Dense triangle-free graphs are four-colorable: A solution to the Erdős-Simonovits problem.

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Abstract

In 1972, Erdős and Simonovits [9] asked whether a triangle-free graph with minimum degree greater than $n/3$, where n is the number of vertices, has chromatic number at most three. Hajnal provided examples of triangle-free graphs with arbitrarily large chromatic number and minimum degree greater than $(1/3 - \varepsilon)n$, for every $\varepsilon > 0$. Häggkvist [10] gave a counterexample to the Erdős-Simonovits problem with chromatic number four based on the Grötzsch graph. Thomassen [15] proved that for every $c > 1/3$, if the minimum degree is at least cn, the chromatic number is bounded by some constant (depending on c). We completely settle the problem, describing the class of triangle-free graphs with minimum degree greater than $n/3$. All these graphs are four-colorable.

1 Introduction.

For every $\varepsilon > 0$, Hajnal (see [9]) provided examples of triangle-free graphs with arbitrarily large chromatic number and minimum degree greater than $(1/3 - \varepsilon)n$, where n is the number of vertices. The construction is as follows: Fix integers $k \ll m \ll l$, where $2m + k$ divides l. Start with a disjoint union of a complete bipartite graph $K_{l,2l}$ and a Kneser graph $KG(2m + k, m)$. Partition the vertices of the part of size $2l$ of $K_{l,2l}$ into subsets $X_1, \dots, X_{2m+k}$ of equal size. Every vertex v of the Kneser graph corresponds to an m-subset I_v of $\{1, \dots, 2m + k\}$ where v and w are adjacent if and only if $I_v \cap I_w = \emptyset$. For every such v, we add all the edges between v and X_i, whenever $i \in I_v$. Observe that this graph is triangle-free, and its minimum degree can be made arbitrarily close (but strictly less) to $n/3$. Moreover, from the celebrated result of Lovász [12], the chromatic number of the Kneser graph, and thus of the whole graph, is at least $k + 2$. Any further attempt to find triangle-free graphs with unbounded chromatic number and minimum degree greater than $n/3$ failed.

It was shown by Andrásfai [1] that triangle-free graphs with minimum degree greater than $2n/5$ are bipartite, this result being sharp because of the 5-cycle. Motivated by Hajnal's construction, Erdős and Simonovits [9] conjectured that every triangle-free graph with chromatic number at least four has minimum degree at most $(1 + o(1))n/3$.

Using a suitable weight function on the set of vertices of the Grötzsch graph, Häggkvist [10] gave a counterexample to this question with minimum degree $10n/29$. He proved also that every triangle-free graph with minimum degree greater than $3n/8$ is 3-colorable. Later on, Jin [11] proved that $3n/8$ could be replaced by $10n/29$, achieving the exact value. Moreover, in 1997, Chen, Jin and Koh [6] proved that every triangle-free graph with minimum degree greater than $n/3$ is either homomorphic with a graph Γ_i, and therefore 3-colorable, or contains the Grötzsch graph as an induced subgraph and therefore its chromatic number is at least 4. The gap to fill-in was to describe the triangle-free graphs between minimum degree $n/3$ and $10n/29$ with chromatic number at least 4.

The finiteness of the chromatic number was proven by Thomassen [15] in 2002, showing that for every $\varepsilon > 0$, if the minimum degree is at least $(1/3 + \varepsilon)n$, the chromatic number is bounded by some constant (depending on ε). Using the Regularity Lemma, Łuczak [14] recently proved that not only the chromatic number is bounded, but there exists a finite graph G_ε such that every triangle-free graph with minimum degree at least $(1/3+\varepsilon)n$ has a homomorphism into G_ε. Also in 2002, Brandt [3] proved that every maximal triangle-free graph which is regular of degree

$> n/3$ has chromatic number at most four and conjectured that the regularity requirement is not needed.

We prove that this is indeed the case, i.e., every triangle-free graph with minimum degree $\delta > n/3$ is 4-colorable by describing the structure of these graphs in detail.

Let us start with some definitions and notations. We generally adopt the terminology from standard graph theory literature like [8] and [16]. We say that a vertex v is *joined to* (or *dominates*) a set of vertices X if v is adjacent to every vertex of X. When there is no edge from v to X, we say that v is *independent* of X. A graph G is *triangle-free* if it does not contain a triangle. Moreover G is *maximal triangle-free* if adding any missing edge to G creates a triangle. Observe that a triangle-free graph is maximal triangle-free if and only if its diameter is at most two.

We denote by N_v the neighborhood of a vertex v. Two vertices v, w are *twins* if $N_v = N_w$. In particular, twins must be non-adjacent. A *weighted graph* is a pair (G, ω), consisting of a graph G together with a *weight function* $\omega : V(G) \to [0, 1]$ such that $\omega(V(G)) := \sum_{v \in V(G)} \omega(v) = 1$. The *degree* of a vertex v of a weighted graph is $\omega(N_v)$, and we denote the minimum degree by $\delta(G, \omega)$, or simply δ. When $\omega(N_v)$ is constant for every vertex v, we say that ω is *regular*. A *good weighted graph* is a pair (G, ω) where G is a maximal triangle-free, twin-free graph with chromatic number at least 4, and ω is a weight function such that $\delta > 1/3$. A graph G is *good* if there exists a weight function ω such that (G, ω) is a good weighted graph. If this function ω is regular, G is a *regular good graph*. Note that this does not mean at all that the graph G itself is regular.

A weighted graph (G, ω) with rational weights > 0 corresponds to the graph H, whose order n is the lowest common multiplier of the denominators of the vertex weights in (G, ω), where each vertex v_i of G is replaced by a set V_i of $\omega(v_i)n$ twins of v_i. Conversely, replacing each class V_i of twins in a graph H of order n by a vertex of weight $|V_i|/n$ gives a weighted graph (G, ω). Note that the minimum degree of H is cn if and only if the minimum degree of (G, ω) is c. A weighted graph can have different optimal weight functions and vertices of weight 0. But as a by-product of our main result we obtain that every good weighted graph has a unique optimal regular weight function where all weights are non-zero and rational.

Our goal is to characterize the class of good weighted graphs.

We can concentrate on the case of graphs with chromatic number at least 4, since the 3-colorable case is solved. To see this, let us denote by Γ_i, for every integer $i \geq 1$, the graph on vertex set $\{1, 2, \ldots, 3i - 1\}$ where the vertex j has neighbors $j + i, \ldots, j + 2i - 1$, these values taken modulo $3i - 1$. Now translating a result of Chen, Jin and Koh [6] into the language of weighted graphs we have the following result.

Theorem 1 *A weighted maximal triangle-free, twin-free graph with minimum degree $\delta > 1/3$ is either isomorphic to a graph Γ_i, for some $i \geq 1$, and therefore 3-colorable, or it contains the Grötzsch graph as an induced subgraph and therefore its chromatic number is at least 4.*

On the other hand, Pach [13] found an interesting characterization of the graphs Γ_i.

Theorem 2 *The graphs Γ_i are just the triangle-free, twin-free graphs, where every maximal stable set is the neighborhood of a vertex.*

Let us now turn to 4-chromatic graphs. A class of good weighted graphs, the Vega graphs were introduced by Brandt and Pisanski [5]. The construction is as follows: For some integer $i \geq 2$, start with a graph Γ_i on vertex set $\{1, \ldots, 3i-1\}$ and add an edge xy and an induced 6-cycle (a, v, c, u, b, w) such that x is joined to a, b, c and y is joined to u, v, w. The set of neighbors of a, u on the Γ_i graph is $\{1, \ldots, i\}$. The set of neighbors of b, v on the Γ_i graph is $\{i+1, \ldots, 2i\}$. The set of neighbors of c, w on the Γ_i graph is $\{2i+1, \ldots, 3i-1\}$. This is the sole Vega graph on $3i+7$ vertices. We denote it by Υ_i.

There are two Vega graphs on $3i+6$ vertices, obtained from Υ_i by a simple vertex deletion. The first one is $\Upsilon_i - \{y\}$, the second $\Upsilon_i - \{2i\}$ (the latter sequence does not occur in [5], perhaps due to the fact that for $i = 2$ both graphs are isomorphic). Finally, the last Vega graph, on $3i+5$ vertices, is $\Upsilon_i - \{y, 2i\}$. Observe that the Vega graph $\Upsilon_2 - \{y, 4\}$ is exactly the Grötzsch graph. Moreover, every vertex in a Vega graph is a neighbor of x, a, b or c, in other words, $N_x \cup N_a \cup N_b \cup N_c$ is the whole vertex set, this makes the Vega graphs 4-colorable, and even 4-chromatic since they contain the Grötzsch graph.

Let us start with the fact that Vega graphs are good (see [5]):

Theorem 3 *Every Vega graph is a regular good weighted graph with positive weight function.*

Proof. We already know that Vega graphs are 4-chromatic and it is routine to check that Vega graphs are twin-free and maximal triangle-free. We have moreover to exhibit a regular weight function with minimum degree $\delta > 1/3$. To avoid fractions, we display integer weights. The normalization for the weight function is obtained by dividing these weights by the total weight.

Υ_i: Assign weight 1 to the vertices $x, y, 1, 2i$, weight $3i-3$ to c, w, weight $3i-2$ to u, v, a, b and weight 3 to all the other vertices. The degree is $9i-6$, and the total weight is $27i-19$.

$\Upsilon_i - \{y\}$: Assign weight 1 to the vertices 1 and $2i$, 2 to x, $3i-4$ to w, $3i-3$ to u, v, c, $3i-2$ to a, b and 3 to all the other vertices. The degree is $9i-7$, and the total weight is $27i-22$.

$\Upsilon_i - \{2i\}$: Assign weight 1 to x, y, 2 to $1, i$, $3i-3$ to b, v, c, w, $3i-2$ to u, a and 3 to all the other vertices. The degree is $9i-7$, and the total weight is $27i-22$.

$\Upsilon_i - \{y, 2i\}$: Assign weight 2 to $x, 1, i$, weight $3i-4$ to v, w, $3i-3$ to u, b, c, $3i-2$ to a and 3 to all the other vertices. The degree is $9i-8$, and the total weight is $27i-25$. ■

The main result of this paper is that the good graphs are just the Vega graphs.

Theorem 4 *A graph is good if and only if it is a Vega graph.*

The necessity follows immediately from Theorem 3, so it is left to prove the sufficiency. Combining this result with Theorem 1 gives the following characterization result:

Corollary 4.1 *The twin-free, maximal triangle-free, weighted graphs with $\delta > 1/3$ are the 3-colorable graphs Γ_i for $i \geq 1$ and the 4-chromatic Vega graphs.*

Concentrating on the chromatic number we immediately get the following statement, conjectured by Brandt [3].

Corollary 4.2 *Every triangle-free graph with minimum degree $> n/3$ is 4-colorable.*

There are several further consequences of our result, the first of which is a strengthening of the previous corollary.

Corollary 4.3 *(1) Every maximal triangle-free graph with minimum degree $> n/3$ has a regular weight function ω.*

(2) Every maximal triangle-free graph with minimum degree $> n/3$ has a dominating $K_{1,t}$, $t \leq 3$.

(3) Every triangle-free graph with minimum degree $> in/(3i-1)$ is homomorphic to Υ_{i-1}, for $i \geq 3$.

(4) Every maximal triangle-free, twin-free weighted graph with minimum degree $\delta > i/(3i-1)$ has at most $3i-4$ vertices, for $i \geq 2$.

Statements (1) and (4) were conjectured in [3, Conjectures 3.8 and 5.1] and (3) refines the result of Łuczak [14].

Proof.

(1) This is immediate from Theorems 1, 3, and 4.

(2) This follows from (1) combined with the main result of [3, Theorem 1.3].

(3) It is easy to verify that the graphs under consideration are Γ_j, $j < i$ and Vega graphs with at most i vertices. By definition, the Vega graph Υ_{i-1} has order $3i+4$ and contains Γ_j and all smaller order Vega graphs.

(4) Again, the graphs under consideration are the graphs Γ_j, $j < i$, with at most $3i-4$ vertices and Vega graphs with at most i vertices. ∎

Our main result can be interpreted in the way that the triangle-free graphs with minimum degree $\delta > n/3$ have a simple structure. So it is not surprising that certain parameters, that are hard to compute for general graphs can be computed in polynomial time. This was shown by Brandt [2] for the independence number and we will prove it here for the chromatic number at the end of this paper.

From now on, let (G, ω) be a good weighted graph with minimum weighted degree $\delta > 1/3$. Among the weight functions ω we choose one which maximizes the minimum degree and if there is a regular weight function ω we choose this. Linear programming yields that if there is a regular weight function, it maximizes δ. Among the vertices of minimum weight of G let x be one of minimum (unweighted) degree in G.

The proof of Theorem 4 proceeds in three steps: The first step is already completed in Theorem 3 by showing that the Vega graphs form a family of good weighted graphs. The final steps are the following statements.

Theorem 5 *If G is a good weighted graph and $G - x$ is not a good weighted graph then $G = \Upsilon_i - \{y, 2i\}$.*

Theorem 6 *If G is a good weighted graph such that $G - x$ is a Vega graph then G is a Vega graph.*

So there cannot be a minimal good weighted graph which is not a Vega graph and therefore the good weighted graphs are just the Vega graphs.

The proof of Theorem 5 is performed in the next two sections while the proof of Theorem 6 is completed in the final section.

2 The structure of regular good graphs.

Here we investigate the structure of (G, ω) when ω is regular. To simplify the notation, we frequently replace the weight $\omega(x)$ simply by x. As usual, $\omega(A)$ (or just A) denotes the sum of the weights of the elements of A. We start with some preliminary results that do not require the fact that the chromatic number is at least 4. Let Q_3 be the graph of the 3-dimensional cube.

Lemma 1 (Brandt [3]) *Every good weighted graph does not contain an induced Q_3 subgraph.*

Lemma 2 (Brandt [3]) *Every stable set S of (G, ω) is such that $\omega(S) \leq \delta$.*

We sketch the proof of this result as a warm-up for the typical reasoning in this paper.

Proof. Consider a stable set S with maximum weight. If S is dominated by a vertex, its weight is clearly at most δ. So assume it is not. The key-fact here is that there exists a subset T of S such that $\bigcap_{x \in T} N_x$ is empty, just consider $T = S$. Now, choose T minimum in size. We claim that $|T| \leq 3$, otherwise G contains an induced cube (i.e. a $K_{4,4}$ minus a perfect matching). Thus T has two or three elements. If $T = \{a, b\}$, since S, N_a and N_b are disjoint, we have $S + 2\delta \leq 1$, and since $\delta > 1/3$, we have $S \leq \delta$. If $T = \{a, b, c\}$, we denote by S' the set $(N_a \cap N_b) \cup (N_a \cap N_c) \cup (N_b \cap N_c)$. Note that S' is a stable set, in particular $S' \leq S$. But if we sum the weights of the neighborhoods of a, b, c, we get $N_a + N_b + N_c - S' + S \leq 1$. Thus $S \leq 1 - 3\delta + S'$, which contradicts the fact that $S' \leq S$. ■

In fact, the stronger result was shown that every maximal weight stable set is the neighborhood of a vertex. The following observation will be frequently used.

Lemma 3 *For every induced C_6 of (G, ω) there is a vertex in G dominating three of its vertices.*

Proof. Assume not, then summing the degrees of the vertices of C_6 we get

$$6\delta \leq \sum_{v \in C_6} d(v) \leq 2,$$

and hence $\delta \leq \frac{1}{3}$, a contradiction. ■

2.1 Global structure.

Here we refine the reasoning in [3] and describe the global structure of a regular good graph in more detail. Recall that x is among the vertices of minimum weight in (G, ω) one of minimum degree in G.

Claim 1 *In the neighborhood of x, there exist three vertices a, b, c such that $N_a \cap N_b \cap N_c = \{x\}$.*

Proof. First of all, $\{x\} = \bigcap_{y \in N_x} N_y$, otherwise N_x would be a subset of some N_z, contradicting the fact that G is maximal triangle-free and twin-free. Now consider a smallest subset T of N_x such that $x = \bigcap_{y \in T} N_y$. By the minimality, for each vertex v of T there is a vertex in G being adjacent to every vertex of T but v. Since Q_3 is isomorphic to $K_{4,4}$ minus a perfect matching, T has at most three vertices by Lemma 1. If $T = \{a\}$, we have $N_a = \{x\}$, and by the fact that G is maximal triangle-free, x would dominate every vertex, contradicting our assumption that G is not 3-colorable. If $T = \{a, b\}$, denoting by Q the vertices which are not neighbors of x, a or b, we have $1 - Q = N_a + N_b + N_x - x > 1 - x$. This gives $Q < x$, and since x has minimum weight, Q is empty. But this would mean that $V(G) = N_a \cup N_b \cup N_x$, and thus G is 3-colorable. ■

We denote by R the (stable) set of vertices that are neighbors of at least two vertices of a, b, c. Note that R contains x. Two cases can occur: there can be a vertex y in G dominating R or not. We will show that if y does not exist, G is just a Vega graph $\Upsilon_i - \{y, 2i\}$. This is the first and major step in the proof of Theorem 5. For the moment, we still assume that y may or may not exist.

Claim 2 *$R = 4\delta - 1 - x$.*

Proof. If we sum all the degrees of a, b, c, x, we get $4\delta - R - x = 1 - Q$, where Q is the set of vertices which are not neighbors of a, b, c, x. It gives that $Q = 1 - 4\delta + R + x$. Since R is a stable set, its weight is at most δ, and thus $Q \leq 1 - 3\delta + x < x$. Since x has minimum weight, the set Q is empty, implying $4\delta - R - x = 1$. ■

Observe that R is a maximal stable set, since adding any vertex gives weight at least $4\delta - 1 > \delta$. Observe also that since Q is empty, every vertex of G is a neighbor of x, a, b, or c. In particular G is 4-colorable. This is how the first author proved in [3] that every regular triangle-free graph with $\delta > n/3$ is 4-colorable.

Let us now partition R into x, U, V, W, where $U = (N_b \cap N_c) \setminus \{x\}$, $V = (N_a \cap N_c) \setminus \{x\}$ and $W = (N_a \cap N_b) \setminus \{x\}$. Note that U (as well as V and W) is not empty since otherwise we would have $N_b \cap N_c = \{x\}$, against the proof of Claim 1. Moreover we denote by X the set $N_a \setminus R$, also $Y := N_b \setminus R$ and $Z := N_c \setminus R$.

Claim 3 *The sets N_x, R, X, Y, Z partition $V(G)$.*

Proof. This is just a reformulation of $V(G) = N_x \cup N_a \cup N_b \cup N_c$. ■

Let $u \in U$ such that $\omega(N_u \cap X)$ is as large as possible, similarly pick $v \in V$ and $w \in W$ such that $\omega(N_v \cap Y)$ and $\omega(N_w \cap Z)$ are as large as possible.

Claim 4 *If $p \in N_u \cap N_x$ then $U \subseteq N_p$. In other words, whenever a neighbor of x is joined to u, it is also joined to U.*

Proof. Assume not, and pick a non-neighbour $u' \in U$ of p. Since G is maximal triangle-free there is a vertex x' such that $p, u' \in N_{x'}$. Note that such a vertex x' is necessarily in X and is not joined to u. Since u has maximum weighted degree in X, there is a vertex x'' of X such that $x'' \in N_u$ but $x'' \notin N_{u'}$. Now let q be a common neighbor of u', x'', observe that $q \in N_x$. We have formed an induced 6-cycle $upx'u'qx''$ such that no vertex of G dominates three of its vertices, a contradiction to Lemma 3. ■

Claim 5 *$X \subseteq N_u$, $Y \subseteq N_v$ and $Z \subseteq N_w$.*

Proof. Assume that a vertex x' of X is not a neighbor of u. Since R is a maximal stable set, there exists a vertex $u' \in R$ joined to x'. Observe that u' is necessarily in U. There exists a common neighbor $p \in N_x$ of u, x'. Since p is joined to u, it is joined to U by the previous claim. This is a contradiction since $px'u'$ is a triangle. The two other inclusions are verified analogously. ■

Claim 6 *Every vertex of $N_u \cap N_v \cap N_w$ dominates R.*

Proof. Observe that N_u is disjoint from $R \cup Y \cup Z$. Thus $N_u \cap N_v \cap N_w$ is disjoint from $R \cup X \cup Y \cup Z$, and falls into N_x by Claim 3. By Claim 4, every vertex of $N_x \cap N_u \cap N_v \cap N_w$ must be a neighbor of $\{x\} \cup U \cup V \cup W$. Finally, $N_u \cap N_v \cap N_w$ dominates R. ■

When R is not dominated, this simply mean that $N_u \cap N_v \cap N_w$ is empty. Otherwise $N_u \cap N_v \cap N_w$ is a (unique) vertex y which dominates R. Let S be the set of vertices that are adjacent to at least two of u, v, w.

Claim 7 *We have $S = N_x$.*

Proof. Observe that $S \subseteq N_x$. We need now two distinct proofs.

If R is not dominated, we let $Q = N_x \setminus S$. We have $R + N_u + N_v + N_w - N_x + Q \leq 1$. Thus $6\delta - 2 - x + Q \leq 0$, and then $Q < x$. In particular, Q is empty.

If R is dominated by some vertex y, it suffices to show that every vertex of N_x is adjacent to at least two vertices of u, v, w. Assume without loss of generality that $p \in N_x$ is not adjacent to u, v. Then p has common neighbors x' with u and y' with v. Note that $x' \in X$ and $y' \in Y$. But then $px'uyvy'$ is an induced C_6 in which no vertex of G has three neighbors, a contradiction to Lemma 3. ■

By Claim 7, when R is dominated by some vertex y, every neighbor of x has at least two neighbors in u, v, w. We will also need that every vertex has a neighbor in u, v, w or y which is a consequence of Claim 3 and 5.

We consider the partition A, B, C of $N_x \setminus \{y\}$ (i.e. N_x if y does not exist), where A is the set of vertices that are neighbors of v, w, B is the set of vertices that are neighbors of u, w and C is the set of vertices that are neighbors of u, v.

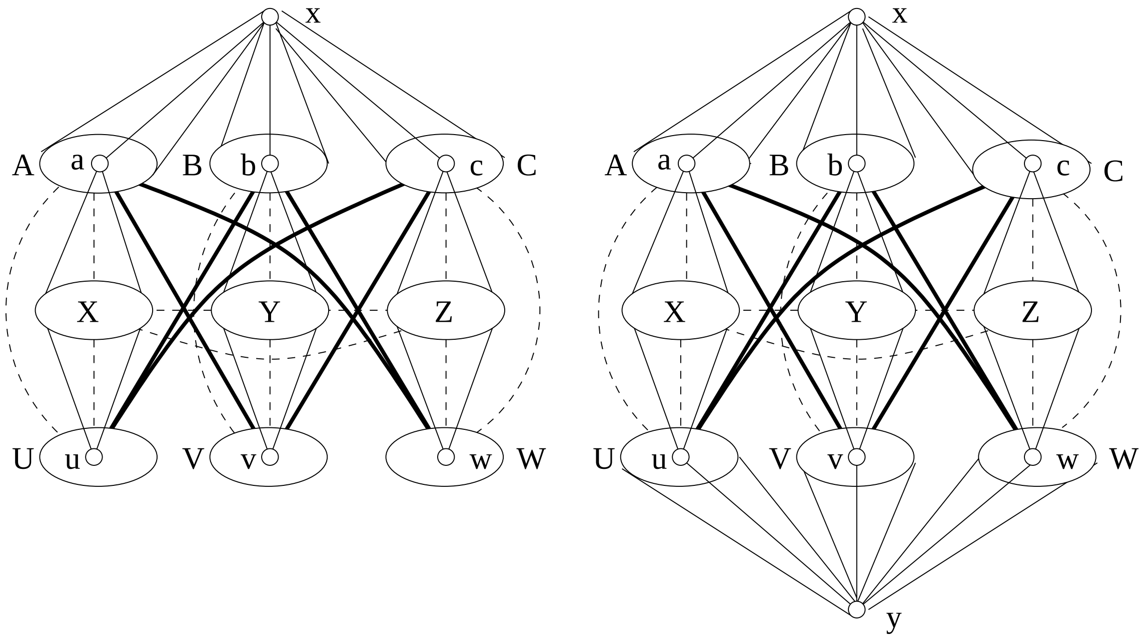

Figure 1: The global structure of regular good weighted graphs (dashed lines indicate that some edges may be present while bold edges indicate that all edges are present between the respective sets).

Claim 8 *There is no edge between B and X, V and X, C and X, W and X, A and Y, U and Y, C and Y, W and Y, A and Z, U and Z, B and Z, V and Z.*

Proof. Each such edge would belong to a triangle, for instance in the case of B and X any edge would form a triangle with u. ■

Claim 9 *There are all edges between A and V, A and W, B and U, B and W, C and U and C and V.*

Proof. Assume that two elements of A and V are not joined by an edge, since G is maximal triangle-free, there should be a vertex dominating them. There is no room for such a vertex in the graph. ■

Observe now that the global structure of a regular good weighted graph is as depicted in Figure 1.

Now the local structure of the subgraphs induced by XYZ, AXU, BYV, and CZW is left to be determined. We will do that for the case that R is not dominated and show that the local structure is such that a Vega graph arises. By the final result, the local structure in the non-dominated case is very much alike. Before starting to determine the local structure we will establish two useful facts concerning the weights of vertices and sets.

Claim 10 *The weight of x is $6\delta - 2$ if R is not dominated and $x = y = 3\delta - 1$ if R is dominated.*

Proof. If R is not dominated, summing the degrees of a, b, c, u, v, w we get $6\delta = 2+x$. Otherwise we get $6\delta = 2 + x + y$ and $R = \delta$ and by Claim 2 we obtain $x = 3\delta - 1$ implying $y = 3\delta - 1$. ■

Claim 11 *If R is not dominated we have $A = X = U + x/2$, $B = Y = V + x/2$, and $C = Z = W + x/2$.*

Proof. Comparing the neighborhoods of u, v, w with x, we get $A = X$, $B = Y$, and $C = Z$. Comparing the neighborhoods of a and u, and b and v, we get $B + C = V + W + x$ and $A+C = U+W+x$. Thus $A+B+C+C = U+V+W+x+x+W$, that is $N_x+C = R+x+W$. Since $R = 4\delta - 1 - x$, we have $\delta + C = 4\delta - 1 + W$. Finally $C = W + x/2$. We prove similarly that $A = U + x/2$ and $B = V + x/2$. ■

It is now left to determine the internal structure of the tripartite subgraphs induced by XYZ, AXU, BYV, and CZW.

2.2 Simple tripartite graphs.

A graph is $2K_2$*-free* if it does not contain two independent edges as an induced subgraph. We call a tripartite graph *simple*, if each pair of its partite sets induces a $2K_2$-free subgraph. It turns out that each of the indicated subgraph must be simple. We call a tripartite triangle-free graph *maximal* if any two non-adjacent vertices from different sets have a common neighbor, which clearly must belong to the third set. Still the graph may not be maximal triangle-free since vertices from the same set can have no common neighbor.

Lemma 4 *The induced tripartite graphs on XYZ, AXU, BYV and CZW are simple and maximal.*

Proof. Consider the graph induced by XYZ and assume for instance that its bipartite subgraph induced by YZ contains a $2K_2$. Since G is maximal triangle-free the graph induced by XYZ must be maximal tripartite triangle-free since by Claim 8, vertices from different sets can only have a common neighbor in the third set. In particular, any common neighbor of a vertex in Y and a vertex in Z belongs to X. Assume now that there exist two independent edges y_1z_1 and y_2z_2 between Y and Z. There exists a vertex $x_1 \in X$ dominating y_1, z_2 and a vertex $x_2 \in X$ dominating y_2, z_1. Now $x_1y_1z_1x_2y_2z_2$ is an induced 6-cycle. But no vertex of G can dominate x_1, z_1, y_2 nor y_1, x_2, z_2, a contradiction to Lemma 3. All the other possible cases follow analogously. ■

Lemma 5 *Every $2K_2$-free bipartite graph with bipartition (X, Y) has a vertex in X dominating Y or an isolated vertex in Y.*

Proof. Choose a vertex $x \in X$ of maximum degree. If x is not dominating Y then there is a vertex $y \in Y$ which is not adjacent to x. Assume that y is not isolated. It has a neighbor $x' \in X$. Since the degree of x is maximum, there must be a neighbor y' of x, which is not adjacent to x'. So $xy', x'y$ is an induced $2K_2$. ■

In a tripartite graph, a *central* vertex is a vertex which set of neighbors consists exactly of one part.

Lemma 6 *Every maximal simple tripartite graph XYZ has at least two central vertices.*

Proof. We just have to prove that there exists a vertex which is independent of one part which is not its own part, it will be then joined to the third part by maximality. Consider now all ordered pairs $(X, Y,), (Y, Z), (Z, X)$. To avoid triangles, one of them have a vertex in the first set which does not dominate the second set. Thus, by Lemma 5, one of the second sets must have a vertex independent of the first set. This is our first central vertex. Observe that by reversing the order of the three pairs, we get a second central vertex. ■

A tripartite graph is *twin-free* if it has no twins in the same partite set.

Lemma 7 *Let XYZ be a maximal tripartite triangle-free graph. If $x, x' \in X$ are not twins, there exists a path of length four with endvertices x, x', and thus with inner vertices in Y and Z.*

Proof. Since the graph is twin-free, there is a edge say xy, with $y \in Y$ such that x' is independent of y. By maximality, there exists a vertex $z \in Z$ which is both joined to x' and y. Now, $xyzx'$ is our path. ■

The next result is the inductive key of the characterization result:

Lemma 8 *If H is a twin-free maximal simple tripartite graph and v is a central vertex of H, then $H - v$ is also a twin-free maximal simple tripartite graph.*

Proof. The graph $H - v$ is obviously simple. Since v is a central vertex, any two twins in the same part of $H - v$ were twins in H. Thus $H - v$ is twin-free. Moreover, any two non-adjacent vertices in different parts of $H - v$ that have no common neighbor had no common neighbor in H. Thus $H - v$ is maximal triangle-free. ■

First we need to recall a useful tool observed by the first author as a variant of Pach's above mentioned result Theorem 2.

Theorem 7 (Brandt [4]) *The only graphs that are maximal triangle-free and without an induced C_6 are the graphs Γ_i, $i \geq 1$, with some vertices duplicated.*

We will now give a characterization of the twin-free maximal simple tripartite graphs. Observe that the graphs Γ_i with the tripartition $X = \{1, \ldots, i\}$, $Y = \{i + 1, \ldots, 2i\}$ and $Z = \{2i + 1, \ldots, 3i - 1\}$ have this property. Indeed, they are maximal triangle-free and twin-free. Assuming that x_1y_1, x_2y_2 is an induced $2K_2$ between two partite sets X, Y, there must be common neighbors z_1 of x_1 and y_2 and z_2 of x_2 and y_1. Since z_1, z_2 must belong to the third partite set Z, this gives an induced C_6, contradicting Theorem 7. So they are simple as well. Moreover every two tripartitions are isomorphic by an isomorphism mapping partite sets onto partite sets.

Fix our initial labelling of Γ_i and the tripartition X, Y, Z. There are two central vertices 1 and $2i$. Deleting the vertex $2i$ from the set Y, we get a twin-free maximal simple tripartite graph Γ'_i (though not maximal triangle-free in the graph sense). The central vertices of this graph are 1 joined to Y and i joined to Z. Deleting 1, we get another twin-free maximal simple tripartite graph Γ''_i (note that this graph contains the twins $i, i+1$ in different sets). Observe that the graphs Γ_i and Γ''_i have an automorphism exchanging X and Y and fixing Z, and Γ'_i has an automorphism exchanging Y and Z and fixing X.

We now show that these are (up to isomorphisms mapping partite sets to partite sets) the only maximal simple tripartite graphs without twins in the same set.

Theorem 8 *A graph H is maximal simple tripartite without twins in the same set, if and only if it is isomorphic to a graph Γ_i, Γ'_i, or Γ''_i, by an isomorphism mapping partite sets to partite sets.*

Proof. We have already seen that the graphs Γ_i, Γ'_i, Γ''_i have this property. So it suffices to prove that every maximal simple tripartite graph without twins in the same set has an isomorphism to one of these graphs mapping partite sets to partite sets. We proceed by induction.

Let H be a maximal simple tripartite graph without twins in the same set. If $|H| \leq 1$ the result holds, so assume that $|H| \geq 2$. By Lemma 6, H has a central vertex v, and by induction and Lemma 8, $H - v$ is isomorphic to a graph Γ_i, Γ'_i, or Γ''_i by an isomorphism respecting the partite sets.

If $H - v$ is isomorphic to Γ_i, then v was deleted from Z. Otherwise by the symmetry of X and Y we may assume that v was deleted from X. Now if v is joined to Y and independent of Z then it is a twin of 1 in X and if v is joined to Z and independent of Y, the neighborhood of v would be a proper subset of the neighborhood of $i \in X$, contradicting the maximality requirement. So, by symmetry, we may assume that $v \in Z$ was joined to Y and independent of X. This gives an isomorphism of H into Γ''_{i+1} mapping j to $j+1$ in $X \cup Y$, j to $j+2$ in Z and v to $3i+2$.

By an analogous reasoning we get, when $H - v$ is isomorphic to Γ'_i, that v was deleted from Y or Z and is joined to X. By symmetry, we get that H is isomorphic to Γ_i. Finally, if $H - v$ is isomorphic to Γ''_i, we get that v was deleted from X and is joined to Y or v was deleted from Y and is joined to X. By symmetry, we get that H is isomorphic to Γ'_i. ■

In the next part we determine the structure inside the tripartite subgraphs in more detail.

2.3 Twins in tripartite graphs.

Here we assume that G is a good graph such that the set R is not dominated (i.e., no vertex y exists) and we investigate the structure of the simple tripartite graphs induced by XYZ, AXU, BYV, and CZW. Note that although G is twin-free there can be twins with respect to the tripartite graphs.

Lemma 9 *All vertices of Y are twins in XYZ or in BYV.*

Proof. Let $y, y' \in Y$ be two vertices. If they are not twins in XYZ, there is a vertex $x' \in X$ which is joined, say, to y' but not to y. Now x' and y must have a common neighbor z in Z. Observe that z is not joined to y'. If y and y' are not twins in BYV then, similarly, there is a path of length three $yv'b'y'$ where $v' \in V$ and $b' \in B$. Now $yv'b'y'x'z$ is a 6-cycle, and no vertex can dominate $yb'x'$ nor $v'y'z$, a contradiction by Lemma 3.

Since G is twin-free, every pair of vertices can be twins only in one of the graphs XYZ or BYV. Consider the auxiliary complete graph H on the vertex set Y. Since being twins is an equivalence relation, it follows that the twins in XYZ and the twins in BYV both induce subgraphs of H consisting of vertex-disjoint cliques. Therefore one of them must contain all edges of H. In particular, all vertices of Y are twins in XYZ or in BYV. ■

Claim 12 *In the subgraph induced by BYV, no vertex of B or V is independent of Y. Moreover, if all the vertices of Y are twins in BYV, then B and V are the singletons $\{b\}$ and $\{v\}$.*

Proof. Observe that every vertex of V has a neighbor in Y, otherwise its neighborhood would be included in N_x. Hence, to avoid triangles, b is independent of V, and thus central. Conversely, no vertex from B can dominate V since R is not dominated. So, by Lemma 5, there must be a vertex in V which is independent of B. Since the neighborhood of this vertex would be included in N_v, this vertex is indeed v. Observe that b and v are the central vertices of BYV both dominating Y. By Lemma 4 and Theorem 8, the graph induced by BYV is isomorphic to Γ''_i (maybe containing twins), since only in Γ''_i the central vertices dominate the same set.

Finally, if all vertices in Y are twins in BYV, it follows that this graph is Γ''_2 (up to twins), i.e. the path on three vertices with center vertex Y. In particular, all the vertices of B and of V are twins in BYV. But this means that they are also twins in G, and thus equal to b and v, respectively. ■

Analogous statements hold for the graphs AXU and CZW.

Claim 13 *All vertices of X, Y, Z are twins in AXU, BYV and CZW, respectively.*

Proof. Observe first that there exists an edge between any two parts of the graph induced by XYZ. If not, say X, Z are independent, pick two vertices $x' \in X$ and $z' \in Z$. Then there is no vertex dominating three vertices of the induced 6-cycle $ax'ucz'wa$, a contradiction to Lemma 3. In particular, for one of the parts in XYZ, say X, not all vertices are twins, otherwise we would get either a triangle, or two independent parts. Thus, by Claim 12, $A = \{a\}$ and $U = \{u\}$.

Assume for a contradiction that the vertices of, say, Z are not twins in CZW. Then the vertices of Z are twins in XYZ by Lemma 9. By Theorem 8, the graph induced by XYZ is (up to twins) isomorphic to some graph $\Gamma_i, \Gamma'_i, \Gamma''_i$. Since Z is a singleton in XYZ, we have $i = 2$. Moreover, the case Γ''_2 is impossible since X has at least two vertices that are not twins. So the graph induced by XYZ is Γ'_2 or Γ_2 (up to twins). In particular $X = \{x_1, x_2\}$, where x_1 dominates Z, and x_2 is independent of Z and therefore dominates Y.

First assume that the vertices of Y are not twins in BYV. Since by Claim 12, no vertex in B or V is independent of Y there are, by Lemma 5, distinct vertices $y_1 \in Y$ dominating B and $y_2 \in Y$ dominating V. By Lemma 7, there is a path $y_1b_1v_2y_2$ in the subgraph BYV. Analogously there is a path $z_1c_1w_2z_2$ in CZW. Observe also that Y is completely joined to Z, and thus x_1 is independent of Y. Let S be the set $\{x_1, x_2, y_1, y_2, z_1, z_2, b_1, v_2, c_1, w_2\}$. Every vertex of B, C, V, W has at most three neighbors in S (for instance a vertex of B has neighbors y_1, w_2 and at most one vertex of the edge y_1v_2). Every vertex of A, Y, Z, U has at most four neighbors in S (this is clear for A and U, and a vertex of Y is joined to x_2, z_1, z_2 and at most one vertex in the edge b_1v_2). Finally, every vertex $x_i \in X$ has exactly two neighbors in S. Thus, summing the degrees of the elements of S and using Claim 11, we get:

$$10\delta \leq 3 + Y + Z + A + U - X - x = 3 + A + B + C + U - X - x = 3 + \delta - \frac{3}{2}x,$$

which implies $\frac{3}{2}x \leq 3 - 9\delta < 0$, a contradiction.

So we may assume that all the vertices of Y are twins in BYV, and thus $B = \{b\}$ and $V = \{v\}$. Recall that the graph induced by XYZ is isomorphic to Γ_2 or Γ'_2 (possibly with twins).

First assume that XYZ induces Γ_2. Then $Y = \{y_1, y_2\}$, where y_1 dominates Z and y_2 dominates X and is independent of Z. Summing the degrees of x_1 and y_1 we get by Claim 11

$$2\delta = Z + a + u + y_2 + Z + b + v + x_2 = 2a + 2b + 2C - x + x_2 + y_2 = 2\delta - x + x_2 + y_2$$

implying $x_2 + y_2 = x$, a contradiction to x having positive minimal weight $6\delta - 2$ (Claim 10).

Finally, assume that XYZ induces Γ'_2. Here we have $Y = \{y_1\}$, dominating Z and x_2. Again, summing the degrees of x_1 and y_1 we get by Claim 11

$$2\delta = Z + a + u + Z + b + v + x_2 = 2a + 2b + 2C - x + x_2 = 2\delta - x + x_2$$

implying $x_2 = x$. Since CZW is a Γ''_i for some $i \geq 3$ (indeed, $|Z| \geq 2$ since the elements of Z are not twins in CZW), there are at least two vertices in C. So the (unweighted) degree of x in G is at least four while the neighborhood of x_2 is a, u, y_1, a contradiction to the degree minimality assumption for x among the minimum weight vertices. ■

In particular, by Claim 12, U, V, W, A, B, C are the singletons $\{u\}, \{v\}, \{w\}, \{a\}, \{b\}, \{c\}$. Our graph G has then the structure depicted in Figure 2.

2.4 The characterization step.

Here we complete the characterization of the regular good graphs when R is not dominated. Since each of the sets A, B, C, U, V, W consists of a single vertex, it is left to show that the graph induced by XYZ is a graph Γ'_i.

Proposition 1 *If R is not dominated, then G is isomorphic to $\Upsilon_i - \{y, 2i\}$, for some $i \geq 2$.*

Proof. We just have to show that the graph H induced by XYZ is Γ'_i, with its usual tripartition.

Assume that this is not the case. Then, by Theorem 8, H must be isomorphic to Γ_i or Γ''_i. In the latter case, there are central vertices $x_1 \in X$ independent of Y, and $y_1 \in Y$ independent of X. Now, ax_1uby_1v is an induced C_6 contradicting Lemma 3.

So we may assume that $H = \Gamma_i$. Then G is isomorphic to $\Upsilon_i - \{y\}$. By complementary slackness G has a unique regular weighting where all vertices have the same degree and the total weight is 1, which is given in Theorem 3, up to normalization. But now the vertices 1 and $2i$ have smaller weight than x, a contradiction to the choice of x. ■

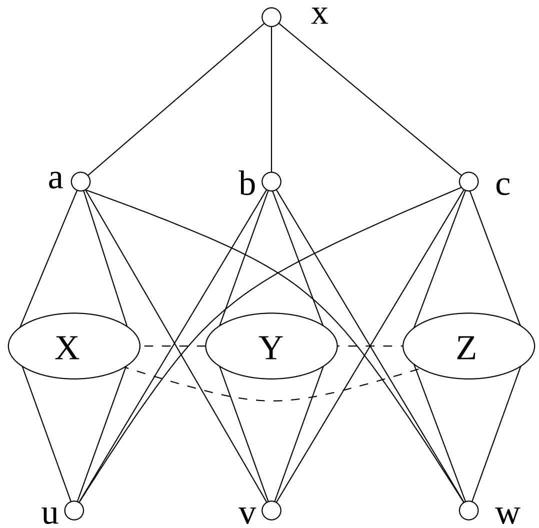

Figure 2: Refined structure of good regular weighted graphs if R is not dominated.

3 The deletion step.

If $G = \Upsilon_i - \{y, 2i\}$, then $G - x$ is not a good weighted graph by Lemma 3 applied to the C_6 induced by a, b, c, u, v, w. Our aim here is to show that for all other good weighted graphs G, regular or not, the graph $G - x$ is again good.

Proposition 2 *If $G \neq \Upsilon_i - \{y, 2i\}$ is a good weighted graph then $(G - x, \omega')$ is a good weighted graph for a suitable weight function ω'.*

Proof. Observe first that $G - x$ is twin-free. If there were twins in $G - x$, one of them would have in G its neighborhood included in the second. This is impossible since G is maximal triangle-free and twin-free. Observe also that if $G - x$ has a weight function with $\delta > 1/3$ and is maximal triangle-free, it cannot have chromatic number less than 4. Otherwise, by Theorem 1, $G - x$ is exactly a Γ_i graph, and thus x, by Theorem 2, would be joined to a maximal stable set of this Γ_i graph, and therefore would be a twin of some other vertex.

By the two previous observations it is left to show that $G - x$ is maximal triangle-free and has a weight function with $\delta > 1/3$. The arguments differ depending on whether the graph has a regular weight function or not.

Let (G, ω) be regular. In particular, since $G \neq \Upsilon_i - \{y, 2i\}$, the set R is dominated. Observe that $G - x$ is maximal triangle-free, since by Claim 7 every pair of vertices of N_x has a common neighbor in $\{u, v, w\}$.

By Claim 10, x has weight $3\delta - 1$. We just have to find a weight function for $G - x$. Delete x and add $3\delta - 1$ to the weights of u, v, w, y. The key observation here is that the weight of every neighborhood increases by $3\delta - 1$ since every neighbor of x has at least two neighbors in u, v, w, and every vertex has a neighbor in u, v, w, y. At the same time, the total weight increases by $9\delta - 3$, so this new weight function ω' still achieves $\delta' > 1/3$, when renormalized.

Let (G, ω) be not regular. Here we use the duality theorem of linear programming. If ω achieves the maximum minimum degree δ, there exists a dual weight function ω^d with the same total weight 1, such that $\omega^d(N_v) \leq \delta$ for every vertex v of G. By complementary slackness, we also know that if a vertex v has weight nonzero for ω, it satisfies $\omega^d(N_v) = \delta$. In particular, it is necessary that ω has a vertex x which has weight zero since if not, ω^d would be regular with degree δ, and we would consider this weight function instead of ω.

So the restriction of ω to $G - x$ has minimum degree $\delta > \frac{1}{3}$, and we just have to show that $G - x$ is maximal triangle-free. If this is not the case, there are two nonadjacent vertices z, z' of $G - x$ such that the unique vertex of G joined to both is exactly x. The sum of the degrees of x, z, z' is 3δ. Meanwhile, the only vertex which is counted twice is x, which has weight zero. This would give $1 \geq 3\delta$, a contradiction. ■

From this point, we know that every good graph contains a $\Upsilon_i - \{y, 2i\}$. In particular, we reproved that every good graph contains a Grötzsch graph.

4 The induction step.

Our final goal is now to prove Theorem 6, saying that if a good weighted graph (G, ω) has a vertex t such that $G - t$ is a Vega graph, then G is also a Vega graph.

Now, let G be a graph having a vertex t such that $G - t$ is a Vega graph. So G consists of 7 or 8 vertices x, a, b, c, u, v, w and possibly y and a graph Γ_i or $\Gamma_i - \{2i\}$, such that the vertex sets $X = N(a) \cap N(u)$, $Y = N(b) \cap N(v)$, $Z = N(c) \cap N(z)$ form a proper 3-coloring of Γ_i. Note that the possibly deleted vertex $2i$ has the property that its set of neighbors in Γ_i is precisely one of the sets X, Y, Z. In order to use the symmetries of the situation we will only use this property of $2i$, and will not prescribe to which set it is joined. Now we are prepared to prove Theorem 6.

Proof. We consider the 6-cycle C induced by $\{a, b, c, u, v, w\}$ and $T = N(t) \cap V(C)$. We will show that in any case:

(1) t is a twin of some vertex of $G - t$, contradicting that G is twin-free,

(2) t belongs to a C_6 in which no vertex in G has more than 2 neighbors, a contradiction to Lemma 3,

(3) t is the missing vertex $2i$ or y of $G - t$, and hence G is a Vega graph, or

(4) there is an independent set containing vertices from all three sets X, Y, Z, which contradicts Pach's characterization (Theorem 2). This is due to the fact that X, Y, Z is (or can be extended to) a proper 3-coloring of Γ_i, but there is no vertex in Γ_i that dominates vertices from three different color classes.

If $T = \{a, b, c\}$ then t is a twin of x. Analogously, if $T = \{u, v, w\}$ then t is either a twin of y or t is the missing vertex y.

Otherwise, $|T| \leq 2$. First assume $T = \emptyset$. Then, since t can be adjacent to at most one vertex of $\{x, y\}$, it must be adjacent to at least one vertex of each of the sets X, Y, Z since it must have distance 2 to each vertex of C. As indicated above, this contradicts Pach's characterization Theorem 2.

Now assume that T has exactly two neighbors in one partite set of the bipartition of C. W.l.o.g., t is adjacent to u, v (or a, b, resp.). But then it must be adjacent to each vertex in Z and to x (y, resp., if y is present) and hence t is a twin of c (w, resp.). So t has at most one neighbor in each of the partite sets.

If t has a neighbor in both partite sets, w.l.o.g., $T = \{a, u\}$, then t cannot be adjacent to x nor to y. Moreover, t must be adjacent to a maximal independent set S of $Y \cup Z$, which contains vertices from both Y and Z, since t has distance 2 to both b and c. Since no independent set can contain vertices from all three sets X, Y, Z, the set S is maximal independent in the graph induced by $X \cup Y \cup Z$ as well. By Pach's characterization there must be a vertex $x' \in X$ dominating S, unless S is the neighborhood of the deleted vertex $2i$. In the first case, t is a twin of x', in the latter case, t is the missing vertex $2i$, and hence G is a Vega graph.

Finally, we have to treat the case where T consists of a single vertex, which we may assume to be a or u. If $T = \{a\}$ then t must be adjacent to a maximal independent set S of $Y \cup Z$, containing vertices $y' \in Y$ and $z' \in Z$, and to y, if y is present in $G - t$. But now t, y', b, x, c, z' is a C_6 in which no vertex of G has more than two neighbors. If $T = \{u\}$ then t must be adjacent to a maximal independent set S of $Y \cup Z$, containing vertices $y' \in Y$ and $z' \in Z$, and to x. Since Γ_i and $\Gamma_i - 2i$ are both twin-free, one of the vertices y', z', say z', has a non-neighbor $x' \in X$. Since G is maximal triangle-free, x' and z' have a common neighbor y'', which must belong to Y. So t, z', y'', x', a, x is a C_6 in which no vertex of G has more than two neighbors. ■

5 Computing the chromatic number.

Here we show that the chromatic number and even a minimal coloring of a triangle-free graph with $\delta > n/3$ can be computed in polynomial time. This question was asked by Brandt [2], who proved that a maximum independent set of a triangle-free graph with minimum degree $\delta > n/3$ can be computed in polynomial time, and, in fact, some tools can be reused here.

Observe first that Theorem 1 combined with our main Theorem 4 immediately gives a trivial $\mathcal{O}(n^{11})$ algorithm that decides the chromatic number of the triangle-free graph with minimum degree $> n/3$. First test whether the graph is bipartite and, if not, test every 11 vertex subset whether it it induces a Grötzsch graph. If yes, the chromatic number is 4 and if none of the sets induces a Grötzsch graph the chromatic number is 3. Next we sketch an algorithm that actually finds a minimal coloring in much faster running time.

The first useful observation from [2] is that a triangle-free graph $G = (V, E)$ of order n with minimum degree $\delta > n/3$ has a unique maximal triangle-free supergraph $G' = (V, E')$ (in the labeled sense), which can be computed in time needed to multiply two $n \times n$-matrices with $0,1$-entries [2]. The currently best upper bound for the running time $M(n)$ due to Coppersmith and Winograd [7] is $\mathcal{O}(n^{2.3677})$, while the best lower bound seems to be $\Omega(n^2)$.

Next we identify twins in G'. For this purpose we order the vertices by the characteristic vector of their neighborhoods. This can be performed in time $\mathcal{O}(n^2 \log n)$, using an $\mathcal{O}(n \log n)$ sorting algorithm, since comparing two n-bit vectors takes time $\mathcal{O}(n)$. Find a vertex x of minimum degree in the resulting twin-free graph G'' and check whether G'' is regular. If $d(x) = 1$

then $G'' = K_2$ is 2-chromatic. If $d(x) \geq 2$ and G is regular then the graph is 3-chromatic, in fact, $G = \Gamma_i$ for an $i \geq 2$. A proper 3-colouring can be found in the following way. Take any neighbor p of x, choose a neighbor p' of x which has the smallest number of common neighbors with p, and choose another neighbor p'' that has the smallest number of common neighbors with p'. Now the neighborhoods $N(x), N(p'), N(p'')$ cover $V(G'')$. If G is not regular then it must be a Vega graph and $d(x)$ is 3 or 4. If $d(x) = 3$ then $\{x\} \cup N(x)$ is a set of 4 vertices whose neighborhoods cover $V(G'')$ since G'' is maximal triangle-free. If $d(x) = 4$ then one of the 3-element subsets S of $N(x)$ has the property that the neighborhoods of $\{x\} \cup S$ cover $V(G'')$. So we easily get a proper 3-coloring of G'' in the regular case and a proper 4-coloring in the non-regular case. All this can be performed in time $\mathcal{O}(n^2)$.

Finally, coloring deleted twins of G' with the color of its remaining twin in the graph in reverse order of deletion we get a proper coloring of G' with the same number of colors. This coloring is also a proper coloring of its spanning subgraph G. So the total running time is $\mathcal{O}(M(n) + n^2 \log n)$.

Concerning the independence number it has been shown that computing the independence number in the class of triangle-free graphs with $\delta > (\frac{1}{4} - \varepsilon)n$ is NP-hard for every $\varepsilon > 0$ [2]. A corresponding hardness statement for the chromatic number of triangle-free graphs with linear degree $\delta > cn$ seems not to be known for any $c > 0$.

6 The very end.

Let us sum-up the results in the following way. Assume that c belongs to $[0, 1/2]$ and denote by χ_c the supremum of the chromatic number of a triangle-free graph with minimum degree at least cn. We have:

- For $c \in]0, 1/3[$, Hajnal proved $\chi_c = +\infty$
- For $c \in]1/3, 10/29[$, the bound is $\chi_c = 4$
- For $c \in]10/29, 2/5]$, Jin proved $\chi_c = 3$
- For $c \in]2/5, 1/2]$, Andrásfai, Erdős and Sós proved $\chi_c = 2$

So the following question remains open.

Problem 1 *What is the maximum chromatic number of a triangle-free graph with minimum degree equal to $n/3$?*

It can be any answer between 4 and $+\infty$. Looking at the question from the other side we can ask the following question:

Problem 2 *Find for every $t \geq 5$ the smallest function $s_t(n)$ such that every triangle-free t-chromatic graph has minimum degree $\delta \leq \frac{1}{3}n - s_t(n)$.*

We know that $0 \leq s_t(n) = o(n)$ for $t \geq 5$ but finding tight upper and lower bounds may be difficult.

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