Density conditions for triangles in multipartite graphs

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Abstract

We consider the problem of finding a large or dense triangle-free subgraph in a given graph G. In response to a question of P. Erdős, we prove that, if the minimum degree of G is at least $17|V(G)|/20$, the largest triangle-free subgraphs are precisely the largest bipartite subgraphs in G. We investigate in particular the case where G is a complete multipartite graph. We prove that a finite tripartite graph with all edge densities greater than the golden ratio has a triangle and that this bound is best possible. Also we show that an infinite-partite graph with finite parts has a triangle, provided that the edge density between any two parts is greater than 1/2.

1 Introduction.

Let G be a n-partite graph on the finite sets $X_1, \ldots, X_n$. Suppose that every vertex x of G has a positive weight $w(x)$. The weight $w(S)$ of a subset S of vertices is defined as the sum of the weights of the vertices in S. The weight of an edge xy is defined as $w(xy) := w(x)w(y)$. The *edge density* of the restriction of G to the bipartite subgraph of G induced by $X_i \cup X_j$ is the sum of the weights of the edges between X_i and X_j divided by $w(X_i)w(X_j)$. The *total density* of G is the sum of all edge densities. Clearly, a graph may be regarded as a weighted graph in which all weights are equal to 1. All graphs are simple, that is, without multiple edges, and also finite, except in Theorem 2.

A *transversal* of G is a subgraph of G induced by a set X such that $|X \cap X_i| = 1$ for each i, $1 \leq i \leq n$.

Lemma 1 *If r is the total density of the multipartite graph G, some transversal of G has at least $\lceil r \rceil$ edges.*

Proof. The sum, taken over all the transversals X of G, of $|E(X)|$ is equal to $r|X_1||X_2|\ldots|X_n|$, since each edge between X_i and X_j is counted $|X_1||X_2|\ldots|X_n|/(|X_i||X_j|)$ times. Thus, some transversal has at least $\lceil r \rceil$ edges. $\square$

Let H be a graph. Denote by $\chi(H)$ the chromatic number of H and by $\text{ex}(H, n)$ the maximum number of edges in a graph on n vertices containing no copy of H. For all $n \geq 1$, denote by

$d_n(H)$ the minimum d such that every n-partite graph in which all edge densities are strictly greater than d contains a copy of H. The definition of $d_n(H)$ does not depend on the graph G being weighted or unweighted, since every real number can be approximated by a rational number. Clearly $d_{n+1}(H) \leq d_n(G)$ for all $n \geq \chi(H)$, so $d(H) := \lim_{n\to\infty} d_n(H)$ exists.

Corollary 1 *Let G be an n-partite graph with total density strictly greater than* $\mathrm{ex}(H,n)$. *Then some transversal of G contains a copy of H.*

Corollary 2

$$d_n(H) \leq \frac{\mathrm{ex}(H,n)}{\binom{n}{2}}.$$

Theorem 1

$$d(H) = \frac{\chi(H)-2}{\chi(H)-1}.$$

Proof. By Corollary 2,

$$d(H) \leq \lim_{n\to\infty} \frac{\mathrm{ex}(H,n)}{\binom{n}{2}}.$$

But, by the Erdős-Stone-Simnovits theorem (see Erdős and Simonovits [2]),

$$\lim_{n\to\infty} \frac{\mathrm{ex}(H,n)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Thus

$$d(H) \leq \frac{\chi(H)-2}{\chi(H)-1}.$$

On the other hand, the n-partite graph G obtained from the empty graph on a set V of n vertices by splitting each vertex v of V into $\chi(H)-1$ vertices $v_1, v_2, \ldots, v_{\chi(H)-1}$, and joining two vertices x_i and y_j if and only if $x \neq y$ and $i \neq j$, has all edge densities equal to $\frac{\chi(H)-2}{\chi(H)-1}$. Moreover, this graph does not contain H because it is a $(\chi(H)-1)$-partite graph (with parts $X_i := \{v_i : v \in V\}$, $1 \leq i \leq \chi(H)-1$). This establishes the opposite inequality

$$d(H) \geq \frac{\chi(H)-2}{\chi(H)-1}.$$

□

2 Infinite-partite graphs.

Define $t_n := d_n(K_3)$, where K_3 is the triangle. By Theorem 1, the limit of t_n, as n tends to infinity, is 1/2. However, this result does not imply the existence of t_ω, where t_ω is the smallest real number such that every infinite-partite graph with finite parts and all edge densities strictly greater than t_ω has a triangle. One needs an additional argument.

Lemma 2 *Let G be a 4-partite graph, one of whose parts is a singleton $\{x\}$. If the minimal edge density is greater than 1/2, G has a triangle.*

Proof. Suppose that G has no triangle. Let A, B and C be the parts distinct from $\{x\}$ and let A_1, B_1 and C_1 be the sets of neighbours of x in A, B and C, respectively. Set $A_2 = A \setminus A_1$, $B_2 = B \setminus B_1$ and $C_2 = C \setminus C_1$. Note that there is no edge between A_1, B_1 and C_1; observe also that $|A_2| < |A_1|$, $|B_2| < |B_1|$ and $|C_2| < |C_1|$. Consider now the 4-partite subgraph induced by A_1, A_2, B_1, B_2. Its total density is greater than 2. Indeed, if we fix the number of edges, the minimal total density is achieved by minimizing the number of edges between A_2 and B_2. Since $|A_1||B_2| + |B_1||A_2| < |A||B|/2$, the total density must exceed 2. Thus, by Lemma 1, there is a transversal with at least seven edges in the 6-partite subgraph induced by $A_1, A_2, B_1, B_2, C_1, C_2$. Since the nine possible types of edges can be partitioned into the three tripartite subgraphs: $A_1B_2C_2$, $B_1C_2A_2$ and $C_1A_2B_2$, one of them yields a triangle in the transversal. $\square$

Theorem 2 *t_ω exists and $t_\omega = 1/2$.*

Proof. Let G be an infinite-partite graph on finite classes with all edge densities greater than 1/2. Let X be any part of G. At least one vertex x of X is joined to more than one half of the vertices of infinitely many classes. Pick three of these classes A, B and C, and apply lemma 2 to the subgraph of G induced by $A, B, C, \{x\}$. $\square$

3 Tripartite graphs and the golden ratio.

Let (G, w) be a weighted tripartite graph G with parts A, B and C. We denote the edge density between A and B by γ, the edge density between B and C by α and the edge density between C and A by β. We say that α, β, γ is a *cyclic triple* if the following conditions are satisfied:

$$\alpha\beta + \gamma > 1,\ \ \beta\gamma + \alpha > 1 \text{ and } \gamma\alpha + \beta > 1.$$

If the edge densities of G form a cyclic triple, then G is said to be *cyclic*. To every triple which is not cyclic, for instance with $\beta\gamma + \alpha \leq 1$, we associate a weighted triangle-free graph on sets A', B' and C' with weight function w' in the following way: $A' = \{a\}$ with $w'(a) = 1$, $B' = \{b_1, b_2\}$ with $w'(b_1) = \gamma$ and $w'(b_2) = 1 - \gamma$, $C' = \{c_1, c_2\}$ with $w'(c_1) = \beta$ and $w'(c_2) = 1 - \beta$, the edges of G forming the 5-circuit $ab_1c_2b_2c_1a$. Observe that the edge densities α', β', γ' of this graph satisfy $\alpha' = 1 - \beta\gamma \geq \alpha$, $\beta' = \beta$, and $\gamma' = \gamma$. The *golden ratio*, denoted by τ, is the positive solution of the equation $x^2 + x = 1$. Note that $\alpha = \beta = \gamma = \tau$ is not a cyclic triple, and thus these densities can be realized by a triangle-free tripartite graph. This shows that $t_3 \geq \tau$.

Theorem 3 *If G is cyclic, it contains a triangle.*

Proof. Without loss of generality, we can assume that the total weights of A, B and C are each equal to 1. Let G be a weighted counterexample with as few vertices as possible. For each vertex a_i of A, we let γ_i (resp. β_i) denote the weight of the neighbourhood of a_i in B (resp. C). By definition, $\beta = \sum \omega(a_i)\beta_i$ and $\gamma = \sum \omega(a_i)\gamma_i$. Let (β', γ') be an element of the boundary of the convex hull $\{\sum x_i(\beta_i, \gamma_i)\ :\ \sum x_i = 1 \text{ and } x_i \geq 0\}$ which is in the positive cone pointed at

(β, γ). We then have $\beta' \geq \beta$ and $\gamma' \geq \gamma$. Moreover $\sum x_i(\beta_i, \gamma_i) = (\beta', \gamma')$ in such a way that only two values x_i are non-zero. Thus, since G is minimal, we may suppose that $|A| \leq 2$, because the vertices a_i of A for which $x_i = 0$ can be deleted and the weight of the remaining vertices can be modified so that the resulting graph is a counterexample to Theorem 3. Furthermore, in the case that $|A| = 2$, say $A = \{a_1, a_2\}$, we may suppose that

$$\text{either} \quad \beta_1 < \beta_2 \text{ and } \gamma_1 > \gamma_2 \quad \text{or} \quad \beta_1 > \beta_2 \text{ and } \gamma_1 < \gamma_2, \tag{1}$$

for otherwise we could have transferred the weight of one vertex of A to the other vertex of A to get a counterexample with fewer vertices. The same condition holds for B and C. If one of the classes has only one vertex, we immediately conclude that G has a triangle since it is cyclic. So $|A| = |B| = |C| = 2$.

Suppose first that every vertex has at least one neighbour in each of the other classes. Then each vertex must have exactly one neighbour in each of the other classes. Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be ordered in such a way that $\omega(x_1) \leq 1/2 \leq \omega(x_2)$ for $x = a, b, c$. Observe that a_1b_2 and a_1c_2 cannot both be edges, otherwise we could transfer the weight of a_2 to a_1. So a_1 is joined to one of b_1, c_1. Without loss of generality, suppose $a_1b_1 \in E(G)$. Then $a_1c_2 \in E(G)$ by condition (1), and $b_1c_2 \in E(G)$ by a condition analogous to (1) on the two vertices of B. Thus (a_1, b_1, c_2) forms a triangle. Now suppose some vertex in some part, say $b_2 \in B$, is not joined to another part, say C. Let $N_A(b_1)$ (resp. $N_C(b_1)$) be the neighbour set of b_1 in A (resp. C). Let $w(b_1) = b$, and let the total weights of vertices in $N_A(b_1)$ (resp. $N_C(b_1)$) be a (resp. c). Then $\alpha = bc$, $\beta \leq 1 - ac$ (since there is no edge between $N_A(b_1)$ and $N_C(b_1)$) and $\gamma \leq 1 - b(1-a)$ (since there is no edge between b_1 and $A \setminus N_A(b_1)$). Then

$$\begin{aligned}
& c(1-b)(\alpha\beta + \gamma - 1) + b(1-c)(\gamma\alpha + \beta - 1) \\
\leq\ & c(1-b)(bc(1-ac) - b(1-a)) + b(1-c)(bc(1-b+ab) - ac) \\
=\ & bc(1-b)(1-c)(ac - ab + b - 1) \\
\leq\ & bc(1-b)^2(1-c)(a-1) \leq 0.
\end{aligned}$$

Thus either $\alpha\beta + \gamma - 1 \leq 0$ or $\gamma\alpha + \beta - 1 \leq 0$, a contradiction. □

Corollary 3 *$t_3 = \tau \approx 0.618$, the golden ratio.*

Problem 1 *Determine t_n, for $4 \leq n < \omega$.*

We suspect that t_n is strictly greater than 1/2 for all $n \geq 3$. The following construction shows that $t_4 > 1/2$. Take a circuit $a_1b_2d_1a_2c_1a_3b_1d_2a_1$. Then add the edges a_1c_1 and d_1b_1. Finally add a vertex c_2 and join it to b_2, a_2, a_3, d_2. We assign weights to the vertices as follows: $w(a_1) = r_1$, $w(a_2) = w(a_3) = (1-r_1)/2$, $w(b_1) = r_2$, $w(b_2) = 1 - r_2$, $w(c_1) = r_3$, $w(c_2) = 1 - r_3$, $w(d_1) = r_2$, and $w(d_2) = 1 - r_2$. The partition consists of $\{a_1, a_2, a_3\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ and $\{d_1, d_2\}$. Choosing $r_1 = 0.66$, $r_2 = 0.3$ and $r_3 = 0.26$, the minimal edge density is 0.51.

4 Locally balanced tripartite graphs.

A multipartite graph G is *locally balanced* if, for every i, $1 \le i \le n$, each vertex $x \in X_i$ has the same number of neighbours in every subset X_j, $j \neq i$.

Theorem 4 *Let G be a locally balanced tripartite graph with edge densities α, β, γ, where*

$$\max\{\alpha+\beta, \beta+\gamma, \gamma+\alpha\} > 1.$$

Then G has a triangle.

Proof. We prove the contrapositive. Suppose that G has no triangle. Let G have parts A, B, C, and consider an edge xy with $x \in A$ and $y \in B$. Since xy belongs to no triangle of G,

$$d_C(x) + d_C(y) \le |C|,$$

where $d_C(x) := |N_C(x)|$. Summing over all edges xy with $x \in A$ and $y \in B$, we have

$$\sum_{xy \in E \cap (A \times B)} d_C(x) + d_C(y) \le \gamma |A||B||C|,$$

that is,

$$\sum_{x \in A} d_C(x) d_B(x) + \sum_{y \in B} d_C(y) d_A(y) \le \gamma |A||B||C|.$$

Because G is locally balanced, $d_B(x) = d_C(x)$ for all $x \in A$, and $d_A(y) = d_C(y)$ for all $y \in B$. Making these substitutions, and applying the Cauchy-Schwartz inequality, yields

$$\frac{(\sum_{x \in A} d_C(x))^2}{|A|} + \frac{(\sum_{y \in B} d_C(y))^2}{|B|} \le \gamma |A||B||C|.$$

Noting that

$$\sum_{x \in A} d_C(x) = \beta |A||C|, \quad \text{and} \quad \sum_{y \in B} d_C(y) = \alpha |B||C|,$$

we obtain

$$\beta^2 |A||C| + \alpha^2 |B||C| \le \gamma |A||B|.$$

But $\beta|A||C| = \alpha|B||C| = \gamma|A||B|$, so $\alpha + \beta \le 1$. Likewise, $\beta + \gamma \le 1$ and $\gamma + \alpha \le 1$. □

Corollary 4 *Let G be a locally balanced tripartite graph. If all edge densities α, β, γ are greater than 1/2, G has a triangle.*

Corollary 5 (Mantel, Turán) *If a graph G on n vertices has more than $n^2/4$ edges, it contains a triangle.*

Proof. Take three copies V_1 V_2 and V_3 of the vertex set of G, and add the edges $x_i y_j$ whenever xy is an edge of G and $i \neq j \in \{1, 2, 3\}$. The resulting tripartite graph is locally balanced. By Corollary 4, this graph has a triangle, which corresponds to a triangle of G. □

5 Bipartite subgraphs versus triangle-free subgraphs.

We have been investigating dense triangle-free multipartite graphs, and in particular tripartite graphs. We now consider triangle-free subgraphs of graphs in general. Denote by $f_t(G)$ the largest number of edges in a triangle-free subgraph of G. This parameter was introduced by Erdős [1], who also defined $f_b(G)$ to be the largest number of edges in a bipartite subgraph of G. Clearly $f_t(G) \geq f_b(G)$, and Turán's theorem for triangles says that equality holds when G is a complete graph. V.T. Sós asked (see [1]) if this is so for every chordal graph, or perhaps even for graphs in which every odd circuit has a chord. The answer is negative.

Consider the complete graph with vertices x_1, x_2, x_3, x_4, x_5. Add five vertices y_1, y_2, y_3, y_4, y_5 and the 10-circuit $C = x_1y_1x_2y_2x_3y_3x_4y_4x_5y_5x_1$. The resulting graph G is chordal. Now $f_t(G) \geq 15$ because C together with the 5-circuit $x_1x_3x_5x_2x_4x_1$ is triangle-free, whereas $f_b(G) \leq 14$ since G is the disjoint union of five triangles and one 5-circuit. (In fact, both inequalities are equalities.)

Erdős[1] raised the question of giving a general condition implying that $f_t(G) = f_b(G)$. In Theorem 5, we give a sufficient condition in terms of the minimum degree.

Theorem 5 *If G is a graph with n vertices and minimum degree at least $17n/20$, then $f_t(G) = f_b(G)$.*

Proof. Let B be a largest bipartite subgraph of G. Erdős observed that B has at least $17n^2/80$ edges, since the degree of every vertex of B is at least the half of its degree in G. Let H be a largest triangle-free subgraph in G. We claim that $|E(H)| = |E(B)|$. Clearly $|E(H)| \geq |E(B)| \geq 17n^2/80$. If H has a vertex x_1 of degree at most $2n/5$, we delete it. If $H \setminus x_1$ has a vertex x_2 of degree at most $2(n-1)/5$, we delete it. We continue in this way until we obtain a graph M with m vertices and minimum degree greater than $2m/5$. By a theorem of Häggkvist [3], M is bipartite. The claim $|E(H)| = |E(B)|$ follows trivially if $m = n$. Now suppose $m < n$. As

$$\frac{m^2}{4} \geq |E(M)| \geq |E(H)| - \frac{2}{5}\left(\binom{n}{2} - \binom{m}{2}\right) \geq \frac{17n^2}{80} - \frac{n^2 - m^2}{5},$$

it follows that $m \geq n/2$. Let $V(G) \setminus V(M) = \{x_1, x_2, \ldots, x_{n-m}\}$. Define $F = \{e \in E(G) : e \text{ is incident to at least one } x_i, 1 \leq i \leq n-m\}$. Then

$$|F| \geq \sum_{i=1}^{n-m} d_G(x_i) - \binom{n-m}{2} \geq \frac{17n(n-m)}{20} - \binom{n-m}{2}.$$

Now one can construct a bipartite graph by adding vertices $x_1, \ldots, x_{n-m}$ together with at least half the edges in F to M, using the Erdös observation that each x_i is added to the partite set so that it has at least half the edges (in G) incident to the opposite partite set. So, the number of edges in the resulting bipartite graph is at least

$$|E(H)| - \frac{2}{5}\left(\binom{n}{2} - \binom{m}{2}\right) + \frac{1}{2}|F| \geq |E(H)| + \frac{(n-m)(2m-n)}{40} \geq |E(H)|.$$

This implies that $|E(B)| \geq |E(H)|$ and thus $|E(B)| = |E(H)|$. □

Let c denote the smallest real number such that, for every sufficiently large graph with minimum degree greater than cn, we have $f_t(G) = f_b(G)$. Theorem 5 shows that $c \leq 0.85$. In Theorem 6, we establish the lower bound $c \geq 27/40 = 0.675$. We need the following proposition.

Proposition 1 *Let G be the lexicographic product $C_5[K_m]$. Then*

(i) $f_t(G) = 5m^2$.

(ii) $f_b(G) = 17m^2/4$.

Proof. Denote by V_i, $1 \leq i \leq 5$, the vertex sets of the five copies of K_m in G (in cyclic order).

(i). Let H be a largest triangle-free subgraph of G. Consider a multi-covering of G by triangles of the form $xyzx$ with $x, y \in V_i$ and $z \in V_{i+1}$, $1 \leq i \leq 5$. Let $\{T_i : 1 \leq i \leq t\}$ be the set of such triangles, where $t = 5m\binom{m}{2}$. Each edge of G appears in either $m-1$ or m of these triangles, according to whether it corresponds to an edge of C_5 or an edge of K_m. Therefore,

$$\sum_{i=1}^{t} |E(H \cap T_i)| \geq (m-1)|E(H)| = (m-1)f_t(G).$$

On the other hand, because H is triangle-free, $|E(H \cap T_i)| \leq 2$, $1 \leq i \leq t$, and so

$$\sum_{i=1}^{t} |E(H \cap T_i)| \leq 2t = 5m^2(m-1).$$

These two inequalities yield the bound $f_t(G) \leq 5m^2$. Since the subgraph $C_5[\overline{K_m}]$ of G is triangle-free and has $5m^2$ edges, we conclude that $f_t(G) = 5m^2$.

(ii). Let $B(X, Y)$ be a largest bipartite subgraph of G. Denote by V_i, $1 \leq i \leq 5$, the vertex sets of the five copies of K_m in G (in cyclic order), and set $x_i := |X \cap V_i|$, $1 \leq i \leq 5$. Then

$$\begin{aligned} e(B) &= \sum_{i=1}^{5} x_i(m - x_i) + \sum_{i=1}^{5} x_i(m - x_{i+1}) + \sum_{i=1}^{5} x_{i+1}(m - x_i) \\ &= 3mx - x^2 + 2\sum_{i=1}^{5} x_i x_{i+2}, \end{aligned}$$

where $x := \sum_{i=1}^{5} x_i = |X|$. We first maximize $\sum_{i=1}^{5} x_i x_{i+2}$ for fixed x; this is clearly equivalent to maximizing $\sum_{i=1}^{5} x_i x_{i+1}$. We assume that the indices are read modulo 5.

Claim 1 *If $x_i > x_{i+1}$, either $x_{i+2} = 0$ or $x_{i+4} = m$.*

Proof. Suppose to the contrary that $x_{i+2} > 0$ or $x_{i+4} < m$. Decreasing x_{i+2} by 1 and increasing x_{i+4} by 1 increase the number of edges by $x_i - x_{i+1}$, a contradiction. □

Corollary 6 1. *If $x_i = x_{i+2} = m$, either $x_{i+1} = m$ or $x_{i+3} = x_{i+4} = m$.*

2. *If $x_i = m$ and $x_j < m$ for all $j \neq i$, then $x_{i+2} = x_{i+3} = 0$.*

3. *If $x_i = x_{i+2} = 0$, either $x_{i+1} = 0$ or $x_{i+3} = x_{i+4} = 0$.*

4. *If $x_i = 0$ and $x_j > 0$ for all $j \neq i$, then $x_{i+2} = x_{i+3} = m$*

5. *If $0 < x_i < m$, $1 \leq i \leq 5$, then $x_i = x_j$, $i \neq j$.*

Set

$$N_0 := |\{i : x_i = 0\}|, \quad N_m := |\{i : x_i = m\}|.$$

Applying the above corollary, it can be shown that the maximal values of $\sum_{i=1}^{5} x_i x_{i+1}$ for fixed x are given by the following table:

N_m	N_0	range of x	max $\sum_{i=1}^{5} x_i x_{i+1}$
0	0	$0 < x < 5m$	$x^2/5$
0	1	none	none
0	2	$0 \leq x \leq 2m$	$x^2/4$
0	3	$0 \leq x \leq 2m$	$x^2/4$
0	4	$0 \leq x \leq m$	0
0	5	$x = 0$	0
1		$m \leq x \leq 3m$	$m(x-m)$
2		$2m \leq x \leq 3m$	$m(x-m)$
2		$x \geq 3m$	$m^2 + (x-m)^2/4$
3		$x \geq 3m$	$m^2 + (x-m)^2/4$
4		$x \geq 4m$	$m(2x-5m)$
5		$x = 5m$	$5m^2$

Thus,

$$\sum_{i=1}^{5} x_i x_{i+1} \;\leq\; \begin{cases} x^2/4, & 0 \leq x \leq 2m \\ m(x-m), & 2m \leq x \leq 3m \\ m^2 + (x-m)^2/4, & 3m \leq x \leq 5m \end{cases}$$

and finally,

$$3mx - x^2 + 2\sum_{i=1}^{5} x_i x_{i+1} \;\leq\; \begin{cases} 4m^2, & 0 \leq x \leq 2m \\ 17m^2/4, & 2m \leq x \leq 3m \\ 4m^2, & 3m \leq x \leq 5m \end{cases}$$

The construction of a bipartite subgraph of G with $17m^2/4$ edges is based on the values $x_1 = m, x_2 = 0, x_3 = m, x_4 = m/2, x_5 = 0$. □

Theorem 6 *For $m \equiv 0\,(mod\,16)$, there is a regular graph H of degree $(27m/8) - 3$ on $5m$ vertices such that $f_t(H) \geq 5m^2$ and $f_b(H) \leq 5m^2 - 4m$.*

Proof. Let G be the lexicographic product $C_5[K_m]$. Then G is a regular graph of degree $3m-1$ on $5m$ vertices such that $f_t(G) \geq 5m^2$ and, by the above discussion, $f_b(G) = 17m^2/4$. We form H by adding $(3m^2/16) - m$ edge-disjoint 5-circuits to G in such a way that each vertex of G lies in $(3m/16) - 1$ of them. Thus H is a regular graph of degree $(3m-1) + (3m/8 - 2) = (27m/8) - 3$ such that $f_t(H) \geq f_t(G) \geq 5m^2$. Moreover,

$$f_b(H) \leq f_b(G) + 4((3m^2/16) - m) = 5m^2 - 4m,$$

because no more than four of the five edges of each added 5-circuit can belong to a bipartite subgraph of H. □

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