Realizing disjoint degree sequences of span two: a solvable discrete tomography problem

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Abstract

We consider the problem of coloring a grid using p colors with the requirement that each row and each column has a specific total number of entries of each color. Ryser [16], and independently Gale [8], obtained a necessary and sufficient condition for the existence of such a coloring when two colors are considered. This characterization yields a linear time algorithm for constructing the coloring when it exists. Chrobak and Dürr [5] showed that the problem is NP-hard when $p \geqslant 4$. The complexity of the case $p = 3$ remains open.

The *span* of a function is the difference between its maximum and its minimum values. In the case $p = 3$, this grid coloring problem is equivalent to find disjoint realizations of two degree sequences in a complete bipartite graph. This kind of question is well-studied when one of the degree sequence (or equivalently color) has span zero or one, see for instance [15], [12], [11], [13] and [3]. Chen and Shastri [4] showed a necessary and sufficient condition for the existence of a coloring when one color has span at most one. However, this condition fails when the span is two.

We introduce a new natural condition - the *saturation condition* - which we prove to be necessary and sufficient when one of the colors has span at most two. Our proof yields a polynomial time algorithm which either finds the coloring or exhibits a non existence certificate.

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1 Introduction

Discrete tomography is devoted to the reconstruction of a finite object from its projections. Since its introduction, discrete tomography has shown deep connections with some classical problems in combinatorics (see for instance [10]). One of these problems involves the coloring of a grid using p colors with the requirement that each row and each column has a specific total number of entries of each color. The case $p = 2$ is the well-known problem of reconstructing a matrix of zeros and ones given each row and column sum. This problem was widely studied by Ryser [16], who gave a necessary and sufficient condition for the existence of a solution. More recently, Gardner, Gritzmann and Prangenberg [9] studied the general case. They proved that this reconstruction problem is NP-hard when considering $p \geqslant 7$ colors. Later, Chrobak and Dürr [5] improved this result by showing that it remains NP-hard when $p \geqslant 4$. The complexity of the case $p = 3$ is still open.

There is a natural equivalence between a $|X| \times |Y|$ grid and the complete bipartite graph $K_{X,Y}$, where each cell of the grid corresponds to an edge of the graph. Hence, each color represents a subgraph. In addition, we can represent the color restrictions in the previous grid-coloring problem by p functions $d_0, \ldots, d_{p-1} : X \cup Y \to \mathbb{N}$, which assign to each row and column their respective color requirement. Each of these functions d_i represent the prescribed degree sequence of the subgraph corresponding to color i.

Formally, the *degree* of a vertex v of a graph $G = (V, E)$, written $d_G(v)$, is the number of edges incident to v in G. We denote $d_G : V \to \mathbb{N}$ the function which assigns to every vertex its degree in G. For a subset F of edges, we denote by d_F the degree function of the graph $H = (V, F)$. The function $d : V \to \mathbb{N}$ is *realizable* in G if there exists $F \subset E$ such that $d_F = d$. We refer to F as a *realization* of d in G. We say that d is *uniquely realizable in* G if it has only one realization.

Given $d_0, \ldots, d_{p-1} : V \to \mathbb{N}$, a $(d_0, \ldots, d_{p-1})$-*decomposition* of G is a partition $(F_0, \ldots, F_{p-1})$ of E such that F_i is a realization of d_i, for every $i = 0, \ldots, p-1$. Thus the discrete tomography problem can be restated as to find a $(d_0, \ldots, d_{p-1})-$decomposition of $K_{X,Y}$. In this context, the result by Chrobak and Dürr shows that deciding the existence of a $(d_0, d_1, d_2, d_3)-$decomposition of $K_{X,Y}$ is NP-hard and hence no good characterization can be expected. As for the tomography problem, the only open case is $p = 3$. From now on, we

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will mainly focus on $(d_0, d_1, d_2)-$decomposition of $K_{X,Y}$.

Being a decomposition necessarily means that $d_0 + d_1 + d_2 = d_G$, we then only need to find disjoint realizations F_1, F_2 of d_1, d_2 in G since the edge set $F_0 = E \setminus (F_1 \cup F_2)$ is indeed a realization of $d_0 = d_G - d_1 - d_2$. When d_1, d_2 have disjoint realization, they are *disjointly realizable in* G. Our main purpose in this paper is to find some necessary and sufficient conditions for d_1, d_2 to be disjointly realizable in G. First note that we need that both d_1 and d_2 are realizable in G. We also need that $d_1 + d_2 \leqslant d_G$, this condition being called the *degree condition in* G. Another natural necessary condition is that $d_1 + d_2$ is realizable in G.

The conditions cited above are easy to check, and can be deduced from a well-known characterization of realizable functions in bipartite graphs which is due to Ore [14]. We denote by $G = (X, Y, E)$ the bipartite graph with parts X and Y and edge set E. For $S \subset X, T \subset Y, F \subset E$, we write $\overline{S} = X - S$, $\overline{T} = Y - T$, $\overline{F} = E - F$ and $F(S, T)$ the set of edges in F with ends in S and T. In addition, for $d : X \cup Y \to \mathbb{N}$ we write $d(S) = \sum_{x \in S} d(x)$ and $d(T) = \sum_{y \in T} d(y)$.

Lemma 1 *Let $G = (X, Y, E)$ be a bipartite graph and $d : X \cup Y \to \mathbb{N}$. Then, d is realizable in G if and only if $d(X) = d(Y)$ and $d(S) \leqslant d(\overline{T}) + |E(S, T)|$, for each $S \subset X$ and $T \subset Y$.*

The following result is a straightforward corollary of Lemma 1 (see [2]). It will be one of the central tool of the proof of our main result.

Lemma 2 *Let $G = (X, Y, E)$ be a bipartite graph and let $d : X \cup Y \to \mathbb{N}$ be realizable in G. Suppose there exist a realization F_0 of d and $S \subset X, T \subset Y$ such that $F_0(S, T) = E(S, T)$ and $F_0(\overline{S}, \overline{T}) = \emptyset$. Then every realization F of d satisfies $F(S, T) = E(S, T)$ and $F(\overline{S}, \overline{T}) = \emptyset$.*

2 Functions with bounded span

For every fixed integer k, it was conjectured by Rao and Rao [15] that if $d, d_1 : X \to \mathbb{N}$ are realizable functions in K_X satisfying $d(x) = d_1(x) + k$ for all x in X, then there exists a realization of d containing a spanning $k-$regular subgraph. In [12], Kundu solved the conjecture, showing that if d, d_1 are realizable functions in K_X satisfying $d = d_1 + d_0$, where the span of d_0 at most one, then d can be realized by a graph containing a realization of d_0. An algorithmic method for finding these realizations was given by Kleitman and Wang in [11] and a very simple proof when $d_0(x) = 1$ for every x, was given by Lovász in [13]. In [3], Chen noticed that when considering the integer function $d_2 = |X| - 1 - d$, an even shorter proof could be obtained. Observe that d_2 is realizable in K_X by taking the complement in K_X of a realization of d. In addition, $d_1 + d_2 = |X| - 1 - d_0$ clearly has span at most one. Finally, Chen's approach of Kundu's result can be stated as follows.

Theorem 3 *Let $d_1, d_2 : X \to \mathbb{N}$ be such that the span of d_1+d_2 is at most one. Then d_1, d_2 are disjointly realizable in K_X if and only if d_1, d_2 are realizable in K_X and $d_1 + d_2 \leqslant |X| - 1$.*

Note that the last requirement simply says that the pair d_1, d_2 satisfies the degree condition in K_X. Later, Chen and Shastri [4] showed that the same argument used in the proof of Theorem 3 also works for the complete bipartite graph $K_{X,Y}$.

Theorem 4 *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ and assume that $(d_1 + d_2)|_Y$ has span at most one. Then d_1, d_2 are disjointly realizable in $K_{X,Y}$ if and only if d_1, d_2 are realizable and satisfy the degree condition in $K_{X,Y}$, that is, $(d_1 + d_2)_{|X} \leqslant |Y|$ and $(d_1 + d_2)_{|Y} \leqslant |X|$.*

The main idea of Chen's proof is the following lemma.

Lemma 5 *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ be realizable functions in $K_{X,Y}$. Assume that for given realizations F_1, F_2 of d_1, d_2, respectively, there exist $x, \overline{x} \in X$ and $y \in Y$ such that $xy \in F_1 \cap F_2$, $\overline{x}y \notin F_1 \cup F_2$ and $d_1(\overline{x})+d_2(\overline{x}) > d_1(x)+d_2(x)-2$. Then there exist realizations F_1', F_2' of d_1, d_2 such that $|F_1' \cap F_2'| < |F_1 \cap F_2|$.*

PROOF. Let $H = (X, Y, F_1 + F_2)$ be the bipartite graph with parts X and Y, and edge set $F_1 + F_2$, the disjoint union of F_1 and F_2. Observe that $F_1 \cap F_2$ is exactly the set of double edges of H. Since xy is a double edge of H and there is no edge in H between $\overline{x}$ and y, there exists a vertex $y' \in Y$ such that the number of edges between $\overline{x}$ and y' in H is strictly greater than the number of edges between x and y'. Without loss of generality, we assume that $\overline{x}y' \in F_1$ and $xy' \notin F_1$. Thus $F_1' = F_1 \cup \{\overline{x}y, xy'\} \setminus \{xy, \overline{x}y'\}$ is a realization of d_1 such that $|F_1' \cap F_2| < |F_1 \cap F_2|$. □

Note that when the span of $(d_1 + d_2)_{|Y}$ is at most one, the proof of Lemma 5 yields a polynomial time algorithm which starts with two realizations of d_1 and d_2 and computes two disjoint realizations. Hence Theorem 4 is a straightforward corollary of this lemma.

In [7] Costa *et al.* solved a particular case of disjoint realizations of two degree sequences in bipartite graphs. Furthermore, Costa *et al.* studied in [6] the problem when the functions d_1, d_2 are restricted to have values in $\{0, 2\}$, hence satisfying that $d_0 + d_1$ has span at most two. Unfortunately, when the function

$d_1 + d_2$ has span larger than one, the realizability of d_1, d_2 and the degree condition are not sufficient for d_1, d_2 to be disjointly realizable in $K_{X,Y}$, as shown in Figure 1. Observe that even asking for $d_1 + d_2$ to be realizable in $K_{X,Y}$ is still not a sufficient condition.

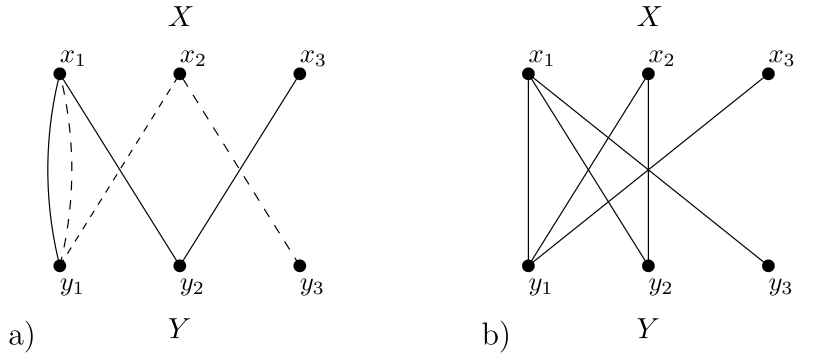

Fig. 1. a) Realizations of functions d_1 (continuous line) and d_2 (dashed line) in $K_{X,Y}$. b) Realization of $d_1 + d_2$ in $K_{X,Y}$. Observe that d_1, d_2 and $d_1 + d_2$ are uniquely realizablen in $K_{X,Y}$. In particular, x_1y_1 belongs to the unique realization of both d_1 and d_2. Hence d_1, d_2 are not disjointly realizable in $K_{X,Y}$. We remark that both $(d_1 + d_2)|_X$ and $(d_1 + d_2)|_Y$ have span exactly two.

Our goal is to provide a new condition which allows us to extend Theorem 4 when the span of both $(d_1 + d_2)|_X$ and $(d_1 + d_2)|_Y$ is at most two. In the following section we introduce this condition and we present our main result, namely Theorem 7. In section 4, we present the proof of Theorem 7.

3 The saturation condition

Let $G = (X, Y, E)$ be a bipartite graph and $d : X \cup Y \to \mathbb{N}$ be a realizable function in G. For $S \subset X$ and $T \subset Y$, we define $m_d(S, T)$ as the minimum number of edges joining S and T among all realizations of d. Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ be realizable functions in G. We say that d_1, d_2 *saturate* $E(S, T)$ if $m_{d_1}(S, T) + m_{d_2}(S, T) > |E(S, T)|$. Clearly, if there exists S and T such that d_1, d_2 saturate $E(S, T)$ then d_1, d_2 are not disjointly realizable in G. We say that d_1, d_2 satisfy the *saturation condition* in G if they do not saturate $E(S, T)$, for each $S \subset X$ and $T \subset Y$.

Theorem 6 *Let $d : X \cup Y \to \mathbb{N}$ be a realizable function in $G = (X, Y, E)$. For fixed $S \subset X$ and $T \subset Y$, $m_d(S, T)$ can be calculated in polynomial time.*

PROOF. We reduce this calculation to a minimum cost flow problem with lower and upper capacities in an auxiliary digraph D. Hence $m_d(S, T)$ is computable in polynomial time (see for instance [1]). We define $D = (V, A)$ as the digraph with vertex set $V = X \cup Y \cup \{s, t\}$ and arcs (s, x) for each $x \in X$, (y, t) for each $y \in Y$ and (x, y) for each $xy \in E$ with $x \in X$ and $y \in Y$.

Let $u, l : A \to \mathbb{N}$ be the lower and upper capacity functions given by $u(s, x) = l(s, x) = d(x)$ for each $x \in X$, $u(y, t) = l(y, t) = d(y)$ for each $y \in Y$, and $u = 1$, $l = 0$ otherwise. For $S \subset X$ and $T \subset Y$, we define a cost function $w = w(S, T) : A \to \{0, 1\}$ by $w(x, y) = 1$ if and only if (x, y) is an arc with both $x \in S$ and $y \in T$. The cost of an $(s, t)-$flow z is defined by $w(z) = \sum_{a \in A} z(a)w(a)$.

Given a realization F of d in G we define $z_F : A \to \mathbb{N}$ by $z_F(s, x) = d(x)$ for every $x \in X$, $z_F(y, t) = d(y)$ for $y \in Y$, and $z_F(x, y)$ with value 1 or 0 depending if xy belongs to F or $\overline{F}$. Note that $l \leqslant z_F \leqslant u$ and hence z_F is a feasible $(s, t)-$flow with value $|z_F| = d(X)$. Moreover, $w(z_F) = \sum_{a \in A} z_F(a)w(a) = |F(S, T)|$ and thus $w(z_F) \leqslant w(z_{F'})$ if and only if $|F(S, T)| \leqslant |F'(S, T)|$.

Furthermore, since l, u and w are integer valued functions the integrality theorem for minimum cost flows ensures the existence of an integer minimum cost $(s, t)-$flow z which is feasible with value $|z| = d(X)$. Define $F(z) = \{xy : (x, y) \in A \text{ with } x \in X,\ y \in Y \text{ and } z(x, y) > 0\}$. Note that z takes only values 0 or 1 for each $(x, y) \in A$ with $x \neq s$ or $y \neq t$, since for these arcs $0 \leqslant l \leqslant u \leqslant 1$. As the value of z is $d(X)$, $F(z)$ is a realization of d. By our previous observation and since z is a minimum cost $(s, t)-$flow, we have $m_d(S, T) = |F(z)(S, T)|$. □

Note that in Figure 1, the calculation for $S = \{x_1\}$ and $T = \{y_1\}$ gives $m_{d_1}(S, T) + m_{d_2}(S, T) = 2 > |S||T|$, thus d_1, d_2 do not satisfy the saturation condition in $K_{X,Y}$. Our main result is the following theorem.

Theorem 7 *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ such that both $(d_1 + d_2)|_X$ and $(d_1 + d_2)|_Y$ have span at most two. Then, d_1, d_2 are disjointly realizable in $K_{X,Y}$ if and only if d_1, d_2 are realizable and satisfy the saturation condition in $K_{X,Y}$.*

We will see that the proof of Theorem 7 yields a polynomial time algorithm which either finds two disjoint realizations of d_1 and d_2 or exhibits two sets S and T which violate the saturation condition. Thus, by Theorem 6, the pair (S, T) is a non existence certificate which can be checked in polynomial time.

It would be tempting to propose the realizability and the saturation condition as a necessary and sufficient condition for the general case of two functions in $K_{X,Y}$. We do not have any example of d_1, d_2 which satisfy these conditions and are not disjointly realizable. We let this as an open question. A problem of independent interest would be to polynomially check if two realizable functions d_1, d_2 satisfy indeed the saturation condition. Let us now motivate a little bit more the introduction of the saturation condition by presenting some particular cases in which it is indeed the required condition.

Theorem 8 illustrates how the saturation condition can provide in some cases a necessary and sufficient condition. The proof follows easily from Lemma 1.

Theorem 8 *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ be realizable in $G = (X, Y, E)$ and assume that d_1 is uniquely realizable. If d_1, d_2 satisfy the saturation condition in G then they are disjointly realizable.*

PROOF. For the sake of contradiction, assume that d_1, d_2 are not disjointly realizable in G and let $F_1 \subset E$ be the unique realization of d_1. Clearly, d_2 is not realizable in the graph $H = (X, Y, \overline{F_1})$. Since $d_2(X) = d_2(Y)$, by Lemma 1, there exist $S \subset X$ and $T \subset Y$ such that $d_2(S) > d_2(\overline{T}) + |\overline{F_1}(S, T)|$.

We consider a realization F_2 of d_2 in G such that $m_{d_2}(S, T) = |F_2(S, T)|$. Then, $d_2(S) = |F_2(S, T)| + |F_2(S, \overline{T})| \leqslant m_{d_2}(S, T) + d_2(\overline{T})$. From the two previous inequalities we obtain $m_{d_2}(S, T) \geqslant d_2(S) - d_2(\overline{T}) > |\overline{F_1}(S, T)|$. But d_1 is uniquely realizable and hence $m_{d_1}(S, T) = |F_1(S, T)|$. Finally, we obtain $m_{d_1}(S, T) + m_{d_2}(S, T) > |F_1(S, T)| + |\overline{F_1}(S, T)| = |E(S, T)|$. □

Figure 1 shows that only asking for realizability is not enough, even when d_1, d_2 and $d_1 + d_2$ are all uniquely realizable.

In Theorem 9, we show that the realizability of $d_1 + d_2$ easily follows from the realizability of d_1, d_2 when the saturation condition holds.

Theorem 9 *Let d_1 and d_2 be realizable in a bipartite graph G. If d_1, d_2 satisfy the saturation condition in G, then $d_1 + d_2$ is realizable in G. In particular d_1, d_2 do satisfy the degree condition.*

PROOF. The realizability of d_1, d_2 in G gives $(d_1+d_2)(X) = d_1(X)+d_2(X) = d_1(Y)+d_2(Y) = (d_1+d_2)(Y)$. Let F_i be a realization of d_i in G, where $i = 1, 2$. It is clear that d_i is realizable in $G_i = (X, Y, F_i)$. Thus Lemma 1 shows that $d_i(S) \leqslant d_i(\overline{T}) + |F_i(S, T)|$ for each $S \subset X$ and $T \subset Y$. Since this holds for every realization of d_i we obtain $d_i(S) \leqslant d_i(\overline{T}) + m_{d_i}(S, T)$. Thus,

$$\begin{aligned}(d_1 + d_2)(S) &= d_1(S) + d_2(S) \\ &\leqslant d_1(\overline{T}) + d_2(\overline{T}) + m_{d_1}(S, T) + m_{d_2}(S, T) \\ &\leqslant (d_1 + d_2)(\overline{T}) + |E(S, T)|\end{aligned}$$

for each $S \subset X$ and $T \subset Y$. Again by Lemma 1, $d_1 + d_2$ is realizable in G. The last remark is straightforward. □

4 The proof of Theorem 7

Consider $d_1, d_2 : X \cup Y \to \mathbb{N}$ such that both $(d_1 + d_2)_{|X}$ and $(d_1 + d_2)_{|Y}$ have span at most two. From the discussion in Section 3, the realizability and the saturation condition are necessary for d_1, d_2 to be disjointly realizable in $K_{X,Y} = (X, Y, E)$. Conversely, assume that d_1 and d_2 are both realizable in $K_{X,Y}$ but not disjointly realizable. We will prove that there exist two sets $S \subset X$ and $T \subset Y$ such that d_1, d_2 saturate $E(S, T)$. As we have seen in Section 3, if d_1, d_2 do not satisfy the degree condition then the saturation condition does not hold. Thus, we can assume that $\max(d_1 + d_2)_{|X} \leqslant |Y|$ and $\max(d_1 + d_2)_{|Y} \leqslant |X|$.

Let F_1, F_2 be some respective realizations of d_1, d_2, chosen in such a way that $|F_1 \cap F_2|$ is as small as possible. Such a pair of realizations is called *minimal.*. Since d_1, d_2 are not disjointly realizable, the set $F_1 \cap F_2$ is not empty. We consider the bipartite graph $H = (X, Y, F_1 + F_2)$, where $F_1 + F_2$ denotes the disjoint union of F_1 and F_2. For $x \in X$ and $y \in Y$, we will refer to xy as an $i-$*edge* in H if xy belongs to F_i, where $i = 1, 2$. We say that xy is a *double edge* if it is both a 1−edge and a 2−edge of H. Note that F_1 and F_2 are disjoint if and only if H is a graph without double edges. We denote $N_H(z)$ the set of neighbors of z in H. In addition, we write $N^i_H(z)$ the set of neighbors of z in the graph (X, Y, F_i), for $i = 1, 2$. Note that $N_H(z) = N^1_H(z) \cup N^2_H(z)$, while $|N_H(z)| = |N^1_H(z)| + |N^2_H(z)| - |N^1_H(z) \cap N^2_H(z)|$. Finally, we write $\overline{H} = (X, Y, \overline{F_1 \cup F_2})$, where $\overline{F_1 \cup F_2}$ is the set of non edges of H, and $N_{\overline{H}}(z)$ accordingly.

Let xy be a double edge of H, where $x \in X$ and $y \in Y$. Observe that $d_H(x) = d_1(x) + d_2(x) \leqslant |Y|$ and $d_H(y) = d_1(y) + d_2(y) \leqslant |X|$. Then $|N_H(x)| < d_H(x) \leqslant |Y|$ and $|N_H(y)| < d_H(y) \leqslant |X|$ and hence $N_{\overline{H}}(x) \neq \emptyset$ and $N_{\overline{H}}(y) \neq \emptyset$. We denote $S_0 = N_{\overline{H}}(y)$ and $T_0 = N_{\overline{H}}(x)$. From now on, $\overline{x}$ and $\overline{y}$ will be fixed vertices in S_0 and T_0, respectively.

Claim 10 *$\overline{x}\,\overline{y}$ is a non-edge of H.*

PROOF. Assume for contradiction that $\overline{x}\,\overline{y} \in F_1$ and define $F_1' = F_1 \cup \{x\overline{y}, \overline{x}y\} \setminus \{xy, \overline{x}\,\overline{y}\}$. Note that F_1' is a realization of d_1. If $\overline{x}\,\overline{y}$ is double in H then $F_1' \cap F_2 = F_1 \cap F_2 \setminus \{xy, \overline{x}\,\overline{y}\}$. Otherwise $\overline{x}\,\overline{y}$ is a simple edge in H and then $F_1' \cap F_2 = F_1 \cap F_2 \setminus \{xy\}$. In both cases we obtain that $|F_1' \cap F_2| < |F_1 \cap F_2|$. This contradicts the minimality of F_1, F_2 and hence $\overline{x}\,\overline{y}$ is not a 1−edge. The same argument shows that $\overline{x}\,\overline{y}$ does not belong to F_2 either. □

Note that Claim 10 shows that $N_H(\overline{x}) \subset N_H(x) \setminus \{y\}$. In Claims 11, 12 and 14 we only include the proof for x and $\overline{x}$. The result for y and $\overline{y}$ can be proved analogously.

Claim 11 (Degree property) *We both have*

$$\max(d_1+d_2)_{|X} = d_1(x)+d_2(x) = d_1(\overline{x})+d_2(\overline{x})+2 = \min(d_1+d_2)_{|X}+2$$

$$\max(d_1+d_2)_{|Y} = d_1(y)+d_2(y) = d_1(\overline{y})+d_2(\overline{y})+2 = \min(d_1+d_2)_{|Y}+2$$

PROOF. Since $(d_1+d_2)|_X$ has span at most two we have $\max(d_1+d_2)_{|X} \leqslant \min(d_1+d_2)_{|X}+2$. By Lemma 5, $d_1(x)+d_2(x) \geqslant d_1(\overline{x})+d_2(\overline{x})+2$; otherwise there exist realizations F_1', F_2' with $|F_1' \cap F_2'| < |F_1 \cap F_2|$, which contradicts that F_1, F_2 is a minimal pair. Thus $\max(d_1+d_2)_{|X} \geqslant d_1(x)+d_2(x) \geqslant d_1(\overline{x})+d_2(\overline{x})+2 \geqslant \min(d_1+d_2)_{|X}+2$. Hence all the inequalities are equalities. □

The proof of the degree property shows that both $(d_1+d_2)|_X$ and $(d_1+d_2)|_Y$ have span exactly two. We remark that this follows easily from Theorem 4 and our assumption that d_1, d_2 are not disjointly realizable. Note that the degree property shows that every vertex incident to a double edge in a minimal pair is of maximum degree in its part (X or Y). It also shows that no vertex in $N_{\overline{H}}(x) \cup N_{\overline{H}}(y)$ is of maximum degree in its respective part. Hence, the following property holds.

Claim 12 *$\overline{x}$ (resp. $\overline{y}$) does not have incident double edges in H. Therefore, $|N_H(\overline{x})| = d_1(x)+d_2(x)-2$ and $|N_H(\overline{y})| = d_1(y)+d_2(y)-2$.*

Claim 13 *$N_H(\overline{x}) = N_H(x) \setminus \{y\}$ and $N_H(\overline{y}) = N_H(y) \setminus \{x\}$.*

PROOF. By Claims 10 and 12 we have $N_H(\overline{x}) \subset N_H(x) \setminus \{y\}$ and $|N_H(\overline{x})| = d_1(x)+d_2(x)-2$. Since $|N_H(x)| = |N_H^1(x)| + |N_H^2(x)| - |N_H^1(x) \cap N_H^2(x)| \leqslant d_1(x)+d_2(x)-1$ we obtain $|N_H(x) \setminus \{y\}| \leqslant d_1(x)+d_2(x)-2 = |N_H(\overline{x})|$. Therefore, $N_H(\overline{x}) = N_H(x) \setminus \{y\}$. □

Since $N_{\overline{H}}(\overline{x}) = Y \setminus N_H(\overline{x})$ and $N_{\overline{H}}(x) = Y \setminus N_H(x)$ we get

Claim 14 *$N_{\overline{H}}(\overline{x}) = T_0 \cup \{y\}$ and $N_{\overline{H}}(\overline{y}) = S_0 \cup \{x\}$.*

Claim 15 *The set of doubles edges of H forms a matching.*

PROOF. From Claim 13 and the degree property we have $|N_H(x) \setminus \{y\}| = d_1(x)+d_2(x)-2$. Hence, $|N_H(x)| = d_1(x)+d_2(x)-1$ which implies that exactly one double edge is incident to x. □

Claim 16 *If $x'y'$ is a non edge where $x' \in N_H(y)$ and $y' \in N_H(x)$, then $x'\overline{y}$ is an $i-$edge and $\overline{x}y'$ is a $j-$edge with $i \neq j$.*

PROOF. By Claim 13, $x' \in N_H(\overline{y})$ and $y' \in N_H(\overline{x})$. Assume for contradiction that $x'\overline{y}, \overline{x}y'$ are $i-$edges. Then $F_i' = F_i \cup \{x\overline{y}, \overline{x}y, x'y'\} \setminus \{xy, \overline{x}y', x'\overline{y}\}$ and $F_j' = F_j$ where $j \neq i$, are realizations of d_1, d_2 satisfying $|F_1' \cap F_2'| < |F_1 \cap F_2|$. □

Note that $\{N_H^i(y) \cap N_H^j(\overline{y})\}_{i,j=1,2}$ is a partition of $\overline{S_0 \cup \{x\}}$ into four (possibly empty) sets. Similarly, $\{N_H^i(x) \cap N_H^j(\overline{x})\}_{i,j=1,2}$ is a partition of $\overline{T_0 \cup \{y\}}$.

Claim 17 *Let $x'y'$ be a non-edge of H, where $x' \notin S_0$ and $y' \notin T_0$. Then $x'y, x'\overline{y}$ are $i-$edges and $xy', \overline{x}y'$ are $j-$edges, for $i \neq j$.*

PROOF. Since $x' \notin S_0$ and $x'y'$ is a non-edge, then $y' \neq y$. Similarly, we have that $x' \neq x$. Hence x' belongs to $N_H^{i_1}(y) \cap N_H^{j_1}(\overline{y})$ and y' to $N_H^{i_2}(x) \cap N_H^{j_2}(\overline{x})$, with i_1, j_1, i_2 and j_2 in $\{1, 2\}$. Without loss of generality, we assume that $j_2 = 2$. By Claim 16, $j_1 = 1$.

For the sake of contradiction assume that $i_1 = 2$. We define $F_1' = F_1 \cup \{x\overline{y}, x'y\} \setminus \{xy, x'\overline{y}\}$ and $F_2' = F_2 \cup \{\overline{x}y, x'y'\} \setminus \{\overline{x}y', x'y\}$. Then F_1' and F_2' are realizations of d_1 and d_2, respectively, which satisfy $|F_1' \cap F_2'| < |F_1 \cap F_2|$. This contradicts the choice of F_1, F_2 and hence $i_1 = 1$. A symmetric argument shows that $i_2 = j_2 = 2$. □

Claim 18 *Let $x'y' \neq xy$ be a double edge in H. Then $x'y, x'\overline{y}$ are $i-$edges and $xy', \overline{x}y'$ are $j-$ edges of H, for $i \neq j$.*

PROOF. Note that $x' \neq x$ by Claim 15. Similarly, Claim 12 shows that $x' \notin S_0$. Consider $\overline{y}' \in N_{\overline{H}}(x')$. We will show that $\overline{y}' \notin T_0$. Assume by contradiction that $x\overline{y}'$ is a non-edge of H. Then Claim 14 shows that $N_{\overline{H}}(\overline{y}') = S_0 \cup \{x\}$, which is impossible since $x' \notin S_0$. Analogously, we can see that $y' \notin T_0$ and $\overline{x}' \notin S_0$ for each $\overline{x}' \in N_{\overline{H}}(y')$. Note that $\overline{x}'\overline{y}'$ is a non-edge of H by Claim 10.

Applying Claim 17 to the non-edge $x'\overline{y}'$, we obtain that $x'y, x'\overline{y}$ are $i-$edges and $x\overline{y}', \overline{x}\overline{y}'$ are $j-$edges of H, where i, j are distinct indices in $\{1, 2\}$. Applying again Claim 17 to the non-edge $\overline{x}'\overline{y}'$, we deduce that $\overline{x}'y, \overline{x}'\overline{y}$ are $i-$edges. Applying finally Claim 17 to the non-edge $\overline{x}'y'$ shows that $xy', \overline{x}y'$ are $j-$edges. □

We remark that for $x' \notin S_0 \cup \{x\} = N_{\overline{H}}(\overline{y})$ and $y' \notin T_0 \cup \{y\} = N_{\overline{H}}(\overline{x})$, all $x'y, x'\overline{y}, xy'$ and $\overline{x}y'$ are simple edges of H. Similarly, Claim 17 and 18 show that if $x'y'$ is either a non-edge or a double edge of H then $x' \in N_H^i(y) \cap N_H^i(\overline{y})$ and $y' \in N_H^j(x) \cap N_H^j(\overline{x})$, where $i \neq j$.

We define three operations which transform the realizations F_1, F_2 into F_1', F_2':

a) Let xy' be a simple edge with y' in $N_H^j(x) \cap N_H^i(\overline{x})$, where $i \neq j$. An $\overline{\mathbf{x}}$**−switch** is the operation which replaces F_i by $F_i' := F_i \cup \{xy', \overline{x}y\} \setminus \{xy, \overline{x}y'\}$ and leaves F_j unchanged, i.e. $F_j' := F_j$.
b) Let $x'y$ be a simple edge with x' in $N_H^j(y) \cap N_H^i(\overline{y})$, where $i \neq j$. A $\overline{\mathbf{y}}$**−switch** is the operation which replaces F_i by $F_i' := F_i \cup \{x'y, x\overline{y}\} \setminus \{xy, x'\overline{y}\}$ and leaves F_j unchanged, i.e. $F_j' := F_j$.
c) Let $x'y'$ be a simple edge with $x' \in N_H^i(\overline{y})$, $y' \in N_H^i(\overline{x})$ and $x'y' \in F_j$, where $i \neq j$. An $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$**−switch** is the operation which replaces F_i by $F_i' := F_i \cup \{x\overline{y}, \overline{x}y, x'y'\} \setminus \{xy, \overline{x}y', x'\overline{y}\}$ and leaves F_j unchanged, i.e. $F_j' := F_j$.

We refer to these operations as *elementary switches*. We also say that $x'y'$ *is reached from* xy *by an elementary switch*.

Note that the sets F_1' and F_2' defined above are realizations of d_1 and d_2, respectively. In addition, $x'y'$ is a double edge, xy is a simple edge and both $\overline{x}y'$ and $x'\overline{y}$ are non-edges of $H' = (X, Y, F_1' + F_2')$. Thus, the total number of doubles edges in H' and H is the same. That is, an elementary switch in H "changes the position" of a double edge and hence it preserves the minimality of the pair of realizations. Consequently F_1', F_2' is also a minimal pair.

We say that $x'y'$ *is reached from* xy if either $x'y' = xy$ or there exists vertices $x_0, x_1 \ldots, x_t \in X$, $y_0, y_1 \ldots, y_t \in Y$ and realizations F_1^k, F_2^k of d_1, d_2, respectively, for $k = 1, \ldots, t$, satisfying $x_0y_0 = xy$, $x_ty_t = x'y'$ and $F_1^0 = F_1, F_2^0 = F_2$ and such that x_ky_k is reached from $x_{k-1}y_{k-1}$ by an elementary switch in $H_{k-1} = (X, Y, F_1^{k-1} + F_2^{k-1})$. Note that F_1^k, F_2^k is a minimal pair for every k. We will refer to the realizations $F_1' = F_1^t$ and $F_2' = F_2^t$ as a *minimal pair associated to* $x'y'$.

Claim 19 *Let $x'y'$ be reached from xy and F_1', F_2' be a minimal pair associated to $x'y'$. Then $x'y'$ is a double edge and $\overline{x}y', x'\overline{y}$ are non-edges of $H' = (X, Y, F_1' + F_2')$.*

PROOF. This follows from the definition of the elementary switches (see Figure 2). □

Let S be the set of vertices x' in X for which there exists y' in Y such that $x'y'$ is reached from xy. In the same way, let T be the set of vertices y in Y

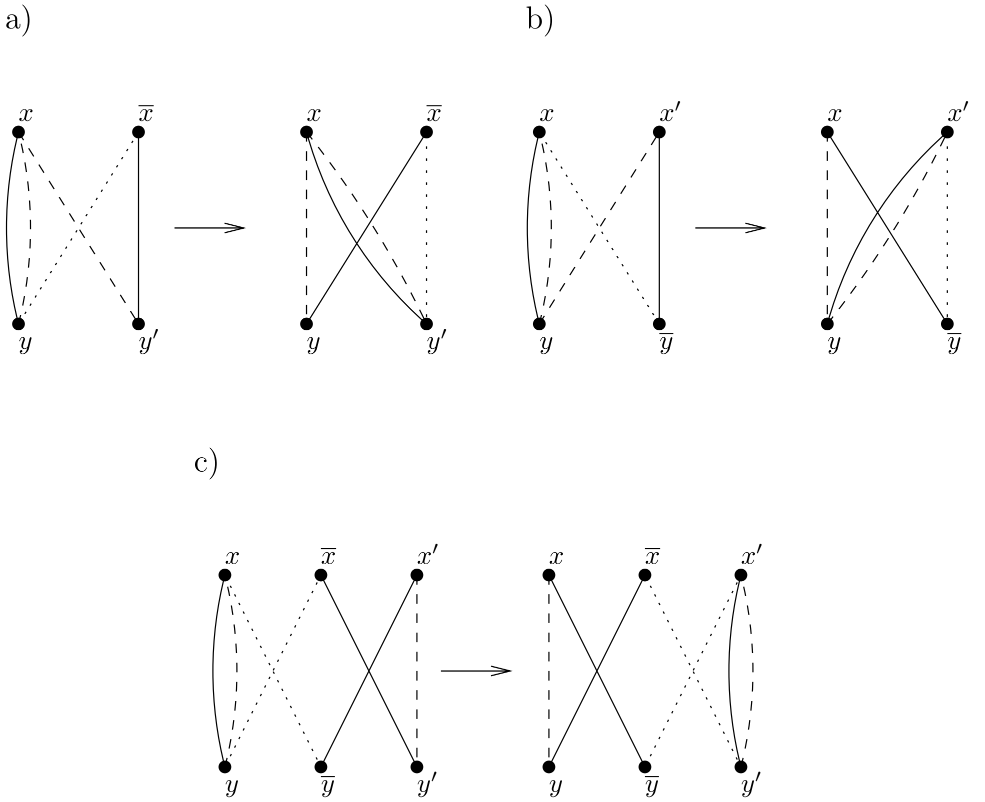

Fig. 2. The three elementary switches. We have represented 1, 2 and non-edges by continuous, dashed and dotted lines, respectively. a) $y' \in N^2_H(x) \cap N^1_H(\overline{x})$ and then xy' is reached from xy by an $\overline{x}-$switch; b) $x' \in N^2_H(y) \cap N^1_H(\overline{y})$ and then $x'y$ is reached from xy by a $\overline{y}-$switch; c) $x' \in N^1_H(\overline{y})$, $y' \in N^1_H(\overline{x})$ and $x'y' \in F_2$. Thus $x'y'$ is reached from xy by an $(\overline{x}, \overline{y})-$switch. For simplicity we have not drawn the non-edge $\overline{x}\,\overline{y}$ and the simples edges xy' and $x'y$.

for which there exists x' in X such that $x'y'$ is reached from xy.

Claim 20 $S \cap S_0 = T \cap T_0 = \emptyset$

PROOF. By the degree condition, no vertex of S_0 has maximum degree in X and no vertex of T_0 has maximum degree in Y. Moreover, by Claim 19 and the degree condition, every vertex of S has maximum degree in X and every vertex of T has maximum degree in Y. □

Claim 21 *Let F'_1, F'_2 be a minimal pair associated to $x'y'$. Then F'_i can only differ from F_i on the edges between $S \cup \{\overline{x}\}$ and $T \cup \{\overline{y}\}$, for $i = 1, 2$.*

PROOF. Consider $x_0, x_1 \ldots, x_t \in X$ and $y_0, y_1 \ldots, y_t \in Y$ as above. Note that F^k_i can only differ from F_i on the edges connecting $\{\overline{x}, x_0 \ldots, x_k\}$ and

$\{\overline{y}, y_0, \ldots, y_k\}$, for every k. Then the result follows by noting that $x_i \in S$ and $y_i \in T$, for each $i = 0, \ldots, t$. □

From Claim 21 we know that for each $u \notin T \cup \{\overline{y}\}$, an edge incident to u belongs to H if and only if it belongs to H'. Similarly, for each $v \notin S \cup \{\overline{x}\}$, an edge incident to v belongs to H if and only if it belongs to H'.

Claim 22 *There are no edges of H between S and $T_0 \setminus \{\overline{y}\}$. Similarly, there are no edges of H between T and $S_0 \setminus \{\overline{x}\}$.*

PROOF. Let $x' \in S$ and y' be such that $x'y'$ is reached from xy. Let F_1', F_2' be a minimal pair associated to $x'y'$ and define $H' = (X, Y, F_1' + F_2')$. For sake of contradiction, let us assume that there exists a vertex u of $T_0 \setminus \{\overline{y}\}$ such that $x'u$ is an edge of H. Since $u \notin T \cup \{\overline{y}\}$ we obtain by Claim 21 that $x'u$ is an edge of H'. From Claim 13 applied to H' we get $N_{H'}(\overline{x}) = N_{H'}(x') \setminus \{y'\}$. Hence, $u\overline{x}$ is an edge of H', since $y' \in T$. But then, $u\overline{x}$ is an edge of H which contradicts the fact that $u \in T_0$. The other part is proved analogously. □

Claim 23 *We have $S = \{x\}$ or $T_0 = \{\overline{y}\}$.*

PROOF. For the sake of contradiction, let us assume that there are $x' \neq x$ in S and $u \neq \overline{y}$ in T_0. By Claim 22, $x'u$ is a non-edge of H. By definition, xu is also a non-edge of H. Then Claim 14 shows that $N_{\overline{H}}(u) \setminus \{x\} = S_0$. We obtain that $x' \in S_0$ and hence $x' \in S \cap S_0$. This contradicts Claim 20. □

We define $S_i = N_H^i(\overline{y}) \setminus S$ and $T_i = N_H^i(\overline{x}) \setminus T$, for $i = 1, 2$. Recall that $d_1(x) + d_2(x) = |N_H^1(x)| + |N_H^2(x)| = |N_H(x)| + 1$. Thus, $\max(d_1 + d_2)_{|X} = |Y| - |T_0| + 1$. Similarly, $\max(d_1 + d_2)_{|Y} = |X| - |S_0| + 1$.

Claim 24 *For $i = 1, 2$, S_i and T_0 (resp. T_i and S_0) are completely connected by $i-$edges.*

PROOF. For the vertex $\overline{y}$ of T_0, the result follows from the definition of S_1 and S_2. Consider now $u \neq \overline{y}$ in T_0 and x_i in S_i. Note that in this case $|T_0| > 1$ and hence $\max(d_1 + d_2)_{|X} = |Y| - |T_0| + 1 < |Y|$. Thus x_i is incident to a non-edge $x_i y'$ of H. Assume that xy' is a non-edge of H. Then $N_{\overline{H}}(y') = S_0 \cup \{x\}$ by Claim 14. Since $x_i \neq x$, we have $x_i \in S_0$. This contradicts that $S_0 \cap S_i = \emptyset$ and hence $y' \notin T_0$.

Thus, we can apply Claim 17 to the non-edge $x_i y'$. Since $x_i\overline{y}$ is an $i-$edge, we have that $x_i y$ is also an $i-$edge. Applying again Claim 17 with $u \in T_0$ in place of $\overline{y}$, we obtain that both x_iu and x_iy are $i-$edges, which concludes the proof. □

Claim 25 *For distinct i, j in $\{1, 2\}$, S_i and $\overline{T_j}$ (resp. T_i and $\overline{S_j}$) are completely connected with $i-$edges and not connected with $j-$edges of H (see Fig. 3).*

PROOF. We will only prove that S_1 and $\overline{T_2}$ are completely connected with simple $1-$edges in H. The other cases can be obtained by a similar argument.

Consider $x_1 \in S_1$ and $y' \in \overline{T_2}$. By Claim 20 and the definition of T_1, $\{T, T_0, T_1\}$ is a partition of $\overline{T_2}$ and then we consider three cases.

- If $y' \in T_0$, Claim 24 shows that x_1y' is a $1-$edge. Then, by Claim 12, x_1y' is not a $2-$edge of H.
- If $y' \in T_1$, Claim 17 asserts that x_1y' is not a non-edge of H since $x_1\overline{y}$ and $\overline{x}y'$ are both 1-edges. Moreover, Claim 18 asserts that x_1y' is neither a double edge of H. By contradiction assume that x_1y' is a $2-$edge. But then x_1y' is reached from xy by an $(\overline{x}, \overline{y})-$switch in H. This contradicts the fact that $x_1 \notin S$.
- If $y' \in T$, consider $x' \in S$ such that $x'y'$ is reached from xy. Let F_1', F_2' be a minimal pair associated to $x'y'$ and define $H' = (X, Y, F_1' + F_2')$. Since $x'y'$ is a double edge of H', Claim 15 implies that x_1y' is not a double edge of H'. Then by Claim 19, $x'\overline{y}$ is a non-edge of H'. In addition, H' can only differ from H on the edges with ends in $S \cup \{\overline{x}\}$ and $T \cup \{\overline{y}\}$. Hence $x_1\overline{y}$ is a $1-$edge of H'. Since $N_{H'}(\overline{y}) \subseteq N_{H'}(y')$, we obtain that x_1y' is an edge in H'. Assume that x_1y' is a $2-$edge in H'. Then x_1 belongs to $N^2_{H'}(y') \cap N^1_{H'}(\overline{y})$ and hence x_1y' is reached from $x'y'$ by a $\overline{y}-$switch in H'. This contradicts that $x_1 \notin S$. Then x_1y' is a $1-$edge of H'. Finally, Claim 19 shows that x_1y' is also a $1-$edge of H.

□

Claim 26 *Let F_1',F_2' be realizations of d_1,d_2, respectively. Then for distinct i, j in $\{1, 2\}$, S_i and $\overline{T_j}$ (resp. $\overline{S_j}$ and T_i) are completely connected with $i-$edges and not connecting with $j-$edges of $H' = (X, Y, F_1' + F_2')$.*

PROOF. By Claim 25, $F_i(S_i, \overline{T_j}) = E(S_i, \overline{T_j})$ and $F_i(\overline{S_i}, T_j) = \emptyset$. Then Lemma 2 implies that $F_i'(S_i, \overline{T_j}) = E(S_i, \overline{T_j})$ and $F_i'(\overline{S_i}, T_j) = \emptyset$. The other part is analogous. □

Claim 27 *Let F_1',F_2' be realizations of d_1,d_2, respectively. Then the total number of edges in $H' = (X, Y, F_1' + F_2')$ between S and T_0 is less than $|S|$.*

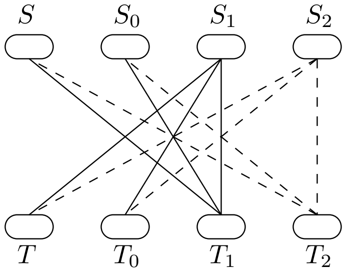

Fig. 3. The edges of F_1 and F_2 connecting the different subsets of X and Y. A continuous line (resp. a dashed line) between two sets means that all the edges connecting them are 1−edges of H (resp. 2−edges). Note that all the edges connecting S_1 and $\overline{T_2} = T \cup T_0 \cup T_1$ are 1−edges and there is no 1−edge connecting $\overline{S_1} = S \cup S_0 \cup S_2$ and T_2 (all of them are 2−edges of H).

PROOF. It is clear that $|F_i'(S, T_0)| \leqslant d_i(T_0) - |F_i'(S_1 \cup S_2, T_0)|$, for $i = 1, 2$. By Claim 26, $|F_i'(S_1 \cup S_2, T_0)| = |S_i||T_0|$ and hence $|F_i'(S, T_0)| \leqslant d_i(T_0) - |S_i||T_0|$. By the degree property, $d_1(\overline{y}') + d_2(\overline{y}') = \min(d_1 + d_2)_{|Y} = |X| - |S_0| - 1$ for each $\overline{y}' \in T_0$. Then

$$\begin{aligned} |F_1'(S, T_0)| + |F_2'(S, T_0)| &\leqslant d_1(T_0) + d_2(T_0) - |S_1||T_0| - |S_2||T_0| \\ &= (|X| - |S_0| - |S_1| - |S_2| - 1)\,|T_0| \\ &= (|S| - 1)|T_0|. \end{aligned}$$

Thus, it is sufficient to show that $(|S| - 1)|T_0| < |S|$. But this is straightforward since $|S| = 1$ or $|T_0| = 1$ by Claim 23. □

We are now ready to prove that S and T violate the saturation condition. Let F_1' and F_2' be realizations of d_1 and d_2, respectively. Note that by Claim 26, $|F_i'(S, T_1 \cup T_2)| = |S||T_i|$. Then $|F_i'(S, T)| = d_i(S) - |F_i'(S, T_1 \cup T_2)| - |F_i'(S, T_0)| = d_i(S) - |S||T_i| - |F_i'(S, T_0)|$. Furthermore, $d_1(x') + d_2(x') = |Y| - |T_0| + 1$ for each $x' \in S$. Then

$$\begin{aligned} |F_1'(S, T)| + |F_2'(S, T)| &= d_1(S) + d_2(S) - |S||T_1| - |S||T_2| - |F_1(S, T_0)| - |F_2(S, T_0)| \\ &= |S|(|Y| - |T_0| + 1) - |S||T_1| - |S||T_2| - |F_1(S, T_0)| - |F_2(S, T_0)| \\ &> |S|(|Y| - |T_0| - |T_1| - |T_2| + 1) - |S| \\ &= |S||T|, \end{aligned}$$

where the inequality follows from Claim 27. Since this holds for each realization of d_1 and d_2 we conclude that $m_{d_1}(S, T) + m_{d_2}(S, T) > |S||T|$. □

We remark that the proof of Theorem 7 yields an algorithm which starts with two realizations of d_1, d_2, respectively, and either finds in polynomial time two realizations with smaller intersection or finds two sets $S \subset X$, $T \subset Y$ which violate the saturation condition.

References

[1] J. Bang-Jensen and G. Gutin. *Digraphs*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2001. Theory, algorithms and applications.

[2] R. A. Brualdi. Matrices of zeros and ones with fixed row and column sum vectors. *Linear Algebra Appl.*, 33:159–231, 1980.

[3] Y. C. Chen. A short proof of Kundu's k-factor theorem. *Discrete Math.*, 71(2):177–179, 1988.

[4] Y. C. Chen and A. Shastri. On joint realization of $(0, 1)$ matrices. *Linear Algebra Appl.*, 112:75–85, 1989.

[5] M. Chrobak and C. Dürr. Reconstructing polyatomic structures from discrete X-rays: NP-completeness proof for three atoms. *Theoret. Comput. Sci.*, 259(1-2):81–98, 2001.

[6] M.-C. Costa, D. de Werra, and C. Picouleau. Using graphs for some discrete tomography problems. *Discrete Appl. Math.*, 154(1):35–46, 2006.

[7] M. C. Costa, D. de Werra, C. Picouleau, and D. Schindl. A solvable case of image reconstruction in discrete tomography. *Discrete Appl. Math.*, 148(3):240–245, 2005.

[8] D. Gale. A theorem on flows in networks. *Pacific J. Math.*, 7:1073–1082, 1957.

[9] R. J. Gardner, P. Gritzmann, and D. Prangenberg. On the computational complexity of determining polyatomic structures by X-rays. *Theoret. Comput. Sci.*, 233(1-2):91–106, 2000.

[10] G. T. Herman and A. Kuba, editors. *Discrete tomography*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 1999. Foundations, algorithms, and applications.

[11] D. J. Kleitman and D. L. Wang. Algorithms for constructing graphs and digraphs with given valences and factors. *Discrete Math.*, 6:79–88, 1973.

[12] S. Kundu. The k-factor conjecture is true. *Discrete Math.*, 6:367–376, 1973.

[13] L. Lovász. Valencies of graphs with 1-factors. *Period. Math. Hungar.*, 5:149–151, 1974.

[14] O. Ore. Studies on directed graphs. I. *Ann. of Math. (2)*, 63:383–406, 1956.

[15] A. R. Rao and S. B. Rao. On factorable degree sequences. *J. Combinatorial Theory Ser. B*, 13:185–191, 1972.

[16] H. J. Ryser. Combinatorial properties of matrices of zeros and ones. *Canad. J. Math.*, 9:371–377, 1957.