

Three min-max theorems concerning cyclic orders of strong digraphs.

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Abstract. In 1963, Tibor Gallai [9] asked whether every strongly connected directed graph D is spanned by α directed circuits, where α is the stability of D . We give a proof of this conjecture using a new structure for digraphs called *cyclic orders*. Three min-max theorems for cyclic orders are derived.

1 A conjecture of Gallai.

In this paper, circuits of length two are allowed. Since loops and multiple arcs play no role in this topic, we will simply assume that our digraphs are loopless and simple. A directed graph (digraph) is *strongly connected*, or simply *strong*, if for all vertices x, y , there exists a directed path from x to y . A *stable set* of a directed graph D is a subset of vertices which are not pairwise joined by arcs. The *stability* of D , denoted by $\alpha(D)$, is the number of vertices of a maximum stable set of D . It is well-known, by the Gallai-Milgram theorem [10] (see also [1] p. 234 and [3] p. 44), that D admits a vertex-partition into $\alpha(D)$ disjoint paths. We shall use in our proof a particular case of this result, known as Dilworth's theorem [8]: a partial order P admits a vertex-partition into $\alpha(P)$ chains (linear orders). Here $\alpha(P)$ is the size of a maximal antichain. In [9], Gallai raised the problem, when D is strongly connected, of spanning D by a union of circuits. Precisely, he made the following conjecture (also formulated in [1] p. 330, [2] and [3] p. 45):

Conjecture 1 *Every strong digraph with stability α is spanned by the union of α circuits.*

The case $\alpha = 1$ is Camion's theorem [6]: Every strong tournament has a hamilton circuit. The case $\alpha = 2$ is a corollary of a result of Chen and Manalastas [7] (see also Bondy [4]): Every strong digraph with stability two is spanned by two circuits intersecting each other on a (possibly empty) path. In [11] was proved the case $\alpha = 3$. In the next section of this paper, we will give a proof of Gallai's conjecture for every α .

2 Cyclic orders.

An *enumeration* E of a digraph D on the vertex set V with $|V| = n$ is a bijection of V into the n -gon $\{1, \dots, n\}$. If $E(x) = k$ for a vertex x of D we simply denote x by v_k and write $E = v_1, \dots, v_n$. An arc $v_i v_j$ of D is a *forward arc* for E if $i < j$, otherwise it is a *backward arc* for E . A *forward path* for E is a directed path of D which only contains forward arcs.

Given a circuit C of the digraph D , the *index of C for E* is the number of turns of $E(C)$ (winding number), that is to say, the number of backward arcs for E contained in C . We denote this index by $i_E(C)$.

Clearly, cyclic permutations of v_1, \dots, v_n leave the index of every circuit of D unchanged. This is also the case when one permute two consecutive vertices of v_1, \dots, v_n which are not joined by an arc of D .

We say that two enumerations of D are *equivalent* if we can obtain one from the other by applying a sequence of the two previous operations. Finally, a *cyclic order of D* is an equivalence class of enumerations. For a circuit C of D , by construction, the index of C in any enumeration of a fixed cyclic order \mathcal{C} is the same and we denote it by $i_{\mathcal{C}}(C)$. A circuit of D with index 1 in \mathcal{C} is called *simple in \mathcal{C}* .

A cyclic order \mathcal{C} is *coherent* if every arc of D belongs to a simple circuit, or equivalently, if for every enumeration E of \mathcal{C} and every backward arc $v_j v_i$ for E , there exists a forward path for E from v_i to v_j . We denote by $\text{cir}(D)$ the set of all directed circuits of D , and for a set of circuits \mathcal{S} we simply denote by $i_{\mathcal{C}}(\mathcal{S})$ the sum of the index of the circuits of \mathcal{S} .

Lemma 1 *Every strong digraph has a coherent cyclic order.*

Proof. Let us consider a cyclic order \mathcal{C} which is minimum with respect to $i_{\mathcal{C}}(\text{cir}(D))$. We suppose for contradiction that \mathcal{C} is not coherent. There exists an enumeration $E = v_1, \dots, v_n$ and a backward arc $a = v_j v_i$ which is not in a simple circuit. Assume moreover that E and a are chosen in order to minimize $j - i$. Let k be the largest integer $i \leq k < j$ such that there exists a forward path from v_i to v_k . Observe that v_k has no out-neighbour in $]v_k, v_j]$. If $k \neq i$, by the minimality of $j - i$, v_k has no in-neighbour in $]v_k, v_j]$. In particular the enumeration $E' = v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_j, v_k, v_{j+1}, \dots, v_n$ is equivalent to E , and contradicts the minimality of $j - i$. Thus $k = i$, and by the minimality of $j - i$, there is no in-neighbour of v_i in $]v_i, v_j[$. In particular the enumeration $E' = v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_i, v_j, \dots, v_n$ is equivalent to E . Observe now that in $E'' = v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_j, v_i, \dots, v_n$, every circuit C satisfies $i_{E''}(C) \leq i_{E'}(C)$, and the inequality is strict if the arc a belongs to C , a contradiction. \square

A direct corollary of Lemma 1 is that every strong tournament has a hamilton circuit, just consider for this any coherent cyclic order.

3 Three min-max theorems.

In this section, we will translate different invariants of graph theory in terms of cyclic order. From now on, \mathcal{C} will be a given coherent cyclic order of some strong digraph D .

The *cyclic stability* of \mathcal{C} is the maximum k for which there exists an enumeration v_1, \dots, v_n of \mathcal{C} such that $\{v_1, \dots, v_k\}$ is a stable set of D . We denote it by $\alpha(\mathcal{C})$, observe that we clearly have $\alpha(\mathcal{C}) \leq \alpha(D)$.

Lemma 2 *For a fixed enumeration of \mathcal{C} , let X be a subset of vertices of D such that there is no forward path for this enumeration between two distinct vertices of X . Then $|X| \leq \alpha(\mathcal{C})$.*

Proof. We consider an enumeration $E = v_1, \dots, v_n$ of \mathcal{C} such that there is no forward path between two distinct vertices of X , and chosen in such a way that $j - i$ is minimum, where v_i is the first element of X in the enumeration, and v_j is the last element of X in the enumeration. Suppose for contradiction that $X \neq \{v_i, \dots, v_j\}$. There exists $v_k \notin X$ for some $i < k < j$. There cannot exist both a forward path from $X \cap \{v_i, \dots, v_{k-1}\}$ to v_k and a forward path from v_k to $X \cap \{v_{k+1}, \dots, v_j\}$. Without loss of generality, we assume that there is no forward path from $X \cap \{v_i, \dots, v_{k-1}\}$ to v_k . Suppose moreover that v_k is chosen with minimum index k . Clearly, v_k has no in-neighbour in $\{v_i, \dots, v_{k-1}\}$, and since \mathcal{C} is coherent, v_k has no out-neighbour in $\{v_i, \dots, v_{k-1}\}$. Thus the enumeration $v_1, \dots, v_{i-1}, v_k, v_i, \dots, v_{k-1}, v_{k+1}, \dots, v_n$ belongs to \mathcal{C} , contradicting the minimality of $j - i$. Consequently, $X = \{v_i, \dots, v_j\}$, and there is no forward arcs, and then no backward arcs, between the vertices of X . Considering now the enumeration $v_i, \dots, v_n, v_1, \dots, v_{i-1}$, we conclude that $|X| \leq \alpha(\mathcal{C})$. \square

Let $P = x_1, \dots, x_k$ be a directed path, we call x_1 the *head* of P and x_k the *tail* of P . We denote the restriction of P to $\{x_i, \dots, x_j\}$ by $P[x_i, x_j]$. The *min vertex cover by circuits* is the minimum index of a family of circuits which spans V . We denote it by $vspan(\mathcal{C})$.

Theorem 1 $\alpha_c(\mathcal{C}) = vspan(\mathcal{C})$.

Proof. Denote by k the cyclic stability of \mathcal{C} and let $E = v_1, \dots, v_n$ be an enumeration of \mathcal{C} such that $S = \{v_1, \dots, v_k\}$ is a stable set of D . Clearly, if a circuit C contains q vertices of S , the index of C is at least q . In particular the inequality $i_C(S) \geq k$ is satisfied for every spanning set of circuits of D and thus, $vspan(\mathcal{C}) \geq \alpha_c(\mathcal{C})$. To prove that equality holds, we will provide a spanning set \mathcal{S} of circuits of D with index $\alpha_c(\mathcal{C})$. Consider the auxiliary acyclic digraph D' on vertex set $V \cup \{v'_1, \dots, v'_k\}$ which arc set consists of every forward arc of E and every arc $v_i v'_j$ for which $v_i v_j$ is an arc of D . We call T' the transitive closure of D' . Let us prove that the size of a maximal antichain in the partial order T' is exactly k . Consider such an antichain A , and set $A_1 := A \cap \{v_1, \dots, v_k\}$, $A_2 := A \cap \{v_{k+1}, \dots, v_n\}$ and $A_3 := A \cap \{v'_1, \dots, v'_k\}$. Since one can arbitrarily permute

the vertices of S in the enumeration E and still remain in \mathcal{C} , we may assume that $A_3 = \{v'_1, \dots, v'_j\}$ for some $0 \leq j \leq k$. Since every vertex is in a simple circuit, there is a directed path in D' from v_i to v'_i , and consequently we cannot both have $v_i \in A$ and $v'_i \in A$. Clearly, the enumeration $E' = v_{j+1}, \dots, v_n, v_1, \dots, v_j$ belongs to \mathcal{C} . By the fact that A is an antichain of T' , there is no forward path joining two elements of $(A \cap V) \cup \{v_1, \dots, v_j\}$ in E' , and thus, by Lemma 2, $|A| = |(A \cap V) \cup \{v_1, \dots, v_j\}| \leq k$. Observe also that $\{v_1, \dots, v_k\}$ are the sources of T' and $\{v'_1, \dots, v'_k\}$ are the sinks of T' , and both are maximal antichains of T' . We apply Dilworth's theorem in order to partition T' into k chains (thus starting in the set $\{v_1, \dots, v_k\}$ and ending in the set $\{v'_1, \dots, v'_k\}$), and by this, there exists a spanning set P_1, \dots, P_k of directed paths of D' with heads in $\{v_1, \dots, v_k\}$ and tails in $\{v'_1, \dots, v'_k\}$. We can assume without loss of generality that the head of P_i is exactly v_i , for all $i = 1, \dots, k$. Let us now denote by σ the permutation of $\{1, \dots, k\}$ such that $v'_{\sigma(i)}$ is the tail of P_i , for all i . Assume that among all spanning sets of paths, we have chosen P_1, \dots, P_k (with respective heads v_1, \dots, v_k) in such a way that the permutation σ has a maximum number of cycles. We claim that if (i_1, \dots, i_p) is a cycle of σ (meaning that $\sigma(i_j) = i_{j+1}$ and $\sigma(i_p) = i_1$), then the paths P_{i_1}, \dots, P_{i_p} are pairwise vertex-disjoint. If not, suppose that v is a common vertex of P_{i_l} and P_{i_m} , and replace P_{i_l} by $P_{i_l}[v_{i_l}, v] \cup P_{i_m}[v, v_{\sigma(i_m)}]$ and P_{i_m} by $P_{i_m}[v_{i_m}, v] \cup P_{i_l}[v, v_{\sigma(i_l)}]$. This is a contradiction to the maximality of the number of cycles of σ . Now, in the set of paths P_1, \dots, P_k , contract all the pairs $\{v_i, v'_i\}$, for $i = 1, \dots, k$. This gives a spanning set \mathcal{S} of circuits of D which satisfies $i_{\mathcal{C}}(\mathcal{S}) = k$. \square

Corollary 11 *Every strong digraph D is spanned by $\alpha(D)$ circuits.*

Proof. By Lemma 1, D has a coherent cyclic order \mathcal{C} . By Theorem 1, D is spanned by a set \mathcal{S} of circuits such that $|\mathcal{S}| \leq i_{\mathcal{C}}(\mathcal{S}) = \alpha(\mathcal{C}) \leq \alpha(D)$. \square

We now establish the arc-cover analogue of Theorem 1. Again, a minimax result holds.

We denote by $\beta(\mathcal{C})$ the maximum k for which there exists an enumeration of \mathcal{C} with k backward arcs. We call k the *maximal feedback arc set* of \mathcal{C} . The *min arc cover by circuits* is the minimum index of a family of circuits which spans the arc set of D . We denote it by $aspan(\mathcal{C})$.

Corollary 12 $\beta(\mathcal{C}) = aspan(\mathcal{C})$.

Proof. Apply Theorem 1 to the line digraph of D . \square

The last min-max theorem consists of a fractional version of a theorem of J.A. Bondy ([5]). Our proof is similar to the classical proof on the circular chromatic number in the non-oriented case, see X. Zhu ([12]) for a survey.

The *cyclic chromatic number* of \mathcal{C} , denoted by $\chi(\mathcal{C})$, is the minimum k for which there exists an enumeration $E = v_1, \dots, v_{i_1}, v_{i_1+1}, \dots, v_{i_2}, v_{i_2+1}, \dots, v_{i_k}$ of \mathcal{C} for which $v_{i_j+1}, \dots, v_{i_{j+1}}$ is a stable set for all $j = 0, \dots, k-1$ (with $i_0 := 0$).

Under the same hypothesis, the *circular chromatic number* of \mathcal{C} , denoted by $\chi_c(\mathcal{C})$ is the infimum of the numbers $r \geq 1$ for which \mathcal{C} admits an r -circular coloration. A mapping $f : V \rightarrow [0, r[$ is called an *r -circular coloration* if f verifies:

- 1) If x and y are linked in D , then $1 \leq |f(x) - f(y)| \leq r - 1$.
- 2) If $0 \leq f(v_1) \leq f(v_2) \leq \dots \leq f(v_n) < r$, then v_1, \dots, v_n must be an enumeration of \mathcal{C} . Such an enumeration is called *related to f* .

As usual, it is convenient to represent such an application as a mapping from V into a circle of the euclidean plane with circumference r . Condition 1) asserts then that two linked vertices have distance at least 1 on this circle. And by condition 2), the vertices of D are placed on the circle according to an enumeration of the cyclic order \mathcal{C} . By compactness of this representation, the infimum used in the definition of χ_c is a minimum, that is to say that there exists a $\chi_c(\mathcal{C})$ -circular coloration of \mathcal{C} .

Note that the enumeration given by 2) is possibly not unique. Indeed, two vertices of V may have the same image by f . In this case, these two vertices are not linked in D (because of 1)) and so, the two enumerations are equivalent. Moreover, two enumerations related to f have same sets of forward arcs and backward arcs.

Lemma 3 *For D a strong digraph and \mathcal{C} a coherent cyclic order of D , we have $\lceil \chi_c(\mathcal{C}) \rceil = \chi(\mathcal{C})$.*

The following Lemma gives a criterion to decide whether an r -circular coloration f is best possible or not. We define an auxiliary digraph D_f with vertex set V and arc set $\{xy \in E(D) : f(y) - f(x) = 1 \text{ or } f(x) - f(y) = r - 1\}$. Observe that the arcs xy of D_f with $f(y) - f(x) = 1$ (resp. $f(x) - f(y) = r - 1$) are forward (resp. backward) in any enumeration related to f .

Lemma 4 *If f is an r -circular coloration of \mathcal{C} with $r > 2$ for which D_f is an acyclic digraph, then we can provide a real number $r' < r$ such that \mathcal{C} admits an r' -circular coloration.*

Proof. First of all, if a vertex x of D has an out-neighbour y with $f(x) - f(y) = 1$, by property 1) of f and coherence of \mathcal{C} , the arc yx must be also in D , and similarly if x has an out-neighbour y with $f(y) - f(x) = r - 1$, the arc yx must be in D . So, a vertex x with in-degree 0 (resp. out-degree 0) in D_f has no neighbour z with $f(x) - f(z) = 1$ modulo r (resp. $f(z) - f(x) = 1$ modulo r). Then, if $E(D_f) = \emptyset$, it is easy to provide an r' -circular coloration f' of \mathcal{C} with $r' < r$. Just multiply f by a factor $1 - \epsilon$ with $\epsilon > 0$ and ϵ small enough.

Now, amongst the r -circular colorations f of \mathcal{C} for which D_f is acyclic, choose one with minimal number of arcs for D_f . Assume that $E(D_f) \neq \emptyset$. We can choose a vertex x of D_f with in-degree 0 and out-degree at least 1. Denote by y_1, \dots, y_p the out-neighbours of x in D_f , we have for all i , $f(y_i) = f(x) + 1$ modulo r . By definition of f , x has no neighbour z such that $f(x) - f(z) < 1$ or

$f(z) - f(x) > r - 1$ and, moreover, since x has in-degree 0 in D_f , by the previous remark, x has no neighbour z with $f(x) - f(z) = 1$ or $f(z) - f(x) = r - 1$. Observe that none of the y_i verifies this, because $r > 2$. So, we can provide an r -circular coloration f' derived from f just by changing the value of $f(x)$: choose $f'(x) = f(x) - \epsilon$ modulo r with $\epsilon > 0$ and such that no neighbour of x has an image by f in $[f'(x) - 1, f'(x)] \cup [r - 1 + f'(x), r[$. We check that $E(D_{f'}) = E(D_f) \setminus \{xy_i : i = 1, \dots, p\}$, which contradicts the choice of f . So, $E(D_f) = \emptyset$ and we provide an r' -circular coloration of \mathcal{C} with $r' < r$ as previously. \square

Finally, we can state a third min-max theorem about cyclic orders. For this, we define, for a fixed cyclic order \mathcal{C} , the *cyclic length* of a circuit C of D , denoted by $l_{\mathcal{C}}(C)$, as the number of vertices of C in D , divided by the index of C in \mathcal{C} . The *max circuit length* in \mathcal{C} is the maximum $l_{\mathcal{C}}(C)$ for a circuit C of D . We denote it by $l_{\mathcal{C}}(\mathcal{C})$.

Theorem 2 $\chi_c(\mathcal{C}) = l_{\mathcal{C}}(\mathcal{C})$.

Proof. Consider an r -circular coloration f of \mathcal{C} with $r = \chi_c(\mathcal{C})$ and $E = v_1, \dots, v_n$ an enumeration of \mathcal{C} related to f . For a circuit C in D , we compute the length l of the image of C by f :

$$l := \sum_{\substack{xy \in E(C) \\ xy \text{ forward in } E}} (f(y) - f(x)) + \sum_{\substack{xy \in E(C) \\ xy \text{ backward in } E}} (r + f(y) - f(x))$$

A straightforward simplification of the sum gives $l = r \cdot i_{\mathcal{C}}(C)$. Furthermore, condition 1) of the definition of f implies that $f(y) - f(x) \geq 1$ if $xy \in E(C)$ and xy is forward in E (i.e. $f(x) < f(y)$) and $r + f(y) - f(x) \geq 1$ if $xy \in E(C)$ and xy is backward in E (i.e. $f(y) < f(x)$). So, we have $c = r \cdot i_{\mathcal{C}}(C) \geq l(C)$, hence $r \geq l_{\mathcal{C}}(\mathcal{C})$, and the inequality $\chi_c(\mathcal{C}) \geq \max\{l_{\mathcal{C}}(C) : C \text{ circuit of } D\}$ holds.

To get the equality, we have to find a circuit C of D such that $l_{\mathcal{C}}(C) = r$. First of all, since \mathcal{C} is coherent, it has a circuit of index 1 and then, its cyclic length is greater or equal to 2. Thus, the previous inequality gives $r \geq 2$ and so, states the case $r = 2$. From now on, assume that $r > 2$, Lemma 4 asserts that there exists a circuit C in the digraph D_f . So, the inequalities provided in the direct sens of the proof are equalities: the image of every arc of C by f has length 1 if the arc is forward or $r - 1$ if the arc is backward. So, we have $l = l(C)$, and $l_{\mathcal{C}}(C) = l(C)/i_{\mathcal{C}}(C) = r$, which achieves the bound. \square

A corollary of Theorem 2 is an earlier result of J.A. Bondy, known since 1976. The *chromatic number* of a digraph D , denoted by $\chi(D)$, is the minimal number k such that the vertices of D admit a partition into k stable sets of D . Clearly, $\chi(D) \leq \chi_c(\mathcal{C})$.

Corollary 21 (Bondy [5]) *Every strong digraph D has a circuit with length at least $\chi(D)$.*

Proof. Consider a coherent cyclic order \mathcal{C} for D and apply Theorem 2 to provide a circuit C with $l_{\mathcal{C}}(C) = \chi_{\mathcal{C}}(C)$. Since by Lemma 3 $\lceil l_{\mathcal{C}}(C) \rceil = \lceil \chi_{\mathcal{C}}(C) \rceil = \chi(C)$, we get $\chi(D) \leq \chi(C) = \lceil l_{\mathcal{C}}(C) \rceil \leq l(C)$. \square

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