Graphs with large girth not embeddable in the sphere

Pierre Charbit* and Stéphan Thomassé†

Abstract

In 1972, Rosenfeld asked if every triangle-free graph could be embedded in the unit sphere S^d in such a way that two vertices joined by an edge have distance more than $\sqrt{3}$ (i.e. distance more than $2\pi/3$ on the sphere). In 1978, Larman [4] disproved this conjecture, constructing a triangle-free graph for which the minimum length of an edge could not exceed $\sqrt{8/3}$. In addition, he conjectured that the right answer would be $\sqrt{2}$, which is not better than the class of all graphs. Larman's conjecture was independently proved by Rosenfeld [7] and Rödl [6]. In this last paper it was shown that no bound better than $\sqrt{2}$ can be found for graphs with arbitrarily large odd girth. We prove in this paper that this is still true for arbitrarily large girth. We discuss then the case of triangle-free graphs with linear minimum degree.

Fix a real $0 < \alpha \leq 1$ and an integer $d \geq 1$. The *Borsuk graph* $Bor(d, \alpha)$ is the (infinite) graph defined on the d-dimensional unit sphere where two points are joined by an edge if and only if the distance on the sphere is at least $(1 + \alpha)\pi/2$. A graph is *α-spherical* if it is a subgraph of $Bor(d, \alpha)$ for some d. It is routine to check that every (loopless) graph is α-spherical for some $\alpha \in]0, 1]$. Just remark for this that for $\varepsilon > 0$ small enough, the matrix $I - \varepsilon A$ is positive semi-definite, where A is the adjacency matrix of the graph. For more results concerning embeddings of graphs on spheres, see Karger, Motwani and Sudan [3]. Observe also that a triangle (i.e. a K_3) is 1/3-spherical, but not more. This motivated Rosenfeld to ask if every triangle-free graph is α-spherical, for some $\alpha > 1/3$. Unfortunately this was not true, and Larman in [4] produced a counter-example. He also conjectured that for every $\alpha > 0$, there exists a triangle-free graph which is not α-spherical. The problem was popularized by Erdős, and was independently proved by Rosenfeld [7] and Rdl [6]. See Nešetřil and Rosenfeld [5] for a survey. In [6] was also proved that for every $\alpha > 0$, there exists a graph with arbitrarily large odd-girth which is not α-spherical. We generalize this result to graphs with arbitrarily large girth. Revisiting the problem twenty years later is much easier: one reason is that the probabilistic method is now widely spread, and the other reason is that the work of Goemans and Williamson on max-cut [2] highlighted the close relationship between sphere-embedding of graphs and cuts.

Here is the key-observation:

Lemma 1 *If G is α-spherical there exists a cut of G which has at least $(1 + \alpha)m/2$ edges, where m is the total number of edges.*

Proof. Embed G in some S^d in such a way that every edge has spherical length at least $(1 + \alpha)\pi/2$. Observe that a random hyperplane cuts an edge of G with probability $(1 + \alpha)/2$. By double counting, there is some hyperplane which cuts at least $(1 + \alpha)m/2$ edges. ■

Now the proof is almost finished, since a graph G satisfying Lemma 1 is certainly far from being random. And indeed, Erdős gave a now classical random-based construction of graphs with large girth

*LIAFA, Université Paris 7 Denis Diderot, 2, place Jussieu, Case 7014, 75251 Paris Cedex 05

†LIRMM, Université Montpellier 2, 161 rue Ada, 34392 Montpellier Cedex 5 - France

and large chromatic number, which, when they have enough vertices, are not α-spherical. This is the next result:

Lemma 2 *For every $\alpha > 0$ and every integer k, there exists a graph G, with girth at least k, in which every cut has less than $(1+\alpha)m/2$ edges, where m is the number of edges of G.*

Proof. We consider for this random graphs on n vertices with independently chosen edges with probability p. We first want to bound the size of a maximum cut.
Let A be a subset of vertices. Let X denotes the number of edges between A and its complement. The worst case being when $|A| = n/2$, X is at most a binomial law $Bin(n^2/4, p)$. Its expectation is then at most $\frac{pn^2}{4}$. Thus:

$$\Pr\left(X \geq (1+\alpha)\frac{pn^2}{4}\right) \leq \Pr\left(Bin(n^2/4,p) - \frac{pn^2}{4} \geq \alpha\frac{pn^2}{4}\right)$$

We use the following form of the Chernoff Bound,
for any $0 \leq t \leq Np$:

$$\Pr(|Bin(N,p) - Np| > t) < 2e^{-t^2/3Np}$$

Thus, with $t = \alpha\frac{pn^2}{4}$, we get

$$\Pr\left(X \geq (1+\alpha)\frac{pn^2}{4}\right) < 2e^{-4\alpha^2p^2n^4/48pn^2} = 2e^{-\alpha^2n^2p/12}$$

Now, the probability that there exists a cut of size more than $(1+\alpha)\frac{pn^2}{4}$ is less than $2^n 2e^{-\alpha^2n^2p/12}$, the value 2^n being a bound to the number of cuts.
Choosing $p = n^{-k/k+1}$, this probability goes to 0 as n tends to infinity.

The other part of the proof is standard and is due to Erdős. Let Y denotes the number of cycles of length at most k.

$$E(Y) = \sum_{i=3}^{k}\binom{n}{i}\frac{(i-1)!}{2}p^i \leq \frac{1}{2}\sum_{i=3}^{k}n^ip^i \leq \frac{1}{2}(k-2)n^kp^k.$$

where the last inequality holds because $np = n^{\frac{1}{k+1}} \geq 1$.
Applying Markov's inequality,

$$\Pr(Y \geq \frac{n}{2}) \leq \frac{E(Y)}{n/2} \leq (k-2)n^{-\frac{1}{k+1}}.$$

Thus for n sufficiently large, we can find a graph on n vertices with $m = (1/2 + o(1))n^{\frac{k+2}{k+1}}$ edges such that the number of cycles of length at most k is less than $n/2$ and the size of a maximum cut is at most $(1+\alpha)m/2$.
We can delete less than $n/2$ edges to make a graph of girth $\gamma > k$ and since $n/2$ is negligible compared to m, we still have that the size of a maximum cut is at most $(1+\alpha)m/2$. ■

Thus for every $\alpha > 0$ there exists graphs with arbitrarily large girth which are not α-spherical. Observe that this alone implies that there exists graphs with large girth and large chromatic number, since every k-chromatic graph has an homomorphism into K_k, which is in turn α_k-spherical for $\alpha_k = \arccos(\frac{-1}{k-1})$.

An interesting case arises when considering triangle-free graphs G with minimum degree δ such that $\delta \geq c.n$ for some fixed constant $c > 0$. Now, there exists a constant α_c, depending on c, such that every triangle-free graph G with minimum degree $\geq cn$ is α_c-spherical. To prove this, enumerate the vertices

$v_1, \dots, v_n$ of G. Fix $c := \delta/n$. For every v_i, we fix a set of neighbours N_i of v_i such that $|N_i| = \delta$. The unit vector associated with v_i is then $V_i = \frac{1}{\sqrt{c(1-c)n}}(x_1^i, \dots, x_n^i)$ where $x_j^i = -c$ if $j \notin N_i$ and $x_j^i = 1-c$ if $j \in N_i$.

Since $N_i \cap N_j$ is empty whenever v_iv_j is an edge of G, we have $V_i.V_j = \frac{-c}{1-c}$. And thus the spherical length of an edge is exactly $\arccos(\frac{-c}{1-c})$. Let us raise the following:

Problem 1 *For every $c \in]0, 1/2]$, what is the largest α_c such that every triangle-free graph with minimum degree at least $c.n$ is α_c-spherical?*

One particular case of the previous construction is when $c = 1/3$, in which case we have $V_i.V_j = -1/2$. This means that if a triangle-free graph has minimum degree $\geq n/3$, its vertices can be positioned on some unit-sphere in such a way that edges have length on the sphere at least $2\pi/3$. Let us mention to conclude that the class of triangle-free graphs with minimum degree $> n/3$ has been recently entirely characterized [1] - they have chromatic number at most four. It would be interesting to try to obtain this bound from a geometric point of view. Moreover, a construction of A. Hajnal shows that minimum degree $> cn$, when $c < 1/3$, does not yield to any bound on the chromatic number, see for instance [1] for the construction. So the last question is to find what could be the largest chromatic number of a triangle-free graph with minimum degree $n/3$. The answer can be anything between four and infinity. Again, a geometric approach could be of use.

References

[1] S. Brandt and S. Thomassé, Dense triangle-free graphs are four-colorable: A solution to the Erdős-Simonovits problem, preprint.

[2] M.X. Goemans and D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *J. Assoc. Comput. Mach.* **42** (1995), 1115–1145.

[3] D. Karger, R. Motwani and M. Sudan, Approximate graph coloring by semidefinite programming, *J. ACM* **45** (1998), 246–265.

[4] D.G. Larman, A triangle-free graph which cannot be $\sqrt{3}$-imbedded in any Euclidean unit sphere, *J. Combinatorial Theory Ser. A*, **24** (1978), 162–169.

[5] J. Nešetřil and M. Rosenfeld, Embedding graphs in Euclidean spaces, an exploration guided by Paul Erdős. *Geombinatorics* **6** (1997), 143–155.

[6] V. Rödl, On combinatorial properties of spheres in Euclidean spaces, *Combinatorica*, **4** (1984), 345–349.

[7] M. Rosenfeld, Triangle-free graphs that are not $\sqrt{3}$-embeddable in S^d, *J. Combin. Theory Ser. B*, **33** (1982), 191–195.