Covering a Strong Digraph by $\alpha - 1$ Disjoint Paths. A proof of Las Vergnas' Conjecture.

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Abstract

The Gallai-Milgram theorem states that every directed graph D is spanned by $\alpha(D)$ disjoint directed paths, where $\alpha(D)$ is the size of a largest stable set of D. When $\alpha(D) > 1$ and D is strongly connected, it has been conjectured by Las Vergnas (cited in [1] and [2]) that D is spanned by an arborescence with $\alpha(D) - 1$ leaves. The case $\alpha = 2$ follows from a result of Chen and Manalastas [5] (see also Bondy [3]). We give a proof of this conjecture in the general case.

In this paper, loops, cycles of length two and multiple arcs are allowed. We denote by $\alpha(D)$ the *stability number* (or *independence number*) of D, that is, the cardinality of a largest stable set of D. A *k-path partition* $\mathcal{P}$ of a digraph D is a partition of the vertex set of D into k directed paths. A *functional digraph* is a digraph in which every vertex has indegree one. An *arborescence* is a connected digraph in which every vertex has indegree one except the *root*, which has indegree zero. The vertices of an arborescence (or a functional digraph) with outdegree zero are the *leaves*. An *arborescence forest* F is a disjoint union of arborescences. We denote by $R(F)$ the set of roots of the arborescences of F, and by $L(F)$ the set of its leaves. A *strong component* of D is a maximal strongly connected subgraph of D. A strong component C of D is *maximal* (resp. *minimal*) if no vertex of C has an out-neighbour (resp. in-neighbour) in $D \setminus C$.

Theorem 1 (Las Vergnas [7], see also Berge [1]) *Let D be a digraph, $m_1, \dots, m_l$ the minimal strong components of D and $x_1, \dots, x_l$ vertices of $m_1, \dots, m_l$, respectively. There exists a spanning arborescence forest F of D with $R(F) = \{x_1, \dots, x_l\}$ and $|L(F)| \leq \alpha(D)$.*

Proof. First observe that there exists a spanning arborescence forest F of D with $R(F) = \{x_1, \dots, x_l\}$. Now let us prove that if a spanning arborescence forest F of D with $R(F) = \{x_1, \dots, x_l\}$ has more than $\alpha(D)$ leaves, there exists a spanning arborescence forest F' of D with $R(F') = \{x_1, \dots, x_l\}$, $|L(F')| = |L(F)| - 1$ and $L(F') \subset L(F)$. Such a forest F' is a *reduction* of F. This statement is easily proved by induction on D: Since $|L(F)| > \alpha(D)$, there exist two leaves x, y of F such that $xy \in E(D)$. Apply a reduction to $D \setminus y$ and $F \setminus y$, and add y to this reduction in order to conclude. To prove Theorem 1, apply successive reductions to a spanning arborescence forest F of D with $R(F) = \{x_1, \dots, x_l\}$. □

Corollary 1.1 (Gallai and Milgram [6]) *Every digraph D admits an $\alpha(D)$-path partition.*

We now prove that every strong digraph with stability number $\alpha > 1$ is spanned by an arborescence with $\alpha - 1$ leaves. This answers a question of Las Vergnas (cited in [1] and [2]) and extends a result of

Chen and Manalastas [5] asserting that every strongly connected digraph with stability number two has a hamiltonian path.

Theorem 2 *Every strong digraph D is spanned by a connected functional digraph with at most $\alpha(D)-1$ leaves.*

Proof. A *disconnecting path* of D is a path Q such that $D\setminus Q$ is not strongly connected. We first prove that either such a path exists or we easily conclude. Consider for this a longest path $Q=u_1,\dots,u_j$ of D. If $D\setminus Q$ is not empty, since there is no arc from u_j to $D\setminus Q$, the path $u_1,\dots,u_{j-1}$ is certainly a disconnecting path. If $D\setminus Q$ is empty and $u_ju_1\notin E(D)$, the path $u_2,\dots,u_{j-1}$ is again a disconnecting path. At last, if $D\setminus Q$ is empty and $u_ju_1\in E(D)$, the digraph D has a hamiltonian circuit and we have our conclusion. A *good path* of D is a disconnecting path $P=v_1,\dots,v_k$ with the following properties: v_1 has an in-neighbour f in a maximal strong component M of $D'=D\setminus P$ and v_k has an out-neighbour in a minimal strong component $m\neq M$ of D'. It is routine to check that a shortest disconnecting path is a good path (indeed, in this case, v_1 has an in-neighbour in every maximal strong component and v_k has an out-neighbour in every minimal strong component). Now, let $P=v_1,\dots,v_k$ be a longest good path of D. Adopting the above notation, we claim that $M=\{f\}$, since if not the path $P=f,v_1,\dots,v_k$, together with a maximal component M' of $M\setminus f$ would be a good path of D. Let $\{m=m_1,\dots,m_l\}$ be the minimal strong components of D. Let x_i be a vertex of m_i which has an in-neighbour y_i on P, $1\leq i\leq l$, where $y_1=v_k$. We apply Theorem 1 in order to span $D'=D\setminus P$ by an arborescence forest F' with $R(F')=\{x_1,\dots,x_l\}$ and $|L(F')|\leq\alpha(D')\leq\alpha(D)$. Note that f is a leaf of this forest. The spanning subgraph of D with edge set $E(F')\cup\{y_1x_1,\dots,y_lx_l\}\cup\{fv_1\}\cup E(P)$ is the functional digraph we are looking for. □

Corollary 2.1 *Every strong digraph D with $\alpha(D)>1$ is spanned by an arborescence with at most $\alpha(D)-1$ leaves.*

Corollary 2.2 *Every strong digraph D with $\alpha(D)>1$ has an $(\alpha(D)-1)$-path partition.*

Remark 1 *The case $\alpha(D)=1$ of Theorem 2 is Camion's Theorem [4]: every strong tournament has a directed hamiltonian cycle.*

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