then we are done by induction. So assume that this is not the case. As $|V(T_1)| = |V(T)| - 2$ this implies that $|X \setminus V(T_1)| + |Y \setminus V(T_1)| \geq 3$.

Without loss of generality assume that $|X \setminus V(T_1)| \geq 2$ and $|Y \setminus V(T_1)| \geq 1$. Let T_2 be the singleton-tree consisting of a vertex in $Y \setminus V(T_1)$ and let T_3 be the singleton-tree $X \setminus (V(T_1) \cup V(T_2))$. Note that

$T - E(C)$ consists of the triangle-trees T_1, T_2 and T_3. By definition, X intersects both T_2 and T_3 and Y intersects T_2. If Y also intersects T_1, we have our conclusion. If not, since $|X| + |Y| \geq |V(T)| + 2$, we have $Y = T_2 \cup T_3$ and $X = V(T)$, and free to rename T_1, T_2, T_3, we have our conclusion. ∎

3 Proof of Theorem 1.

We will need the following results:

Theorem 2 (Tewes and Volkmann [5]) *Let D be a p-partite tournament with partite sets $V_1, V_2, \ldots V_p$. Then there exists a partition $Q_1, Q_2, \ldots, Q_k$ of D such that*

- *each Q_i induces an independent set or a strong component,*
- *there are no arcs from Q_j to Q_i for all $j > i$, and there is an arc from Q_i to Q_{i+1} for all $i = 1, 2, \ldots, k-1$.*

Theorem 3 (Guo and Volkmann [2]) *Let D be a strong p-partite tournament with partite sets $V_1, V_2, \ldots V_p$. For every $1 \leq i \leq p$, there exists a vertex $x \in V_i$ which belongs to a k-circuit for all $3 \leq k \leq p$.*

Now, we assume that D is a strong tournament as otherwise we just consider the terminal strong component. Let T be a maximum size triangle-tree in D, and assume for the sake of contradiction that $|V(T)| < 2\delta^+(D) + 1$. Let D^{MT} be the multipartite tournament obtained from D by deleting all the arcs with both endpoints in $V(T)$. Let $V_1, V_2, \ldots, V_l$ be the partite sets in D^{MT} such that $V_1 = V(T)$ and $|V_i| = 1$ for all $i > 1$.

Let $Q_1, Q_2, \ldots, Q_k$ be a partition of $V(D^{MT})$ given by Theorem 2.

If there is a Q_i with $Q_i \cap V_1 \neq \emptyset$ and $Q_i \not\subseteq V_1$ then we obtain the following contradiction. Since $Q_i \not\subseteq V_1$, we observe that Q_i contains at least two partite set. In addition, note that at least three partite sets intersect Q_i as $D^{MT}\langle Q_i \rangle$ would not be strong if there were only two partite sets since $|V_i| = 1$ for all $i > 1$. By Theorem 3, in the subgraph of D^{MT} induced by Q_i, there is a 3-circuit containing exactly one vertex from V_1. This contradicts the maximality of T. So every set Q_i is either a subset of V_1 or is disjoint from V_1.

Note that $Q_1 \cap V_1 \neq \emptyset$ and $Q_k \cap V_1 \neq \emptyset$, as otherwise D would not be strong. Applying the observation above, we obtain $Q_1 \cup Q_k \subset V_1$. Let $D' = D\langle V_1 \rangle$. If there is a vertex $x \in Q_k$ with $d^+_{D'}(x) \leq \frac{|V_1|-1}{2}$, then $d^+_D(x) \leq \frac{|V_1|-1}{2}$, which implies that $|V(T)| \geq 2\delta^+(D) + 1$, a contradiction. So $d^+_{D'}(x) \geq \frac{|V_1|+1}{2}$ for all $x \in Q_k$, as $|V_1|$ is odd.

Let G_1 denote the bipartite graph with partite sets Q_k and $V_1 - Q_k$, and with $E(G_1) = \{uv \mid u \in Q_k,\ v \in V_1 - Q_k,\ uv \in E(D)\}$. Note that the following now holds by the above.

$$|Q_k| \frac{|V_1| + 1}{2} \leq \sum_{u \in Q_k} d^+_{D'}(u) = \binom{|Q_k|}{2} + |E(G_1)| \tag{1}$$

This implies that $|E(G_1)| \geq \frac{|Q_k|(|V_1| - |Q_k|)}{2} + |Q_k|$, which by Corollary 1 implies that there is a component in G_1 of size at least $|V_1|/2 + \sqrt{2|Q_k|} \geq |V_1|/2 + \sqrt{2}$. As the size of the maximum component in G_1 is an integer it is at least $|V_1|/2 + 3/2$. Two cases can now occur:

- If $|Q_{k-1}| > 1$ or $Q_{k-2} \not\subseteq V_1$ (or both). If $|Q_{k-1}| > 1$ then let $Z = \{z_1, z_2\}$ be any two distinct vertices in Q_{k-1} otherwise let Z be any two distinct vertices in $Q_{k-1} \cup Q_{k-2}$. By the definition of the Q_i's we note that $Z \cap V_1 = \emptyset$ and there are all arcs from $(V_1 - Q_k)$ to Z and from Z to Q_k. We let $X = Y$ be the vertices of a component in G_1 of size at least $(|V_1| + 3)/2$ and use Lemma 2 to find a triangle C in T, such that the three disjoint triangle-trees, T_1, T_2 and T_3, of $T - E(C)$ all intersect

X (as $X = Y$). As X are the vertices of a component in G_1 there are edges, u_1v_1 and u_2v_2, from G_1 such that the following holds. The edge u_1v_1 connects T_3 and T_j, where u_2v_2 connects T_{3-j} and $T_j \cup T_3$. generality assume that $u_1, u_2 \in Q_k$ and $v_1, v_2 \in V_1 - Q_k$. Now $T - E(C)$ together with the vertices z_1 and z_2 as well as the 3-circuits $v_1z_1u_1v_1$ and $v_2z_2u_2v_2$ is a triangle-tree in D with more triangles than T, a contradiction.

- If $|Q_{k-1}| = 1$ and $Q_{k-2} \subseteq V_1$. Note that $k > 3$, as otherwise $|V(D) \setminus V(T)| = 1$ and we have a contradiction to our asumption. This implies that $k > 4$ as $Q_1 \subseteq V_1$, which implies that $Q_2 \not\subseteq V_1$. Now let $Q_{k-1} = \{z_1\}$ and let $z_2 \in Q_{k-3}$ be arbitrary. Let G_2 denote the bipartite graph with partite sets $A = Q_k \cup Q_{k-2}$ and $B = V_1 - A$, and with $E(G_2) = \{uv \mid u \in A,\ v \in B,\ uv \in E(D)\}$. Recall that $d^+_{D'}(x) \geq \frac{|V_1|+1}{2}$ for all $x \in Q_k$. Analogously we get that $d^+_{D'}(y) \geq \frac{|V_1|+1}{2} - 1$ for all $y \in Q_{k-2}$ (as $|Q_{k-1}| = 1$). This implies the following.

$$|A|\frac{|V_1|+1}{2} - |Q_{k-2}| \leq \sum_{u \in A} d^+_{D'}(u) = \binom{|A|}{2} + |E(G_2)| \tag{2}$$

This implies that $|E(G_2)| \geq \frac{|A|(|V_1|-|A|)}{2} + |A| - |Q_{k-2}|$, which by Corollary 1 implies that there is a component in G_2 of size at least $|V_1|/2 + \sqrt{2|Q_k|}$, as $|A| - |Q_{k-2}| = |Q_k|$. Note that $|Q_k| > 1$, as otherwise the vertex in Q_{k-1} only has out-degree one, a contradiction. Therefore there is a component in G_2 of size at least $|V_1|/2 + 2$ and so at least $|V_1|/2 + 5/2$ as V_1 is odd.

Let X be the vertices of a component in G_1 of size at least $|V_1|/2 + 3/2$ and let Y be the vertices in a connected component of G_2 of size at least $|V_1|/2 + 5/2$. Now use Lemma 2 to find a triangle C in T, such that the three disjoint triangle-trees, T_1, T_2 and T_3, of $T - E(C)$ have the following property. The set Y intersects T_1 and T_2 and the set X intersects T_2 and T_3. Due to the definition of X and Y there exists edges, $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, such that the following holds. The edge u_1v_1 connects T_3 and T_j, where $j \in \{1, 2\}$ and u_2v_2 connects T_{3-j} and $T_j \cup T_3$. Without loss of generality assume that $u_1, u_2 \in Q_k$ and $v_1, v_2 \in V_1 - Q_k$. Now $T - E(C)$ together with the vertices z_1 and z_2 as well as the 3-circuits $v_1z_1u_1v_1$ and $v_2z_2u_2v_2$ is a triangle-tree in D with more triangles than T, a contradiction. This completes the proof.

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