Indivisibility and Alpha-Morphisms

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Abstract

A relation R is p-divisible if for any partition of its basis into $p+1$ subsets, R is embedded into the union of p of them. We prove that any countable p-divisible relation embeds two copies of itself intersecting in at most $p-1$ elements.

A relation R is *indivisible* if for any partition of its set of vertices into two subsets V_1 and V_2, there exists $i \in \{1,2\}$ such that R is embedded into $R(V_i)$, where $R(V_i)$ denotes the restriction of R to V_i. Clearly, there is no finite indivisible relation with more than one vertex. In the infinite case, in addition to some trivial examples of indivisible relations such that complete graphs or the order type of natural numbers, one can easily check that Rado's graph and the order type of the rationals are also indivisible. The starting point of this work is the following conjecture of Fraïssé : if R is indivisible, there exist two disjoint embeddings from R into itself. This conjecture has been proved true in the countable case by Pouzet [6] : a countable counterexample would give a non-principal analytic ultrafilter on ω, contrary to a result of Sierpinski. For recent developments on Fraïssé's conjecture in the general case, see Bishop's work on ultrafilters [1].

One possible generalization of indivisibility is to increase the number of blocks of the partition, thus, a relation is *p-divisible* if for any partition of its basis into $p+1$ subsets $\{V_1, V_2, \ldots, V_{p+1}\}$ there exists i such that R is embedded into $R \setminus V_i$ (clearly, 1-divisible is indivisible). Answering a question of Hajnal, Laver proved in [5] that for any countable linear ordering L there exists a p such that L is p-divisible. We prove in this paper a generalization of Pouzet's theorem : if R is a countable p-divisible relation, one can find two copies of R into itself intersecting in at most $p-1$ vertices ($p=1$ gives Pouzet's theorem).

We give a purely combinatorial proof of this theorem which is based on the notion of α*-morphism* (the key idea of Ehrenfeucht-Fraïssé games). A 0-morphism from R into R' is a local isomorphism (an isomorphism from a finite restriction of R into a finite restriction of R'). An $\alpha+1$-morphism is a local isomorphism from R into R' which can be extended on any finite subset of R as an α-morphism. Lastly, if α is a limit ordinal, an α-morphism is a β-morphism for any $\beta < \alpha$. According to this definition, a relation R α*-embeds* a relation R' if the empty morphism is an α-morphism from R' into R. The fundamental theorem of Fraïssé asserts that if R and R' are countable and R ω_1-embeds R' then R embeds R'. In this paper, we use this notion in order to construct inductively the disjoint (resp. nearly disjoint) copies of an indivisible (resp. p-divisible) relation. Here are the two steps of the proof, when R is an indivisible relation :

First we prove that if for any countable α, one can find two disjoint subsets V_1^α and V_2^α such that R is α-embedded into $R(V_1^\alpha)$ and into $R(V_2^\alpha)$, then R embeds two disjoint copies of itself. We then prove that if R is indivisible, one can always find such subsets V_1^α and V_2^α for any countable α.

The easy step is the first one - we just have to define inductively the notion of disjoint α-morphisms. We prove the second step by induction. To initialize the induction, we have to find V_1^1 and V_2^1 (each of these subsets must contain all the finite isomorphism types of R). This can be done easily, as any finite restriction of an indivisible relation R is disjointly embedded into R infinitely many times. The next stage of the proof contains the whole difficulty of the problem : here we have to consider infinite families of finite extensions. The tool we use to handle these families is the notion of Δ-system. We make extensive use of a lemma asserting that any infinite family of finite sets either contains an infinite Δ-system or is "nested" in a sense to be made precise.

Of more note than our slight extension of the indivisibility theorem is the main idea of this paper, that given an α-morphism, either we can extend it in only one direction or we can extend it in infinitely many "disjoint" directions.

1 Disjoint Alpha-Morphisms.

In this part, we recall the main results about α-morphisms. Furthermore, we state the first step of our proof : given two countable relations R and R', if for any countable ordinal α, there is a partition of R' into two subsets such that each one α-embeds R, then one can find two disjoint embeddings from R into R'.

Definition 1 (Fraïssé [3])

- An *n-ary relation* is a pair $R = (V, E)$ where V is a set and E a subset of V^n. The elements of V are *vertices*, those of E are *edges*. A n-ary relation $R = (V, E)$ is *embedded* into another n-ary relation $R' = (V', E')$ if there is an injective mapping f from V into V' such that $(v_1, ..., v_n) \in E$ if and only if $(f(v_1), ..., f(v_n)) \in E'$. If f is a $1-1$ mapping, then R is *isomorphic* to R'. Let Y be a subset of V, $R(Y) = (Y, V^n \cap E)$ is the *induced subrelation* on Y. Hence, the relation R is embedded into R' if and only if R is isomorphic to an induced subrelation of R'.

- A *local isomorphism* from R into R' is an isomorphism f from $R(F)$ into $R'(F')$ where F and F' are finite subsets of V and V', we denote F by $Dom(f)$ and F' by $Im(f)$. If D is a subset of F, the restriction of f to D is denoted by f_D. If G is a finite subset containing F, we will usually denote an extension of f to G by f^G.

- For any ordinal α, we define by induction the notion of *α-morphism* from R into R' :

i) Any local isomorphism is a 0-morphism.

ii) Let f be a local isomorphism from R into R'. If for any finite subset F of V containing Dom(f), we can extend f to F as an α-morphism, then f is an $(\alpha + 1)$-morphism.

iii) If α is a limit ordinal and f is a β-morphism for any $\beta < \alpha$ then f is an α-morphism.

- We say that R' *α-embeds* R if the empty local isomorphism is an α-morphism from R into R'.

Example 1 - If f is an embedding from R into R', then for any finite $F \subseteq V$ and any ordinal α, the restriction f_F is an α-morphism.

- The ordinal ω does not 2-embed $\omega + 1$. More generally, for any countable ordinal α, ω^α does not $(\alpha + 1)$-embed $\omega^\alpha + 1$.

- In the class of countable locally finite connected graphs, if G 2-embeds G' then G embeds G'. (If f is a 1-morphism from G' into G such that $Dom(f)$ is not empty, there are only finitely many ways to extend f to the neighbourhood of $Dom(f)$. By König's infinitary lemma, one of these extensions is also a 1-morphism.)

Throughout this section, $R = (V, E)$ and $R' = (V', E')$ are countable n-ary relations, f is an α-morphism from R into R' which domain is F and image is F'.

Lemma 1 *If $\beta < \alpha$, then f is a β-morphism.*

Lemma 2 *Let D be a subset of F, the restriction of f to D is an α-morphism.*

Proof. By induction on α.

- Trivial if α is limit or equal to 0.

- If $\alpha = \beta + 1$, for any finite G containing D we just have to prove that we can extend f_D to G as a β-morphism. We consider an extension $f^{F \cup G}$ which is a β-morphism, thus, by the induction hypothesis, $(f^{F \cup G})_D = f_D$ is a β-morphism. □

Theorem 1 (Fraïssé [3]) *If R' ω_1-embeds R then R' embeds R.*

Proof. We enumerate the vertices of R, thus $V = \{v_i\}_{i \in \omega}$. For any countable α, as the empty morphism is an $(\alpha + 1)$-morphism, there exists a vertex x_α of V' such that the local isomorphism which maps v_0 into x_α is an α-morphism. Thus, there is an element y_0 which is cofinal in the sequence $\{x_\alpha\}_{\alpha < \omega_1}$. Then, the local isomorphism f_0 which maps v_0 into y_0 is an ω_1-morphism. Similarly, we can extend f_0 to the domain $\{v_0, v_1\}$ in another ω_1-morphism f_1. This process gives a sequence of local isomorphisms $\{f_i\}_{i \in \omega}$ the union of which is an embedding from R into R'. □

Our purpose is to find disjoint copies of a relation, so we need an extension of this theorem.

Definition 2 - Let f and g be two local isomorphisms from R into R' defined on the same domain and let A be a finite subset of V'. We give by induction the definition of *disjoint α-morphisms relative to A*.

i) If $Im(f) \cap Im(g) \subseteq A$ then f and g are disjoint 0-morphisms relative to A.

ii) If f and g are both $(\alpha + 1)$-morphisms then they are disjoint relative to A if for any finite subset F extending their domain, one can find two extensions f^F and g^F which are disjoint α-morphisms relative to A.

iii) If α is limit, disjoint α-morphisms relative to A means disjoint β-morphisms relative to A for any $\beta < \alpha$.

- Now we use the definition for $f = g$. If f and f are disjoint α-morphisms relative to $Im(f)$ then we will simply say that f is a *disjoint* α-morphism from R into R'.

Example 2 - Any local isomorphism from the Rado countable universal graph into itself is a disjoint ω_1-morphism.

- Any 2-morphism from the infinite path into the infinite binary tree is a disjoint ω_1-morphism.

Theorem 2 *If f is a disjoint ω_1-morphism from R into R', then there exists two embeddings g and h from R into R', both extending f, such that $g(V) \cap h(V) = Im(f)$.*

Proof. Same as theorem 1. We enumerate the vertices which do not belong to $Dom(f)$, thus $V \setminus Dom(f) = \{v_i\}_{i \in \omega}$. Now, for any countable α, as f is a disjoint $(\alpha + 1)$-morphism, there are two different vertices (x_α, y_α) of $V' \setminus Im(f)$ such that the extension of f which maps v_0 into x_α and the extension of f which maps v_0 into y_α are disjoint α-morphisms relative to $Im(f)$. There exists a cofinal (x^0, y^0) in the sequence $\{(x_\alpha, y_\alpha)\}_{\alpha < \omega_1}$, thus, the following extensions of f : g_0 which maps v_0 into x^0 and h_0 which maps v_0 into y^0, are disjoint ω_1-morphisms relative to $Im(f)$. Similarly, we can extend g_0 and h_0 to the domain $Dom(f) \cup \{v_0, v_1\}$ in another couple of disjoint ω_1-morphisms relative to $Im(f)$. This process gives a sequence of pair of local isomorphisms $\{(g_i, h_i)\}_{i \in \omega}$ and the unions of these $g = \cup g_i$, $h = \cup h_i$ have the required properties. □

Now, to prove the indivisibility theorem, we just have to check that the empty morphism from a countable indivisible relation into itself is an α-disjoint morphism for all $\alpha < \omega_1$. The case $\alpha = 1$ is easy since any finite restriction of an indivisible relation R can be embedded into R in countably many (and hence two) disjoint ways. In fact, the whole difficulty of the proof is for the case $\alpha = 2$. Here we have to handle infinite families of finite subsets in order to extract disjoint families. This is the aim of the next part :

2 Duality Lemmas on Delta-Systems.

Definition 3 - A Δ-*system* $\mathcal{F}$ is a family of sets such that the intersection of any two distinct elements of $\mathcal{F}$ is a given set F. The set F is the *center* of $\mathcal{F}$. A Δ-system is *disjoint* if its center is empty.

- If $\mathcal{F}$ is a family and X is a set, $\mathcal{F} \setminus X$ (resp. $\mathcal{F} \cup X$) is the family of sets $Y \setminus X$ (resp. $Y \cup X$) where Y belongs to $\mathcal{F}$.

In this paper, "countable" means "countably infinite". All the families are finite or countable families of finite sets. Unless stated otherwise, the Δ-systems are always countable families of finite sets.

Lemma 3 *Let $\mathcal{F}$ be a countable family. One and only one of the following cases occurs :*

i) $\mathcal{F}$ contains a Δ-system.

ii) There is a disjoint Δ-system $\mathcal{D}$ such that any element of $\mathcal{D}$ intersects all but a finite number of elements of $\mathcal{F}$.

Proof. Clearly i) and ii) are mutually exclusive. Suppose i) is false. Inductively define finite families $\mathcal{F}_n$ so that $\mathcal{F}_n$ is a maximal disjoint subfamily of $\mathcal{F} \setminus \bigcup\{F : F \in \bigcup\{\mathcal{F}_i\}_{i<n}\}$. Then ii) holds with $\mathcal{D} = \{\bigcup \mathcal{F}_n : n < \omega\}$. $\square$

Corollary 1 *If the size of the elements of $\mathcal{F}$ is bounded, only case i) occurs.*

Definition 4 - A countable family which satisfies condition i) of lemma 3 is called *wide* otherwise it is *narrow.*

- Let $\mathcal{F}$ be a wide family, if F is a center of a Δ-system included in $\mathcal{F}$ and F is minimal by inclusion for this property, it is called *kernel* of $\mathcal{F}$. The set of kernels of $\mathcal{F}$ is denoted by $K(\mathcal{F})$.

Lemma 4 *Let $\mathcal{F}$ be a wide family such that $K(\mathcal{F})$ is infinite, then $K(\mathcal{F})$ is a narrow family.*

Proof. Suppose for contradiction that $K(\mathcal{F})$ is wide, then any of its kernels is a kernel of $\mathcal{F}$, against the minimality. $\square$

The two following lemmas are immediate from the definitions.

Lemma 5 *Let $\mathcal{F}$ be a wide family partitioned into finitely many subfamilies, namely $\mathcal{F} = \bigcup\{\mathcal{F}_i\}_{i<n}$. Then there exists an i such that $\mathcal{F}_i$ is wide, each kernel of it contains a kernel of $\mathcal{F}$. Moreover, each kernel of $\mathcal{F}$ is the kernel of some $\mathcal{F}_j$.*

Lemma 6 *Let $\mathcal{F}$ be a wide family and F a finite set. Then the family $\mathcal{F} \setminus F$ is wide, moreover for any kernel K of $\mathcal{F} \setminus F$ there is a kernel K' of $\mathcal{F}$ such that $K = K' \setminus F$.*

The following lemma is an extension of lemma 3, in order to check it, one just have to choose an injective rank.

Lemma 7 *Let $\mathcal{F}$ be a family and r, a mapping from $\mathcal{F}$ into the natural numbers called rank. One and only one of the following cases occurs :*

i) $\mathcal{F}$ contains a Δ-system $\mathcal{D}$ which has the following property : for any n one can find $D \in \mathcal{D}$ such that $r(D) > n$.

ii) There is a disjoint Δ-system $\mathcal{D}$ which has the following property : for any $D \in \mathcal{D}$ there is a n such that any element of $\mathcal{F}$ with rank greater than n intersects D.

Proof. Here again, we check easily that those two cases exclude each other.

Now we suppose that ii) does not hold. One can possibly find a finite set F_0 which intersects any element of $\mathcal{F}$ greater than a given rank. Again, one can possibly find a finite set F_1, disjoint from F_0 which intersects any element of $\mathcal{F}$ greater than a given rank. Iterating this process, we construct a sequence F_0, F_1, F_2,... which stops on one F_n. Now let $F = \bigcup\{F_i\}_{0\leq i\leq n}$, for any finite set F' disjoint from F there is an element H of $\mathcal{F}$ with arbitrarily high rank such that $H \cap F' = \emptyset$.

We denote by $\mathcal{F}_i$, the subfamily of $\mathcal{F}$ which contains all the elements with rank greater than i. As ii) does not hold, all of the $\mathcal{F}_i$ are wide. Moreover, for any i, the family $\mathcal{F}_i \setminus F$ has an empty kernel. Thus, following lemma 6, each $\mathcal{F}_i$ has a kernel included in F.

We now consider a subset F' of F which is a kernel of infinitely many $\mathcal{F}_i$. Then any of those $\mathcal{F}_i$ contains a Δ-system which center is F'. By a diagonal argument on this collection of Δ-systems, we extract a Δ-system which satisfies i). $\square$

Here we have the basic tools to start the discussion. We will show in the following section that an analog of lemma 3 can be stated for α-morphisms. This property is a way of measuring how disjoint are the extensions of local isomorphisms. This tool will give the impartibility theorem : either local isomorphisms are not very disjoint and one can divide the relation, or they are very disjoint and one can find two copies thanks to theorem 2.

3 Wide and Narrow Alpha-Morphisms.

Throughout this section, $R = (V, E)$ and $R' = (V', E')$ are countable n-ary relations, f is a local isomorphism from R into R' with domain F and image F'. Moreover, all the Δ-systems are countable families of finite subsets of V'.

Definition 5 - A *realization* of a Δ-system is the union of any infinite subfamily.

- f is *α-narrow* if there is a disjoint Δ-system $\mathcal{D}$, called an *α-narrow system* of f, which has the following property : for any realization Y of $\mathcal{D}$, f is not an α-morphism from R into $R'(V' \setminus Y)$.

- f is *α-wide* if there is a Δ-system $\mathcal{D}$ which has the following property : for any realization Y of $\mathcal{D}$, f is an α-morphism from R into $R'(Y)$. Such a Δ-system is called an *α-wide system* of f. We denote by $\mathcal{D}_\alpha(f)$ the set of α-wide systems of f, and $\mathcal{C}_\alpha(f)$ is the set of centers of the elements of $\mathcal{D}_\alpha(f)$. The elements of $\mathcal{C}_\alpha(f)$ which are minimal for inclusion are called *α-kernels* of f, we denote the set of α-kernels of f by $\mathcal{K}_\alpha(f)$. Clearly, any α-kernel of f contains $Im(f)$.

Example 3 - Any local isomorphism f is 0-wide (consider any Δ-system with center $Im(f)$). Trivially, if f is not an α-morphism, it is α-narrow.

- The empty isomorphism from the infinite path into itself is 2-narrow.

- Any α-morphism from the infinite path into the ω-tree (the acyclic connected graph such that any vertex has countable degree) is α-wide.

- This definition gives us some examples of disjoint α-morphisms : namely, if f is α-wide and $\mathcal{K}_\alpha(f) = \{Im(f)\}$ then f is a disjoint α-morphism.

Lemma 8 *Let $\mathcal{D}$ be a finite or countable set of disjoint Δ-systems. There exists a disjoint Δ-system $\mathcal{D}'$ such that any realization of $\mathcal{D}'$ contains a realization of any of the Δ-systems of $\mathcal{D}$. We will call such a $\mathcal{D}'$ a diagonal Δ-system of $\mathcal{D}$.*

Proof. Direct diagonal argument : let $\mathcal{D} = \{\mathcal{D}_i\}_{i\in\omega}$, the $\mathcal{D}_i$ are not necessarily different. Let D_0 be an element of $\mathcal{D}_0$. Now let F_0 and F_1 be some elements of $\mathcal{D}_0$ and $\mathcal{D}_1$ disjoint from D_0, we choose $D_1 = F_0 \cup F_1$. We construct in a similar way D_2, D_3,... Now $\mathcal{D}' = \{D_i\}_{i\in\omega}$. $\square$

Corollary 2 *If f is α-wide and $\mathcal{K}_\alpha(f)$ is infinite, then $\mathcal{K}_\alpha(f)$ is narrow.*

Proof. Suppose for contradiction that $\mathcal{K}_\alpha(f)$ contains a Δ-system $\mathcal{D} = \{D_i\}_{i\in\omega}$ centered in F. Any D_i is the center of an α-wide system $\mathcal{F}_i$ of f. We apply lemma 8 to the set $\{\mathcal{D} \setminus F, \mathcal{F}_0 \setminus D_0, ..., \mathcal{F}_i \setminus D_i, ...\}$, we denote one of its diagonal Δ-system by $\mathcal{D}'$, now $\mathcal{D}' \cup F$ is an α-wide system of f centered in F, thus F contains an α-kernel of f, against minimality. $\square$

Now, we prove the central theorem.

Theorem 3 *For any countable α, f is either α-narrow or α-wide.*

Proof. Clearly, f cannot be both α-narrow and α-wide.

We prove the result by induction on ω_1. We have already observed (example 3) that f is 0-wide. Note that, if f is β-narrow then it is α-narrow for all $\alpha > \beta$. So assume that f is β-wide for all $\beta < \alpha$. We have to show that f is either α-narrow or α-wide. We may assume that f is an α-morphism, otherwise it is α-narrow.

- If α is a limit ordinal, let $\{\beta_i\}_{i\in\omega}$ be a cofinal sequence in α. We denote by $\mathcal{K}$ the union of all the $\mathcal{K}_{\beta_i}(f)$. Two cases may occur :

a) There is some F which belongs to infinitely many $\mathcal{K}_{\beta_i}(f)$. This means that one can find a sequence $\{\mathcal{D}_j\}_{j\in\omega}$, each $\mathcal{D}_j$ is a β_j-wide system of f centered in F, and $\{\beta_j\}_{j\in\omega}$ is cofinal in α. By lemma 8, one can find a diagonal disjoint Δ-system $\mathcal{D}$ of the set $\{\mathcal{D}_j \setminus F\}_{j\in\omega}$. Now $F \cup \mathcal{D}$ is an α-wide system of f centered in F. Then f is α-wide.

b) If a) does not hold, for any $F \in \mathcal{K}$ we consider $r(F) = max\{i \in \omega : F \in \mathcal{K}_{\beta_i}(f)\}$. Now, we have a ranked family on which we can apply lemma 7.

Suppose that $(\mathcal{K}, r)$ satisfies condition i) of lemma 7, then $\mathcal{K}$ contains a Δ-system $\mathcal{D}$ which has the following property : for any n, one can find $D_n \in \mathcal{D}$ such that D_n is the kernel of a γ-wide system $\mathcal{F}_n$ of f with $\gamma > \beta_n$. Let C be the center of $\mathcal{D}$, then, by a diagonal argument on $\{\{D_i \setminus C : i \in \omega\}, \mathcal{F}_1 \setminus D_1, \ldots, \mathcal{F}_i \setminus D_i, \ldots\}$, C is the center of a β_i-wide system of f for infinitely many i. So, we can find a kernel included in C which belongs to infinitely many $\mathcal{K}_{\beta_i}(f)$, then a) holds. This is a contradiction.

Therefore $(\mathcal{K}, r)$ satisfies condition ii) of lemma 7. Then there is a disjoint Δ-system $\mathcal{D} = \{D_i\}_{i\in\omega}$ which has the following property : for any i there is an n such that any element of $\mathcal{K}$ with rank greater than n intersects D_i. This means that f cannot be β_n-wide from R into $R'(V' \setminus D_i)$, otherwise there would be a kernel disjoint from D_i. By the induction hypothesis, f is therefore β_n-narrow from R into $R'(V' \setminus D_i)$, let $\mathcal{F}_i$ be one of its β_n-narrow systems. We consider a diagonal disjoint Δ-system of $\{\mathcal{D}, \mathcal{F}_0, ..., \mathcal{F}_i, ...\}$, this is an α-narrow system of f. Thus f is α-narrow.

- The next case is $\alpha = \beta + 1$. If there is a finite set $F \supseteq Dom(f)$ such that every extension f^F is β-narrow, then f is α-narrow. For there are only countably many different extensions of f to F and each of them has a β-narrow system. Then a diagonal system of these is an α-narrow system. (Let f_i $(i \in \omega)$ be the different extensions to F, and let $\mathcal{F}_i$ be a witness that f_i is β-narrow. Consider any realization Y of the diagonal system of the $\mathcal{F}_i$. If f is an α-morphism from R into $R'(V' - Y)$, then it has an extension f_i to F which is a β-morphism, and this is a contradiction since Y is also a realization of $\mathcal{F}_i$.) So we can assume, for each finite set F, there is a β-wide extension of f to F.

Let $\{F_i : i \in \omega\}$ be an increasing sequence of finite sets such that $F_0 = Dom(f)$ and $V = \bigcup\{F_i : i \in \omega\}$.

Now, for any F_i, there is an extension f^{F_i} which is β-wide, we denote by $\mathcal{E}_i$ the set of β-wide extensions of f to F_i. Moreover, let $\mathcal{K}_i = \bigcup\{\mathcal{K}_\beta(g) : g \in \mathcal{E}_i\}$ and $\mathcal{K} = \bigcup\{\mathcal{K}_i\}_{i\in\omega}$. Each element of $\mathcal{K}_i$ contains an image of F_i, and so has size at least $|F_i|$. Thus, for any $F \in \mathcal{K}$, one can define $r(F) = max\{i \in \omega : F \in \mathcal{K}_i\}$. Now we have our ranked family $(\mathcal{K}, r)$. We apply lemma 7, two cases may occur :

i) Either $\mathcal{K}$ contains a Δ-system $\mathcal{D}$ which has the following property : for any n one can find $D_n \in \mathcal{D}$ such that D_n is the kernel of a β-wide system $\mathcal{F}_n$ of an extension of f to F_n. Let C be the center of $\mathcal{D}$, by a diagonal argument on $\{\{D_i \setminus C : i \in \omega\}, \mathcal{F}_1 \setminus D_1, \ldots, \mathcal{F}_i \setminus D_i, \ldots\}$, f is α-wide.

ii) Or there is a disjoint Δ-system $\mathcal{D}$ which has the following property : for any $D \in \mathcal{D}$ there is an n such that any kernel of a β-wide extension of f to F_n intersects D. Then, there is no extension of f

to F_n which is β-wide from R into $R'(V' \setminus D)$. We conclude as in the last part of the limit case : f is α-narrow. $\square$

Corollary 3 *If f is α-wide and $Im(f)$ is not an α-kernel of f, then there exists a finite set A disjoint from $Im(f)$ such that f is α-narrow from R into $R'(V' \setminus A)$.*

Proof. If $\mathcal{K}_\alpha(f)$ is finite, then we just choose $A = (\bigcup \mathcal{K}_\alpha(f)) \setminus Im(f)$. Now if $\mathcal{K}_\alpha(f)$ is infinite, following corollary 2, it is narrow. Thus one can find a finite set A, disjoint from $Im(f)$ and intersecting all the elements of $\mathcal{K}_\alpha(f)$. $\square$

And what about the ω_1 case? The problem here is not to find an ω_1-kernel, we just have to extract a constant cofinal sequence of α-kernels. But we do not know how to extract an ω_1-wide system. A positive answer to the following conjecture would help to solve the problem.

Conjecture 1 (Fraïssé [3]) For any countable relation R, there is a countable α such that, given any relation R' embedded into R, if R' α-embeds R then R' embeds R.

4 An Extension of the Indivisibility Theorem

In this section, $R = (V, E)$ is a relation.

Definition 6 - R is *indivisible* if for any partition of V into two subsets V_1 and V_2, R is embedded into $R(V_1)$ or into $R(V_2)$.

- R is *p-divisible* if for any partition of V into $p+1$ subsets V_1, V_2,..., V_{p+1}, there is one $i \in \{1, ..., p+1\}$ such that R is embedded into $R(V \setminus V_i)$. Clearly 1-divisible is indivisible.

- A *critical vertex* of R is an element $v \in V$ such that R is not embedded into $R(V \setminus \{v\})$. The *kernel* of R is its set of critical vertices.

Example 4 - Any infinite empty graph, Rado's graph and the order type of the rationals are indivisible relations.

- The order types $\omega + n$ and $\omega.(n+1)$ are (n+1)-divisible.

Theorem 4 (El-Zahar, Sauer [2]) *The homogeneous K_n-free graphs are indivisible.*

Theorem 5 (Laver [5]) *For any countable linear ordering L, there exists a p such that L is p-divisible.*

Conjecture 2 For any countable homogeneous relation R, there exists a p such that R is p-divisible.

Following the characterization of the homogeneous graphs (Lachlan, Woodrow [4]) and theorem 4, this conjecture holds for the class of homogeneous graphs. It is also true for homogeneous partial orders, the only homogeneous tournament which is not trivially p-divisible for some p is the following :

Example 5 We consider the tournament $T = (V, E)$ constructed on a dense countable set V of the unit circle, provided that if $x \in V$ then $x + \pi \notin V$. The set E is constructed as follows : $(x, y) \in E$ if and only if $0 < (y - x)$ *mod* π. This tournament is 2-divisible. Indeed, another way of constructing this tournament is to consider a linear ordering $<$ on V isomorphic to the rationals. Now let l be a mapping from V into $\{0, 1\}$ such that $l^{-1}(0)$ and $l^{-1}(1)$ are dense in $(V, <)$. The set of edges E is constructed in the following way : $(x, y) \in E$ if and only if ($l(x) = l(y)$ and $x < y$) or ($l(x) \neq l(y)$ and $x > y$). In Fraïssé's terminology, the tournament T is freely interpretable by the rational order type and one unary relation. As $(V, <, l)$ is 2-divisible, T is also 2-divisible.

Lemma 9 *Suppose the kernel K of R is finite. Then :*
i) For any embedding f from R into R, $f(K) = K$.
ii) For any finite subset A disjoint from K, there is an embedding f from R into $R(V \setminus A)$.

Proof. Let $v \in K$, by definition of the kernel, $v \in Im(f)$ so there is an x such that $f(x) = v$. The vertex x is also critical, otherwise, there would be an embedding g such that $x \notin g(V)$, and finally $v \notin f(g(V))$. As K is finite, $f(K) = K$.

We prove now ii) by induction on $card(A)$, this is true if A is the empty set. Now let $y \in A$, as y is not a critical element of R there exists an embedding g such that $y \notin Im(g)$. Now we use induction hypothesis on $R(g(V))$. $\square$

Theorem 6 (Pouzet [6]) *If R is indivisible and countable, there are two disjoint subsets V_1 and V_2 of V such that R is isomorphic to $R(V_1)$ and to $R(V_2)$.*

As a generalization of this last result, we state :

Theorem 7 *If R is p-divisible and countable, there are two subsets V_1 and V_2 of V which satisfy the following properties :*
i) R is isomorphic to $R(V_1)$ and to $R(V_2)$.
ii) $card(V_1 \cap V_2) < p$.

Proof. We denote by K the kernel of R. As R is p-divisible, K has at most $p - 1$ elements. Suppose for contradiction that any local isomorphism from K into K is α-narrow for one countable α. Thus, for any local isomorphism f from K into K, there is a disjoint Δ-system $\mathcal{D}_f$ such that for any realization Y of $\mathcal{D}_f$, f is not an α-morphism and then cannot be extended to an embedding from R into $R(V \setminus Y)$. Let $\mathcal{D}$ be a diagonal disjoint Δ-system of the finite set $\{\mathcal{D}_f : f \in Aut(K)$. Now, following lemma 9 i), for any realization Y of $\mathcal{D}$, R is not embedded into $R(V \setminus Y)$. At last, we consider a partition of V into $p + 1$ subsets, each of them contains a realization of $\mathcal{D}$: this partition does not satisfy the p-divisibility property.

Suppose for contradiction that for any f from K into K which is α-wide for any countable α, there is a particular β_f such that $\mathcal{K}_{\beta_f}(f) \neq \{K\}$. Then, for any $f \in Aut(K)$, following corollary 3, there exists a finite set A_f disjoint from K such that f is β_f-narrow from R into $R(V \setminus A_f)$. We denote by A the (finite) union of the A_f. Now, all the local isomorphisms on domain K are α-narrow from R into $R(V \setminus A)$ for a suitably large $\alpha < \omega_1$. But following lemma 9 ii), there is one embedding g from R into $R(V \setminus A)$, moreover we proved in the previous paragraph that there exists a local isomorphism h from K into K which is α-wide for any countable α. Now, goh is a local isomorphism from R into $R(V \setminus A)$ which is α-wide for any countable α. Contradiction.

At last, there is a local isomorphism f on domain K which is α-wide from R into R and such that $\mathcal{K}_\alpha(f) = Im(f) = \{K\}$ for any countable α. Thus, following the last remark in example 3, f is a disjoint α-morphism for any countable α.

So, following theorem 2, we finally have two copies of R intersecting on K, with $card(K) < p$. $\square$

When we started this work about α-morphisms and indivisibility, our purpose was to prove the following conjecture of Pouzet[6].

Conjecture 3 Let R and S be countable relations. If in any bipartition of R, S is embedded into one side of the partition, then there exists two disjoint copies of S in R.

We are still blocked in this case : for any countable α, the empty morphism from S into R is α-wide with non-empty α-kernel. We cannot even prove that the 2-kernel is empty. But as for the proof of p-divisibility, the whole difficulty (or maybe the falsity) seems to be in this case.

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