Oriented trees in digraphs

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Abstract

Let $f(k)$ be the smallest integer such that every $f(k)$-chromatic digraph contains every oriented tree of order k. Burr proved that $f(k) \leq (k-1)^2$ and conjectured $f(k) = 2n-2$. In this paper, we give some sufficient conditions for an n-chromatic digraphs to contains some oriented tree. In particular, we show that every acyclic n-chromatic digraph contains every oriented tree of order n. We also show that $f(k) \leq k^2/2 - k/2 + 1$. Finally, we consider the existence of antidirected trees in digraphs. We prove that every antidirected tree of order k is contained in every $(5k-9)$-chromatic digraph. We conjecture that if $|E(D)| > (k-2)|V(D)|$, then the digraph D contains every antidirected tree of order k. This generalizes Burr's conjecture for antidirected trees and the celebrated Erdős-Sós Conjecture. We give some evidences for our conjecture to be true.

1 Introduction

All the graphs and digraphs we will consider here are *simple*, i.e. they have no loops nor multiple arcs. We rely on [3] for classical notation and concepts. An *orientation* of a graph G is a digraph obtained from G by replacing every edge uv of G by exactly one of the two arcs uv or vu. An *oriented graph* is an orientation of a graph. Similarly an *oriented tree* (resp. *oriented path*) is an orientation of a tree (resp. path).

A *k-colouring* of a digraph is a mapping c from its vertex into $\{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ for all arc uv. A digraph is *k-colourable* if its admits a k-colouring. The *chromatic number* of a digraph D, denoted $\chi(D)$, is the least integer k such that D is k-colourable. A digraph is *k-chromatic* if its chromatic number equals k.

The celebrated Gallai-Hasse-Roy-Vitaver Theorem [16, 18, 23, 26] states that every n-chromatic digraph contains a directed path of length $n-1$. More generally, one can ask which digraphs are contained in every n-chromatic digraph. Such digraphs are called *n-universal*. Since there exist n-chromatic graphs with arbitrarily large girth [13], n-universal digraphs must be oriented trees. Burr [6] considered the function f such that every oriented tree of order k is $f(k)$-universal. He proved that $f(k) \leq (k-1)^2$

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and conjecture $f(k) = 2k-2$ remarking that $f(k) \geq 2k-2$ since a regular tournament (orientation of a complete graph) of order $2k-3$ has no vertex of out-degree at least $k-1$ and thus does not contain the oriented tree S_k^+ consisting of a vertex dominating $k-1$ leaves.

Conjecture 1 (Burr [6]) $f(k) = 2k-2$ *i. e. every oriented tree of order* k *is* $(2k-2)$*-universal.*

Conjecture 1 is a generalization of Sumner's conjecture which states that every oriented tree of order k is contained in every tournament of order $2k-2$. The first linear bound was given by Häggkvist and Thomason [17]. The best bound so far, $3k-3$, was obtained by El Sahili [12], refining an idea of [20].

Regarding the universality of oriented trees, there is no better upper bound than the one given by Burr for oriented trees. Very few special cases are known, only about universality of paths. El-Sahili proved [11] that every oriented path of order 4 is 4-universal and that the antidirected path of order 5 is 5-universal. Recently, Addario-Berry, Havet and Thomassé [1] showed that every oriented path of order $k \geq 4$ with two blocks is k-universal. In Section 2, we give different results which imply the one of Burr. In particular, we show that every k-chromatic acyclic digraph contains every oriented tree of order n. We then derive $f(k) \leq k^2/2 - k/2 + 1$.

In Section 3, we study the universality of *antidirected trees*, that are oriented trees in which every vertex has in-degree 0 or out-degree 0. Burr [7] showed that every digraph D with at least $4(k-1)|V(D)|$ arcs contains all antidirected trees of order k. He deduces that every antidirected tree of order k is $(8k-7)$-universal. We first improve this bound to $(5k-9)$ (for $k \geq 2$) in Subsection 3.1. Then, in Subsection 3.2, we prove Conjecture 1 for antidirected trees of diameter 3.

We then consider the smallest integer $a(k)$ such that every digraph D with more than $a(k)|V(D)|$ arcs contains every antidirected tree of order k. The above-mentionned result of Burr asserts $a(k) \leq 4k-4$. We conjecture that $a(k) = k-2$.

Conjecture 2 Let D be a digraph. If $|E(D)| > (k-2)|V(D)|$, then D contains every antidirected tree of order k.

The value $k-2$ for $a(k)$ would be best possible. Indeed the oriented tree S_k^+ is not contained in any digraph in which every vertex has outdegree $k-2$. It is also tight because the complete symmetric digraph on $k-1$ vertices $\vec{K}_{k-1}$ has $(k-2)(k-1)$ arcs but does trivially not contains any oriented tree of order k.

Observe that there is no analog of Conjecture 2 for non-antidirected tree. Indeed, a bipartite digraph with bipartition (A, B) such that all the arcs have tail in A and head in B contains no directed paths of length two. Hence for any oriented tree T which is not directed and any constant C, there is a digraph D with at least $C \times |V(D)|$ arcs that does not contain T.

Conjecture 2 for oriented graphs implies Burr's conjecture for antidirected trees. Indeed, every $(2k-2)$-critical digraph D is an oriented graph and has minimum degree at least $2k-3$ and so at least $\frac{2k-3}{2}|V(D)| > (k-2)|V(D)|$ arcs.

Conjecture 2 may be seen as a directed analog of the following well-known Erdös-Sós conjecture reported in [14].

Conjecture 3 (Erdös and Sós, 1963) Let G be a graph. If $|E(G)| > \frac{1}{2}(k-2)|V(G)|$, then G contains every tree of order k .

In fact, Conjecture 2 for symmetric digraphs is equivalent to Conjecture 3. Indeed, consider a graph G and its corresponding symmetric digraph D (the digraph obtained from G by replacing each edge uv by the two arcs uv and vu). Then G has more than $\frac{1}{2}(k-2)|V(G)|$ edges if and only if $|E(D)| > (k-2)|V(D)|$. Furthermore, if T is a tree and $\vec{T}$ one of its (two) antidirected orientations, then it is simple matter to check that G contains T if and only if D contains $\vec{T}$.

Conjecture 3 has been proved in particular cases: when the graph has no C_4 in [24]; and for trees with diameter at most four [21]. Finally, using the Regularity Lemma, Ajtai et al. [2] proved that Conjecture 3 is true for sufficiently large k.

In Subsection 3.2, we settle Conjecture 2 for antidirected trees of diameter at most 3. We then derive that every antidirected tree of order k and diameter at most 3 is $(2k-4)$-universal.

2 General upper bounds

2.1 Constructing the tree iteratively

Let T be an oriented tree. The *in-leaves* (resp. *out-leaves*) of T are the vertices v of T such that $d_T^+(v)=1$ and $d_T^-(v)=0$, (resp. $d_T^+(v)=0$ and $d_T^-(v)=1$). The set of out-leaves (resp. in-leaves) of T is denoted by $Out(T)$ (resp. $In(T)$) and its cardinality is denoted by $out(T)$ (resp. $in(T)$).

An *out-star* is an oriented tree T such that $T-out(T)$ has a single vertex x. Hence x dominates all the other vertices which are out-leaves. The out-star of order k is denoted S_k^+. An *in-star* is the directional dual of an out-star; the in-star of order k is denoted S_k^-. A *star* is either an out-star or an in-star.

Lemma 4 *Let D be a digraph with minimum in- and out-degree $k-1$ and T a tree of order k. For any vertex x of D and vertex t of T, D contains a copy of T in which x corresponds to t.*

Proof. We prove the result by induction on k, the result holding trivially when $k=1$. Assume now that $k\geq 2$. Let v be a leaf of T distinct from t. By directional duality, we may assume that v is an out-leaf. Let u be its in-neighbour in T. By the induction hypothesis, D contains a copy T' of $T-u$ in which x corresponds to t. Let y be the vertex corresponding to u in T'. Since $d^+(v)\geq k-2$, there is an out-neighbour z of y not in $V(T')$. Hence adding the vertez z and the arc yz to T', we obtain the desired copy of T. □

Lemma 5 *Let D be an oriented graph with minimum in- and out-degree $k-2$ and T a tree of order k. If T has two out-leaves which are not dominated by the same vertex, then D contains T.*

Proof. Let v_1 and v_2 be two out-leaves of T which are dominated by the two distinct vertices u_1 and u_2. By Lemma 4, D contains a copy T' of $T-v_1$. Let x_1, x_2 and y be the vertices corresponding to respectively u_1, u_2 and v_2 in T'. If x_1 has an out-neighbour z_1 in $V(D)\setminus V(T")$, then adding z_1 and the arc x_1z_1 to T', we obtain a copy of T.

So we may assume that all the out-neighbours of x_1 are in $V(T")$. Since $d^+(x_1)\geq k-2$, x_1 dominates all the vertices of $T'-x_1$. In particular, it dominates x_2 and y. Hence the tree T'' obtained from T' by removing the arc x_2y and adding the arc x_1y is a copy of $T-v_2$. Now x_2 has out-degree at least $k+2$ and it does not dominated x_1 because D is an oriented graph. So x_2 has an out-neighbour z_2 in $V(D)\setminus V(T'')$. Thus adding z_2 and the arc x_2z_2 to T'', we obtain a copy of T. □

A (di)graph is *k-degenerate* if all its sub(di)graphs have a vertex of degree at most k. It is well-known that every k-degenerate (di)graph is $(k+1)$-colourable.

A (di)graph is *k-critical* if its chromatic number is k and all its proper sub(di)graphs are $(k-1)$-colourable. It is folklore that every k-critical graph has minimum degree at least $k-1$.

The *average degree* of a (di)graph G, denoted $Ad(G)$ is $\frac{\sum_{v\in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$. The *maximum average degree* of G, denoted $Mad(G)$, is $\max\{Ad(H), H \text{ subgraph of } G\}$.

Lemma 6 *Let $k\geq 3$ be an integer and G be a graph of maximum average degree at most k. Then $\chi(G)=k$ or G contains a complete graph on $k+1$ vertices.*

Proof. Assume that $\chi(G) > k$, then G contains a $(k+1)$-critical graph H. This graph has minimum degree at least k and so $Ad(H) \geq k$. Since H is a subgraph of G and $Mad(G) \leq k$, we have that $Ad(H) = k$. So every vertex has degree k and so $\Delta(H) = k$. Because $\chi(H) = k+1$, by Brooks' Theorem, H is a complete graph on $k+1$ vertices. $\square$

Lemma 7 *Let T be an oriented tree of order $k \geq 3$ which is not S_k^+. If $T - Out(T)$ is l-universal then T is $(l+2k-4)$-universal.*

Proof. Since $T \neq S_k^+$, then $T - Out(T)$ has more than one vertex and thus $l \geq 2$. If $Out(T) = \emptyset$, the results holds trivially so we assume that $out(T) \geq 1$.

Let D be an $(l+2k-4)$-chromatic digraph. Without loss of generality, we may assume that D is connected. Let S be the set of vertices of D with out-degree at most $k-2$.

Assume first that $\chi(D - S) \geq l$, then $D - S$ contains a copy T' of $T - Out(T)$. Let $v_1, v_2, \dots, v_p$ be the out-leaves of T and $w_1, w_2, \dots, w_p$ be their respectives in-neighbours in T. Now for $1 \leq i \leq p$, since the out-degree of w'_i, the vertex corresponding to w_i in T', is at least $k-1$ in D, one can find an out-neighbour v'_i of w'_i in $V(D) \setminus (V(T') \cup \{v_j \mid 1 \leq j < i\})$. Hence D contains T.

Assume now that $\chi(D - S) < l$, then $\chi(D[S]) \geq 2k-3$, because $\chi(D) = l+2k-4$. Let H be a subdigraph of $D[S]$. Then $\sum_{v \in V(H)} d(v) = 2E(H) \leq 2\sum_{v \in V(H)} d_D^+(v) \leq (2k-4) \times |V(H)|$. Hence $Mad(D[S]) \leq 2k-4$. Thus by Lemma 6, $D[S]$ contains a tournament R of order $2k-3$. Furthermore, since the out-degree in R is at most the out-degree in $D[S]$ and thus $k-2$, every vertex of R has both in- and out-degree equal to $k-2$ in R. Since all vertices in R have out-degree at most $k-2$ in D, each vertex of R has no out-neighbour in $V(D) \setminus V(R)$. Now, since D is connected, there is an arc xy with $x \in V(D) \setminus V(R)$ and $y \in V(R)$.

If T contains an in-leaf v, then let u be its out-neighbour in T. By Lemma 4, R contains a copy of $T - v$ such that u corresponds to y. This copy together with the vertex x and the arc xy is a copy of T in D.

If T contains no in-leaf, then it contains only out-leaves. Moreover, since $T \neq S_k^+$, then T has two leaves which are dominated by different vertices. Thus by Lemma 5, R contains T. $\square$

Let $st(T)$ be the minimum number of successive removal of the in-leaves or out-leaves after which the oriented tree is reduced to a single vertex. Since such a removal remove one or two edges of a path, we have $\lceil diam(T)/2 \rceil \leq st(T) \leq diam(T)$.

Proposition 8 *Every oriented tree T of order k is $[(2k-3-st(T))st(T)+2]$-universal.*

Proof. Let $T = T_0, T_1, \dots, T_{st(T)}$ be a sequence of oriented trees such that $T_i = T_{i-1} \setminus Out(T_{i-1})$ or $T_i = T_{i-1} \setminus In(T_{i-1})$ and $T_{st(T)-1}$ is an out-star or an in-star and thus is $(2|T_{st(T)-1}| - 2)$-universal. By successive application of Lemma 7, T is contained in every oriented tree of chromatic number at least Σ with $\Sigma = 2|T_0| - 4 + 2|T_1| - 4 + \cdots + 2|T_{st(T)-2}| - 4 + 2|T_{st(T)-1}| - 2 = 2\sum_{i=0}^{st(T)-1} |T_i| - 4st(T) + 2$. Now for all $0 \leq i \leq st(T)-1$, $|T_i| \leq k - i$, so $\Sigma \leq (2k-3-st(T))st(T)+2$. Hence T is $[(2k-3-st(T))st(T)+2]$-universal. $\square$

Proposition 8 implies directly Burr's result.

Corollary 9 (Burr [6]) *Every oriented tree T of order k is (k^2-3k+4)-universal.*

Proof. Let T be a tree of order k. If T is a directed path, then it is k-universal by Gallai-Hasse-Roy-Vitaver Theorem Theorem. If T is not a directed path, then $st(T) \leq k-2$. So, by Proposition 8, it is (k^2-3k+4)-universal. $\square$

In order to find an oriented tree T in digraphs of sufficiently large chromatic number, it would be useful to find a sequence of few removal of the in-leaves or out-leaves after which the tree is reduced to a single vertex. However, we do not know if finding such a sequence with the minimum number of steps can be done in polynomial time.

Problem 10 What is the complexity of determining $st(T)$ for a given an oriented tree T?

2.2 Oriented trees in bikernel-perfect digraphs

Let D be a digraph. A set S of vertices is *dominating* if every vertex v in $V(D) \setminus S$ is dominated by a vertex in S. Similarly, a set S is *antidominating* if every vertex v in $V(D) \setminus S$ dominates a vertex in S. A dominating stable set is called a *kernel* and an antidominating stable set an *antikernel*. If every induced subdigraph of D has a kernel (resp. antikernel), then D is said to be *kernel-perfect* (resp. *antikernel-perfect*). A digraph which is both kernel- and antikernel-perfect is said to be *bikernel-perfect*.

Theorem 11 *Every oriented tree of order k is contained in every k-chromatic bikernel-perfect digraphs.*

Proof. Let us prove the result by induction on k, the result being trivially true if $k = 1$.

Let T be an oriented tree of order k and D be a k-chromatic bikernel-perfect digraph. Let v be a leaf of T and w its unique neighbour in T. By directional symmetry, we may assume that $v \to w$. Since D is bikernel-perfect, T has a kernel S. The digraph $D - S$ has chromatic number at least $(k - 1)$, so by induction it contains a copy T' of $T - v$. Now by definition of kernel, the vertex w' in T' corresponding to w is dominated by a vertex v' of K. Hence D contains T. □

Several classes of bikernel-perfect digraphs are known. It is easy to show that symmetric digraphs are bikernel-perfect. Richardson [22] proved that acyclic digraphs and more generally, digraphs without directed cycles of odd length are also bikernel-perfect. Several extensions of Richardson's Theorem have been obtained [8, 9, 10, 15]. Sands, Sauer and Woodrow [25] showed that a digraph whose arcs may be partitionned into two posets is bikernel-perfect.

An approach to find a better upper bound for $f(k)$ would be to prove that every digraph with not too large chromatic number contains an acyclic (or more generally bikernel-perfect) k-chromatic digraph.

Problem 12 What is the minimum integer $g(k)$ such that every $g(k)$-chromatic digraph has an acyclic k-chromatic subdigraph?

What is the minimum integer $g'(k)$ such that every $g'(k)$-chromatic digraph has a bikernel-perfect k-chromatic subdigraph?

An easy consequence of Theorem 11 is that $f(k) \leq g'(k) \leq g(k)$.

Proposition 13 $g(k) \leq k^2 - 2k + 2$.

Proof. Let D be a $(k^2 - 2k + 2)$-chromatic digraph. Let $v_1, v_2, \dots, v_l$ be an ordering of the vertices of D. Let D_1 and D_2 be the digraphs with vertex set $V(D)$ and edge-sets $E(D_1) = \{v_iv_j \in E(D), i < j\}$ and $E(D_2) = \{v_iv_j \in E(D), i > j\}$. Clearly, D_1 and D_2 are acyclic and $\chi(D_1) \times \chi(D_2) \geq \chi(D) = k^2 - 2k + 2$. Hence either D_1 or D_2 has chromatic number at least $\lceil \sqrt{k^2 - 2k + 2} \rceil = k$. □

The above proposition implies directly that $f(k) \leq k^2 - 2k + 2$. We now give a better upper bound for $f(k)$.

Theorem 14 $f(k) \leq k^2/2 - k/2 + 1$.

Proof. Let us prove that $f(k) \leq f(k-1)+k-1$. Then an easy induction will give the result as $f(1) = 1$.

Let D be an $(f(k-1)+k-1)$-chromatic digraph and T be an oriented tree of order k. Let A be a maximal acyclic induced subdigraph of D. If $\chi(A) \geq k$, then by Theorem 11, A contains T, so D contains T. If $\chi(A) \leq k-1$, then $\chi(D-A) \geq f(k-1)$. Let v be a leaf of T. The digraph $D-A$ contains $T-v$. Now, by maximality of A, for every vertex x of $D-A$, there are vertices y and z of A such that xy and zx are arcs. So we can extend $T-v$ to T by adding a vertex to A. □

Another approach will be to prove the existence of a dominating set with not too large chromatic number in any k-chromatic digraph.

Problem 15 What is the minimum integer $h(k)$ such that every k-chromatic digraph has an $h(k)$-chromatic dominating set?

2.3 Acyclic partition and labelled oriented trees

Let D be a digraph. An *acyclic partition* of D is a partition of its vertex set $(V_1, V_2, \dots, V_p)$ such that the digraph $D[V_i]$ induced by each of the V_i is acyclic. The *acyclic number* of D, denoted $ac(D)$, is the minimum number of parts of an acyclic partition of D. Note that a colouring is an acyclic partition since a stable set is acyclic. So $\chi(G) \geq ac(G)$.

Theorem 16 *Let T be an oriented tree with vertices $v_1, \dots, v_k$ and D a digraph with acyclic number k. Then for any acyclic partition of D in k sets $V_1, \dots, V_k$, D contains a copy of T such that $v_i \in V_i$ for all $1 \leq i \leq k$.*

Proof. We prove the result by induction on k, the result being trivial for $k = 1$. Let v be a leaf of T. Free to relabel the vertices and the sets of the acyclic partition, we may assume that $v = v_k$ and the neighbour of v_k in T is v_{k-1}. Moreover, by directional symmetry, we may assume that $v_{k-1} \rightarrow v_k$. Let us now consider $D' = D[V_1 \cup \dots \cup V_{k-1}]$. Obviously $ac(D') = k-1$, so $(V_1, \dots, V_{k-1})$ is an acyclic partition of D' in $ac(D')$ sets. Hence, by the induction hypothesis, D' contains copies of $T' = T - v_k$ such that $v_i \in V_i$ for all $1 \leq i \leq k-1$.

Let S be the set of vertices of V_{k-1} that correspond to v_{k-1} in such a copy of T' in D'. Let us show that a vertex s of S dominates a vertex t in V_k, which gives the result. Suppose for a contradiction that no vertex of S dominates a vertex of V_k. Then $D[V_k \cup S]$ is acyclic. Let us consider $D'' = D' \setminus S$. Then $ac(D'') = k-1$. Indeed an acyclic partition of D'' in less than $k-1$ sets together with $S \cup V_k$ would be an acyclic partition of D in less than k sets which is impossible. In particular, $S \neq V_{k-1}$. So $(V_1, \dots, V_{k-2}, V_{k-1} \setminus S)$ is an acyclic partition of D'' in $ac(D'')$ sets. Thus, by the induction hypothesis, D'' contains a copy of T' such that $v_i \in V_i$ for all $1 \leq i \leq k-2$ and $v_{k-1} \in V_{k-1} \setminus S$. But this contradicts the definition of S. □

Theorem 16 and Theorem 11 yield that $f(k) \leq (k-1)^2+1$. Indeed let D be a $((k-1)^2+1)$-chromatic digraph D and T be an oriented tree of order k. If $ac(D) \geq k$, by Theorem 16, D contains T. If not, in an acyclic partition in $ac(D) < k$ sets, one of the sets induces a digraph with chromatic number at least k and by Theorem 11, D contains T.

3 Universality of antidirected trees

In [7], Burr proved that every antidirected tree of order k is contained in every digraph D with at least $4(k-1)|V(D)|$ arcs. This implies trivially that every antidirected tree of order k is $(8k-7)$-universal since every $(8k-7)$-critical digraph D has minimum degree at least $8k-8$ and thus has at least $4(k-1)|V(D)|$ arcs.

In this section, we will first improve Burr's result by showing that every antidirected tree of order k is $(5k-9)$-universal. We then settle Conjecture 2 for antidirected trees of diameter at most 3 and deduce that every such tree is $(2k-4)$-universal.

3.1 Improved upper bound

Let T be an antidirected tree. Let $V^+(T)$ (resp. $V^-(T)$) be the set of vertices with in-degree (resp. out-degree) 0 in T. Clearly $(V^-(T), V^+(T))$ is a partition of $V(T)$. We set $m(T) = \max\{|V^+(T)|, |V^-(T)|\}$.

Theorem 17 *Let T be an antidirected tree and $D = (V, E)$ a digraph with at least $(4m(T) - 4)|V|$ arcs. Then D contains T.*

The proof of this theorem is based on the following three lemmas :

Lemma 18 (Burr [7]) *Let $G = (V, E)$ be a bipartite graph and p be an integer. If $|E| \geq p|V|$ then G has a subgraph with minimum degree at least $p + 1$.*

Proof. Let us prove it by induction on $|V|$. If $|V| = 4p$, then G is the complete bipartite graph $K_{2p,2p}$ and we have the result.

Suppose now that $|V| > 4p + 1$. If G has minimum degree at least $p + 1$ then G itself is the desired subgraph. Otherwise, there is a vertex v with degree at most p. Then $G - v$ is bipartite and has at least $p(|V| - 1)$ edges. Then, by the induction hypothesis, it has a subgraph with minimum degree at least $p + 1$. □

Remark 19 This result is tight : for any $\epsilon = \frac{p}{m+p} > 0$, the complete bipartite graph $K_{p,m}$ has $pm = p|V|(1 - \epsilon)$ edges but every subgraph has minimum degree at most p.

Let (A, B) be a bipartition of the vertex set of a digraph D. We denote by $E(A, B)$ the set of arcs with tail tail in A and head in B and by $e(A, B)$ its cardinality.

Lemma 20 (Burr [7]) Every digraph D contains a partition (A, B) such that $e(A, B) \geq |E(D)|/4$.

Proof. Let (A, B) be a partition that maximizes the number of arcs between A and B in any direction. Then every vertex v has at least $d(v)/2$ neighbours in the opposite part. So $e(A, B) + e(B, A) \geq |E(D)|/2$. It follows that either $e(A, B)$ or $e(B, A)$ is at least $|E(D)|/4$. □

Lemma 21 *Let T be an antidirected tree and $D = ((A, B), E)$ be a bipartite graph such that every vertex in A has out-degree at least $m(T)$ and every vertex in B has in-degree at least $m(T)$. Then D contains T.*

Proof. Let us show by induction on $|T|$ that one may find a copy of T such that every vertex of $V^+(T)$ (resp. $V^-(T)$) is in A (resp. B).

Let v be a leaf of T. By directional symmetry, we may assume that v is an out-leaf so $v \in V^+(T)$. Let u be the out-neighbour of v in T. Then u is $V^-(T)$. $T - v$ satisfies $m(T - v) \leq m(T)$ so one can find a copy of $T - v$ such that every vertex of $V^+(T - v)$ (resp. $V^-(T - v)$) is in A (resp. B). In particular, u is in B. Now u has at least $m(T)$ in-neighbours in A, so one of them is not in the copy of $T - v$ since $V^+(T - v) < m(T)$. So adding a vertex in $A \setminus N^-(u)$ to the copy of $T - v$, we get the desired copy of T. □

Proof of Theorem 17. By Lemma 20, it contains a bipartite subdigraph $D' = (V = (A, B), E(A, B))$ with at least $(m(T) - 1)|V|$ edges. By Lemma 18, D' has a bipartite subdigraph such that $D'' =$

$((A'', B''), E'')$ such that every vertex of A'' has out-degree at least $m(T)$ and every vertex of B'' has in-degree at least $m(T)$. Hence, by Lemma 21, D'' (and so D) contains T. □

Corollary 22 *Every antidirected tree T is $(8m(T) - 7)$-universal.*

Proof. Every $(8m(T) - 7)$-chromatic digraph D contains an $(8m(T) - 7)$-critical digraph D' which has minimum degree at least $8m(T) - 8$. So D' has at least $(4m(T) - 4)|V|$ arcs. Hence, by Theorem 17, D contains T. □

Note that Corollary 22 is rather good when $m(T)$ is close to $|T|/2)$. We will now improve Corollary 22 when $m(T)$ is big.

Lemma 23 *Let T be an antidirected tree. Then T has at least $Exc(T) = |V^+(T)| - |V^-(T)|$ out-leaves.*

Proof. Let us prove it by induction on the order of T.

Note that if $Exc(T) \leq 0$, the result is trivial. Suppose now that $Exc(T) > 0$. Let v be a leaf of T.

If v is an out-leaf then $Exc(T - v) = Exc(T) - 1$. By induction $T - v$ has $Exc(T) - 1$ out-leaves. These leaves and v are the $Exc(T)$ out-leaves of T.

If v is an in-leaf then $Exc(T - v) = Exc(T) + 1$. By induction $T - v$ has $Exc(T) + 1$ out-leaves and at most one of them dominates v. So T has at least $Exc(T)$ out-leaves. □

Theorem 24 *Let T be an antidirected tree of order k which is not a star. Then T is $(10k - 8m(T) - 11)$-universal.*

Proof. By directional duality, we may assume that $Exc(T) \geq 0$. Let F be a set of $Exc(T)$ out-leaves and U be the antidirected tree $T - F$. Then $Exc(U) = 0$, so $m(U) = |U|/2 = k - m(T)$. Hence, by Corollary 22, U is $(8k - 8m(T) - 7)$-universal. Now, by Lemma 7, T is $(10k - 8m(T) - 11)$-universal. □

Corollary 25 *Every antidirected tree T of order $k \geq 2$ is $(5k - 9)$-universal.*

Proof. If T is a star, then it is $(2k - 2)$-universal, so we may assume that T is not a star.

Corollary 22 and Theorem 24 yield that T is $(\min\{8m(T) - 7; 10k - 8m(T) - 11\})$-universal. The first function increases with $m(T)$ and the second decreases with $m(T)$. They are equal when $m(T) = \frac{5}{8}k - \frac{1}{4}$. In this case, the value of the two functions is $5k - 9$. □

3.2 Antidirected trees of diameter 3

In this subsection, we give evidence for Conjecture 2. We settle it for antidirected trees of diameter at most 3.

It is easy to show that Conjecture 2 holds for antidirected trees of diameter 2 because there are only two antidirected trees of order k and diameter 2: the tree S_k^+ with a vertex v which dominates the $k - 1$ others and its directional dual S_k^-.

Proposition 26 *Let D be a digraph. If $|E(D)| > (k - 2)|V(D)|$, then D contains S_k^+ and S_k^-.*

Proof. Let D be a digraph with more than $(k - 2)|V(D)|$ arcs. Since $\sum_{v \in V(D)} d^+(v) = E(D) > (k - 2)|V(D)|$, D contains a vertex of out-degree at least $k - 1$. So it comtains S_k^+. Similarly, D contains S_k^-. □

Henceforth, we now restrict our attention on antidirected trees of diameter 3. An antidirected tree of order k and diameter 3 is made of a central arc uv such that u dominates the $in(T) \geq 1$ in-leaves of T and v is dominated by the $out(T) = k-2-in(T)$ out-leaves of T. In particular, $k \geq 4$.

Lemma 27 *Let D be a digraph, T an antidirected tree of diameter 3 and $uv \in E(D)$. If*

a) $d^+(u) \geq k-1$ and $d^-(v) \geq out(T)+1$, or

b) $d^+(u) \geq k-2$, $d^-(v) \geq out(T)+1$ and $N^-(v) \not\subset N^+(u) \cup \{u\}$,

then D contains T.

Proof. Set $out(T) = p$. Since its in-degree is at least $p+1$, the vertex v has at p in-neighbours $v_1, \dots, v_p$ distinct from u with $v_1 \in N^-(v) \setminus N^+(u) \cup \{u\}$ in case b). Since $d^+(u) \geq k-1$ or $v_1 \notin N^+(u)$, the vertex u has $k-2-p = in(T)$ out-neighbours in $V(D) \setminus \{v, v_1, \dots, v_p\}$. Hence D contains T. □

We will now show a statement which is slightly stronger than Conjecture 2 for antidirected trees of diameter 3.

Theorem 28 *Let D be a connected digraph. If $|E(D)| \geq (k-2)|V(D)|$ and $D \neq \vec{K}_{k-1}$, then D contains every antidirected tree of order k and diameter 3.*

Proof. Let T be an antidirected tree of order k and diameter 3. Let us prove the result by induction on $|V(D)|$.

Let V^+ (resp. V^-) be the set of vertices of out-degree (resp. in-degree) at least $k-1$.

Assume first that $V^+ = V^- = \emptyset$. Then every vertex v satisfies $d^+(v) = d^-(v) = k-2$. If D is not $\vec{K}_{k-1}$, then it is not complete symmetric and has at least k vertices. Thus there exists three vertices u, v and v_1 such that $uv \in E(D)$, $v_1v \in E(D)$ and $uv_1 \notin E(D)$. So u and v satisfies the condition b) of Lemma 27. Hence D contains T.

Hence, by symmetry, we may assume that $V^+ \neq \emptyset$. If $V^- = \emptyset$ then every vertex has in-degree $k-2$. Picking a vertex $u \in V^+$ and one of its out-neighbour v, since $k-2 \geq out(T)$, Lemma 27 gives the result.

Hence we may assume that V^+ and V^- are nonempty.

Let u be a vertex of out-degree at least $k-1$ and v an out-neighbour of u. If $d^-(v) \geq out(T)+1$, then Lemma 27 gives the result. So we may assume that every out-neighbour of u has in-degree at most $d^-(v) \leq out(T)$. In particular, the set V_1 of vertices of in-degree at most $out(T)$ has cardinality at least $k-1$.

Analogously, we may assume that the set V_2 of vertices of out-degree at most $in(T)$ has cardinality at least $k-1$.

Suppose first that $V_1 \cap V_2$ has a vertex v. Then $d(v) \leq in(T) + out(T) = k-2$. Hence $|E(D-v)| \geq (k-2)|V(D-v)|$ and by induction hypothesis, T is contained in $D-v$ and so in D unless $D - v = \vec{K}_{k-1}$. But in this case, $d(v) = k-2$ and it is simple matter to check that D contains T. Hence we may assume that $V_1 \cap V_2 = \emptyset$.

Suppose that there is $v_1 \in V_1$ and $v_2 \in V_2$ such that v_1v_2 is not an arc. Then consider the digraph D' obtain by replacing the two vertices v_1 and v_2 by a vertex t dominating the out-neighbours of v_1 and dominated by the in-neighbours of v_2. Then D' has one vertex less than D and at most $d^-(v_1) + d^+(v_2)$ arcs less than D (the $d^-(v_1)$ ingoing v_1, the $d^+(v_2)$ outgoing v_2 and $v_1v_2 \notin E(D)$). Now $d^-(v_1) + d^+(v_2) \leq in(T) + out(T) = k-2$, so $|E(D')| \geq (k-2)|V(D')|$. If $D' \neq \vec{K}_{k-1}$, by induction hypothesis, D' contains a copy of T. This copy may be transformed into a copy of T in D, by replacing t by v_1 (resp. v_2) if t is a source (resp. a sink) in T. If $D' = \vec{K}_{k-1}$ then $D - \{v_1, v_2\} = \vec{K}_{k-2}$. Since $d^+(v_1) \geq in(T)+1$ and $d^-(v_2) \geq out(T)+1$, one can easily check that D contains T.

Hence $V_1 \to V_2$. Then any vertex $u \in V_1$ has degree at least $k-1$ and dominates any vertex $v \in V_2$ which has in-degree at least $k-1$. So by Lemma 27, D contains T. □

Theorem 28 implies that every connected digraph D with minimum degree at least $2k-4$ which is not $\vec{K}_{k-1}$ contains every antidirected tree of order k and diameter 3. In particular, this is the case if D is $(2k-3)$-critical. Hence antidirected trees of order k and diameter 3 are $(2k-3)$-universal. We will now improve slightly this result by showing that such trees are $(2k-4)$-universal.

Proposition 29 *Let D be an oriented graph with minimum degree at least $2k-5$. Then D contains every antidirected tree of order k of diameter 3.*

Proof. Suppose for a contradiction that there is an antidirected tree T of order k and diameter 3 which is not contained in an oriented digraph D with minimum degree $2k-5$.

Assume first that $k=4$.

We claim that D contains a vertex of out-degree 2. Suppose not. Then there is a vertex x of out-degree 3. By Lemma 27, each out-neighbour of x has in-degree 1 and thus out-degree at least 2 and so at least 3. Then the oriented graph induced by the vertices of out-degree at least 3 contains a vertex of in-degree at least 3. So there is an arc uv such that $d^+(u) \geq 3$ and $d^-(v) \geq 3$. This contradicts Lemma 27.

Let a be a vertex of out-degree 2, and b and c its two out-neighbours. We claim that there is no arc between b and c. Indeed suppose there is one, say bc. Since $d(b) \geq 3$, b has a neighbour u distinct from a and c. If u is an in-neighbour then (u,b,a,c) is a copy of T otherwise (a,c,b,u) is.

It follows that $d^-(b) = d^-(c) = 1$ so $d^+(b) = d^+(c) = 2$.

Hence the oriented graph induced by the vertices of out-degree 2 contains a vertex of in-degree at least 2. So there is an arc uv such that $d^+(u) \geq 2$ and $d^-(v) \geq 2$. Moreover by the above claim, $N^-(v) \cap N^+(u) = \emptyset$. This contradicts Lemma 27.

Assume now that $k \geq 5$.

By symmetry, we may assume that $out(T) \leq in(T)$, so $out(T) \leq k-4$. Let V^+ be the set of vertices of out-degree at least $k-1$.

We claim that $V^+ = \emptyset$.

Suppose not. If there is no arc uv with $u \in V^+$ and $v \notin V^+$, then each vertex of V^+ has its out-neighbour in V^+. So the digraph D^+ induced by V^+ has at least $|V^+|(k-1)$ arcs and thus has at least a vertex v of in-degree $k-1$ in D^+. Let u be any in-neighbour of v in D^+. As $d^+(u) \geq k-1$, uv contradicts Lemma 27.

Hence we may assume that there is an arc uv with $u \in V^+$ and $v \notin V^+$, then $d^+(v) \leq k-2$ so $d^-(v) \geq k-3$. Since $out(T) \leq k-4$, uv contradicts by Lemma 27. This proves the claim.

So every vertex has out-degree at most $k-2$ and thus in-degree at least $k-3$. We claim that there is a vertex u of out-degree $k-2$ dominating a vertex of in-degree at least $k-2$. Suppose not. Then every vertex of out-degree $k-2$ has its out-neighbours with in-degree at most $k-3$ and thus out-degree at least $k-2$. So V_2 the set of vertices of out-degree $k-2$ has no outgoing arcs and thus there is a vertex v in V_2 with in-degree at least $k-2$ in $D[V_2]$. Picking any in-neighbour u of v in V_2 we get the desired vertices, a contradiction.

By Lemma 27, for every out-neighbour w of u, $N^-(w) \subset N^+(u) \cup \{u\}$. Since each vertex has in-degree at least $k-3$ and v has in-degree $k-2$, the digraph $D[N^+(u)]$ has at least $(k-2)(k-4)+1 > \binom{k-2}{2}$ arcs which is impossible since D is an oriented graph.

□

Note that Proposition 29 does not holds for digraphs instead of oriented graphs. Indeed there are connected digraphs such that $d(v) \geq 2k-5$ for every vertex v that do not contain every antidirected tree of order k of diameter 3. Indeed let $G = (A,B),E)$ be a regular bipartite graph of degree $k-3$. Let D the digraph obtained from G by orienting all the edges from A to B and adding for each $a \in A$ (resp. $b \in B$) a copy of $\vec{K}_{k-2}$ dominating a (resp. dominated by b). One can easily check that for every vertex

$d^+ + d^- \geq 2k - 5$ and that D does not contain the antidirected tree of order k and diameter 3 with one out-leaf.

Corollary 30 *Every antidirected tree of order k and diameter 3 is $(2k-4)$-universal.*

Proof. Let D be a $(2k-4)$-chromatic digraph. It contains a $(2k-4)$-critical oriented graph D', in which every vertex has degree at least $2k-5$. Hence D', and so D, contains every antidirected tree of order k of diameter 3 by Proposition 29. □

Corollary 30 and Proposition 29 are tight. Indeed a regular tournament of order $2k-5$ is $(2k-5)$-chromatic and is an oriented graph in which each vertex with minimum degree $2k-6$ but does not contain the antidirected tree with $k-3$ out-leaves because no vertex has in-degree $k-2$ or more.

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