Highly connected hypergraphs containing no two edge-disjoint spanning connected subhypergraphs

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Abstract

We prove that there is no degree of connectivity which will guarantee that a hypergraph contains two edge-disjoint spanning connected subhypergraphs. We also show that Edmonds' theorem on arc-disjoint branchings cannot be extended to directed hypergraphs. Here we use a definition of a directed hypergraph that naturally generalizes the notion of a directed graph.

For standard notation and results on digraphs and hypergraphs we refer to [1] and [2].

A **spanning tree** of a graph $G = (V, E)$ is a subtree which contains all vertices of G. A graph $G = (V, E)$ is k**-edge-connected** if and only if there are at least k edges connecting X to $V - X$ for every non empty proper subset X of V. Clearly G is 1-edge-connected if and only if G contains a spanning tree. However, it is not true that every k-edge-connected graph contains k-edge-disjoint spanning trees and hence k-edge-connectivity is not sufficient to ensure that a graph can be decomposed into k edge-disjoint spanning subgraphs. Tutte characterized those graphs which have k edge-disjoint spanning trees. A **partition** of a set S is a collection of disjoint non empty subsets $S_1, S_2, \ldots, S_t \subseteq S$ such that $S = \bigcup_{i=1}^{t} S_i$.

Theorem 1 (Tutte) *[10] A graph $G = (V, E)$ has k edge-disjoint trees if and only if for every partition $\mathcal{P} = \{V_1, V_2, \ldots, V_t\}$ of V, the number of edges in G which connect different sets in $\mathcal{P}$ is at least $k(t - 1)$.*

It is easy to check that Tutte's theorem implies that every $2k$-edge-connected graph can be decomposed into k edge-disjoint spanning subgraphs and we can also use the condition in Theorem 1 to show that $2k$ is best possible.

Tutte's theorem can be proved in at least two different ways: An **out-branching from** s in a digraph is a tree which is oriented in such a way that every vertex other than s has precisely one arc coming in. It is easy to see that a graph G has k edge-disjoint spanning trees if and only if it can be oriented as a digraph D so that D

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contains k arc-disjoint out-branchings from a vertex s (if G has k edge-disjoint trees just pick up s in each tree and orient it away from s). Thus one can prove Theorem 1 by showing that the condition in the theorem guaranties such an orientation [6]. A different way of proving the theorem is to use matroids and Edmonds' theorem on matroid partition [4]. Namely, to prove Theorem 1 it suffices to observe that a graph has k edge-disjoint spanning trees if and only if the matroid M formed as the union (sum) of k copies of the circuit matroid of G has k disjoint bases. Now the theorem follows easily from Edmonds' matroid partition theorem.

A hypergraph $H = (V, E)$ is **k-edge-connected** if the number of hyperedges intersecting X and $V - X$ is at least k for every non empty proper subset X of V. Since hypergraphs generalize graphs, it is natural to ask under what conditions the edges of a hypergraph H can be decomposed into k spanning subhypergraphs of H. This is not an easy problem. In fact, already for $k = 2$, the problem is NP-complete as shown in [7].

In order to obtain some generalization of Tutte's theorem to hypergraphs, Frank et al. [7] introduced the following generalization of edge-connectivity for hypergraphs. A hypergraph $H = (V, E)$ is **k-partition-connected** if for every partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ of V we have

$$\alpha_{\mathcal{P}} \geq k(t-1), \tag{1}$$

where $\alpha_{\mathcal{P}}$ is the number of hyperedges of E which intersect at least two sets in $\mathcal{P}$. Clearly a k-partition-connected hypergraph is k-edge-connected, but the opposite does not hold in general since a hypergraph must have at least $|V|-1$ edges to be 1-partition connected, whereas it needs only one to be connected if it contains the edge $e = V$.

Note that, by Theorem 1, a graph is k-partition-connected if and only if it has k edge-disjoint spanning trees. The following theorem by Frank et al. generalizes Tutte's theorem to partition-connected hypergraphs. They proved this result using matroid theory but it can also be derived from an analogue of Edmonds branching theorem and an orientation theorem concerning a version of directed hypergraphs that we define below (combine Theorem 4 below with Theorem 6.7 in [5] for $l = 0$).

Theorem 2 *[8] A hypergraph H is k-partition-connected if and only if H can be decomposed into k spanning sub-hypergraphs each of which is partition-connected.*

It is an easy corollary of Theorem 2 that if the size of the largest hyperedge in H is q and H is kq-edge-connected, then H admits a partition into k edge-disjoint spanning connected subhypergraphs. However, the following example shows that one cannot hope to find a condition, not involving the size of the largest hyperedge, which still guaranties a decomposition into two spanning connected subhypergraphs.

Theorem 3 *For every natural number k there exists a k-edge-connected hypergraph which contains no two edge-disjoint spanning connected subhypergraphs.*

Proof: Let $t = \binom{2k+1}{k+1}$ and let $I_1, I_2, \ldots, I_t$ be an arbitrary enumeration of the t distinct $(k+1)$-subsets of $S = \{1, 2, \ldots, 2k+1\}$. Let $\mathcal{H} = (V, E)$ be the hypergraph with vertex set $V = \{u_1, u_2, \ldots, u_{2k+1}\} \cup \{v_1, v_2, \ldots, v_t\}$ and edge set $E = \{\{u_i, u_j\} :$

$1 \leq i < j \leq 2k+1\} \cup \{U_1, U_2, \ldots, U_{2k+1}\}$, where U_i is the edge containing u_i and those v_j for which the set I_j contains the element i. Since $\mathcal{H}$ restricted to the vertices $U = \{u_1, u_2, \ldots, u_{2k+1}\}$ is a complete graph and hence $2k$-connected and every vertex in $\{v_1, v_2, \ldots, v_t\}$ has $k+1$ edges to U, it is not difficult to show that $\mathcal{H}$ is $(k+1)$-edge-connected (in fact it is even $(k+1)$-vertex-connected, meaning that we must remove at least $k+1$ vertices to obtain a disconnected hypergraph). Still we claim that the edge set of $\mathcal{H}$ cannot be decomposed into two disjoint spanning connected subhypergraphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$. For suppose H_1 and H_2 where such hypergraphs. Without loss of generality H_1 contains at least $k+1$ of the $2k+1$ edges $\{U_1, U_2, \ldots, U_{2k+1}\}$. Let I be the index set of those edges from $\{U_1, U_2, \ldots, U_{2k+1}\}$ that are in H_1. Since $|I| \geq k+1$ there is some I_j such that $I_j \subseteq I$ and hence the vertex v_j is not incident to any edge in H_2, a contradiction. $\diamond$

Our second aim is to show that our construction above also implies an impossibility result for edge-disjoint in-branchings in directed hypergraphs. One can define a directed hypergraph in many ways. Below we follow Frank et al. [7] and give a definition that straightforwardly generalizes the notion of a directed graph. To make it more clear what is going on we use the name star hypergraph for this kind of orientation. A **star hypergraph** is a hypergraph $H^* = (V, A)$ together with a function $h : A \to V$ that associates one vertex $h(a) \in a$ to each hyperedge $a \in A$. We call $h(a)$ the **head** of a. For each of the definitions below let $H^* = (V, A)$ be a star hypergraph. We always denote by $H = (V, E)$ the underlying hypergraph of H^*, that is, the hypergraph we obtain by ignoring the orientation (thus E and A contain the same edges as subsets of V). By an **arc** of H^* we always mean a hyperedge with a designated head. The arc a **enters** a set $X \subset V$ if $a \cap (V - X) \neq \emptyset$ and $h(a) \in X$. Similarly, a **leaves** X if $h(a) \notin X$ and $a \cap X \neq \emptyset$. The **in-degree** of X, $d^-(X)$, is the number of arcs that enter X and the **out-degree** of X, $d^+(X)$, is the number of arcs that leave X. Note that, as for usual digraphs, we have $d^-(X) = d^+(V - X)$. Note also that an arc a may contribute to the out-degree of up to $|a| - 1$ sets in a partition $\mathcal{P}$ of V but only to the in-degree of at most one set in $\mathcal{P}$.

A **path** in H^* from v_1 to v_k is a sequence $P = v_1, a_1, v_2, a_2, v_3, a_3, \ldots, a_{k-1}, v_k$ such that $v_i \in V$, for $i = 1, 2, \ldots, k$, all v_i are distinct, $a_j \in A$ for $j = 1, 2, \ldots, k-1$, $h(a_i) = v_{i+1}$ and $v_i \in a_i$ for $i = 1, 2, \ldots, k-1$. We call a path P as above an (s,t)**-path** if $s = v_1$ and $t = v_k$.

Let $s \in V$ be a vertex. An **in-branching** rooted at s is a collection of arcs $A' = \{a_1, a_2, \ldots, a_r\}$ with the property that the hypergraph induced by the arcs in A' contains a (t,s)-path for every $t \in V$ and A' is minimal with the property (that is, no proper subset of A' has the properties above). An **out-branching** rooted at s in a star hypergraph on n vertices is a collection of $n-1$ arcs $A' = \{a_1, a_2, \ldots, a_{n-1}\}$ with the property that for all $v \neq s$ there is a path from s to v which uses only arcs from A' (note that in an out-branching every vertex except s is the head of precisely one arc in A').

Theorem 4 *[7, Proposition 1.3](***Edmonds' out-branching theorem for star hypergraphs***) A star hypergraph $H^* = (V, A)$ has k-arc-disjoint out-branchings rooted at s if and only if*

$$d^-_{H^*}(X) \geq k \quad \textit{for every } X \subseteq V - s. \tag{2}$$

This is an easy consequence of Edmonds out-branching theorem for digraphs and the following useful lemma. By **shrinking** an arc a with $|a| > 2$ in a star hypergraph $H^* = (V, A)$ we mean replacing a by $a' = a - \{x\}$ for some $x \in a - \{h(a)\}$ and taking $h(a') = h(a)$. If $H^*_1 = (V, A)$ and $H^*_2 = (V, A')$ are star hypergraphs on the same vertex set, then we say that H^*_1 can be **shrinked** into H^*_2 if there exists a sequence of successive shrinkings of arcs starting from A so that eventually we reach A'. A family $\mathcal{F}$ of subsets of a ground set S is **intersecting** if $X, Y \in \mathcal{F}$ and $X \cap Y \neq \emptyset$ implies that $X \cap Y, X \cup Y \in \mathcal{F}$. The lemma below (which was used without being explicitly stated in [7]) implies that several results for digraphs extend directly to star hypergraphs. It can be proved using the equation [7, Claim 1.2] for the in-degree function of star hypergraphs.

Lemma 5 *Let $H^* = (V, A)$ be a star hypergraph, let $\mathcal{F}$ be an intersecting family of subsets of V. Suppose H^* satisfies that*

$$d^-_{H^*}(X) \geq k \quad \textit{for all } X \in \mathcal{F} \tag{3}$$

then H^* can be shrinked into a digraph D on the same vertex set as H^* such that $d^-_D(X) \geq k$ for every $X \in \mathcal{F}$.

For digraphs it is easy to see, by reversing all arcs and applying the theorem above, that a digraph has k arc-disjoint in-branchings rooted at s if and only if the out-degree of every set not containing s is at least k. This result cannot be extended to star hypergraphs. To see this, consider the star hypergraph $\mathcal{H}^* = (V, A)$ that we obtain from the hypergraph $\mathcal{H}$ from the proof of Theorem 3 by orienting the edges inside U as a k-arc-strong tournament (that is all arcs have size 2 here) and making u_i the head of U_i for $i = 1, 2, \ldots, 2k + 1$ (a digraph D is k**-arc-strong** if it remains strong after deletion of any subset of at most $k-1$ arcs). It is not difficult to check that $\mathcal{H}^*$ satisfies

$$d^+_{\mathcal{H}^*}(X) \geq k \text{ for every } X \in V - u_1. \tag{4}$$

However, since each in-branching rooted at u_1 is connected as an undirected hypergraph and $\mathcal{H}$ cannot be decomposed into two edge-disjoint spanning hypergraphs, it follows that there are no two arc-disjoint in-branchings from u_1 in $\mathcal{H}^*$. This example shows that there is no sufficient condition just in terms of out-degrees of sets not containing s which ensures two arc-disjoint in-branchings rooted at s in a star hypergraph.

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