Anisodyne: an anisotropic dynamo

or the easy dynamo

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The equations of dynamo action

For conducting, non-magnetic materials, Maxwell equations (1864) are the following, when radiative phenomena are not considered

$$\nabla \cdot \mathrm{B} = 0$$
$$\nabla \times \mathrm{B} = \mu \mathrm{j}$$
$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

Ohm's law tells us that $\mathrm{j} = \sigma(\mathrm{E} + \mathrm{u} \times \mathrm{B})$
Taking its curl, we obtain the induction equation

$$\frac{\partial \mathrm{B}}{\partial t} = \nabla \times (\mathrm{u} \times \mathrm{B}) + \frac{1}{\mu\sigma}\nabla^2 \mathrm{B}$$

The dynamo instability

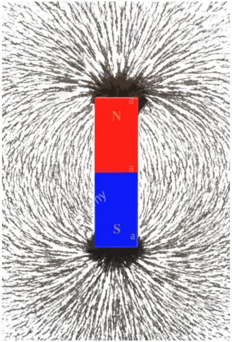
Starting with no magnetic field, a velocity field u is imposed on conducting materials. One then investigates whether a magnetic field develops spontaneously, or not.

$$\frac{\partial \mathsf{B}}{\partial t} = \nabla \times (\mathsf{u} \times \mathsf{B}) + \frac{1}{\mu\sigma}\nabla^2 \mathsf{B}$$

This is the problem of kinematic dynamo.

Anti-dynamo theorems (Cowling)

- A plane flow cannot maintain dynamo action
- In a sphere, a flow with no radial component cannot maintain dynamo action
- An axisymmetric magnetic field cannot be maintained by dynamo action



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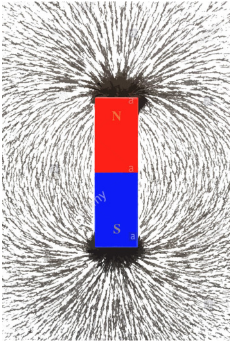
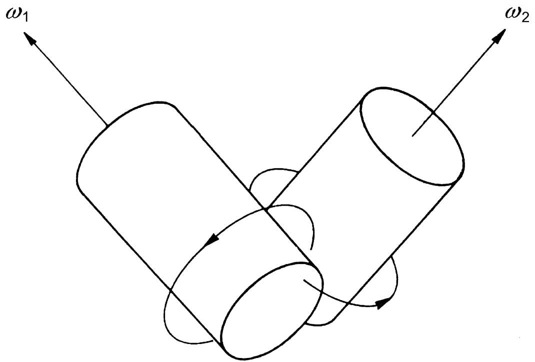
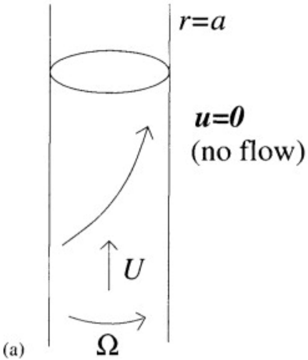


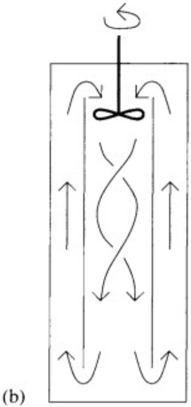
But these theorems do not apply when the electric conductivity tensor is anisotropic

Herzenberg's dynamo



Ponomarenko's dynamo

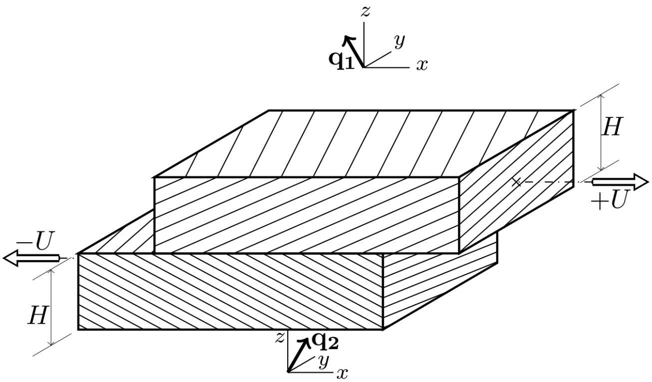




What can be done with

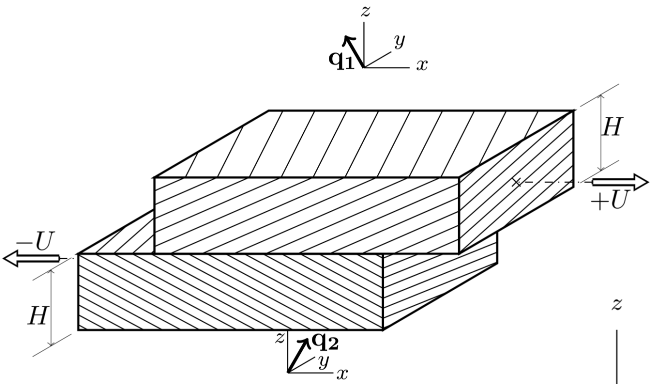
ANISOTROPY

A simple configuration



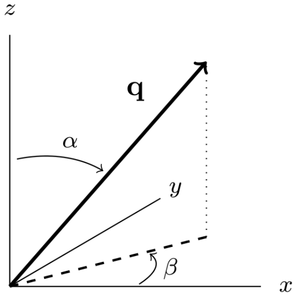
Two plates sliding on top of each other

A simple configuration

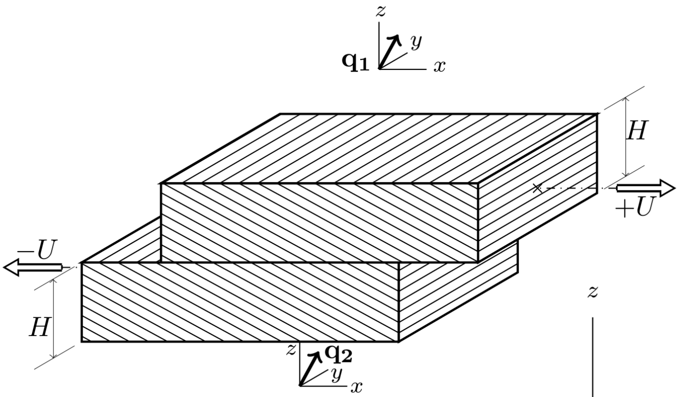


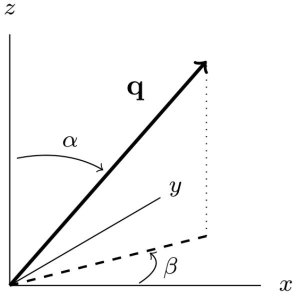
conductivity anisotropy

$$\Sigma_{ij} = \sigma_0 \delta_{ij} + (\sigma_1 - \sigma_0)\, q_i q_j$$



A very simple configuration

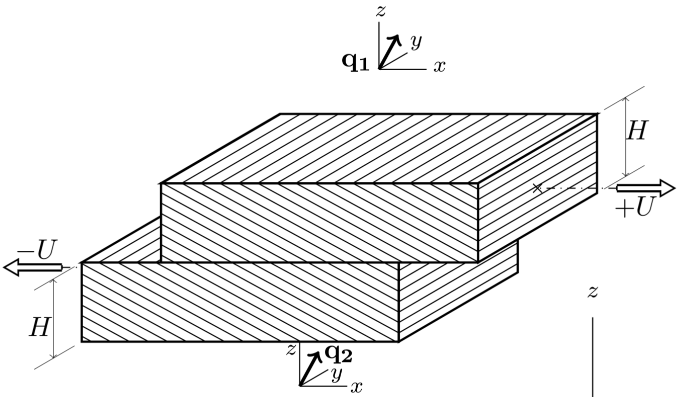




Uniform anisotropy, in the (x, z) plane

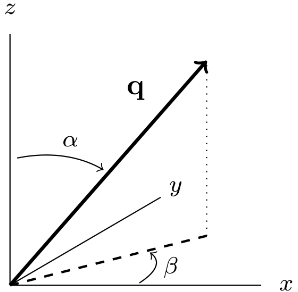
$\beta = 0$

A very simple configuration



Uniform anisotropy, in the (x, z) plane
$\beta = 0$

DYNAMO!



Equations

Resistivity tensor
$$R_{ij} = \frac{1}{\sigma_0}\delta_{ij} + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_0}\right) q_i q_j$$

Induction equation
$$\frac{\partial \mathrm{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathrm{u}\times\mathrm{B}) - \boldsymbol{\nabla}\times(\eta\cdot\boldsymbol{\nabla}\times\mathrm{B})$$

$$\text{with} \qquad \eta_{ij} = \frac{R_{ij}}{\mu_0} = \frac{1}{\mu_0\sigma_0}\left[\delta_{ij} + \eta_1 q_i q_j\right]$$

Poloidal – Toroidal Decomposition
$$\mathrm{B} = \boldsymbol{\nabla}\times(T\mathrm{e}_z) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(P\mathrm{e}_z))$$

Invariance in x and y
$$P = P(z)\exp\left(ik_x x + ik_y y + \gamma t\right)$$
$$T = T(z)\exp\left(ik_x x + ik_y y + \gamma t\right)$$

Equations

The most interesting case is when $k_x = 0$

$$\gamma P = \left(1 + \eta_1 q_x^2\right) \left[P'' - k_y^2 P\right] + i\eta_1 k_y q_x q_z T$$
$$\gamma T = T'' - k_y^2 T - \eta_1 k_y^2 q_z^2 T + i\eta_1 k_y q_x q_z \left[P'' - k_y^2 P\right]$$

On threshold, $\Re(\gamma) = 0$, we also have $\Im(\gamma) = 0$. Hence we get

$$T = i\frac{\frac{1}{\eta_1} + q_x^2}{k_y q_x q_z} \left[P'' - k_y^2 P\right]$$

and substituting for T brings a fourth-order equation for P

$$\boxed{P^{(4)} - [1 + \mathcal{N}]k_y^2 P'' + \mathcal{N} k_y^4 P = 0}$$

$$\text{with} \qquad \mathcal{N} = \frac{\frac{1}{\eta_1} + 1}{\frac{1}{\eta_1} + q_x^2}$$

Solution

$$P = a_1 e^{k_y z} + a_2 e^{-k_y z} + a_3 e^{\sqrt{\mathcal{N}} k_y z} + a_4 e^{-\sqrt{\mathcal{N}} k_y z}$$

with

$$\mathcal{N} = \frac{\frac{1}{\eta_1} + 1}{\frac{1}{\eta_1} + q_x^2}$$

Boundary conditions

- $P' + k_y P = 0$ at the top, $P' - k_y P = 0$ at the bottom
- $T = 0$ at top and bottom
- P, P', T are continuous across the interface between the plates
- continuity of the electric field E_y across the interface

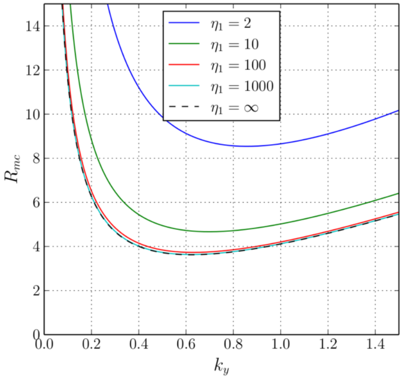
$$iT'_t + k_y U P_t = iT'_b - k_y U P_b$$

Solution

The condition for the existence of a non-zero eigenmode is obtained analytically

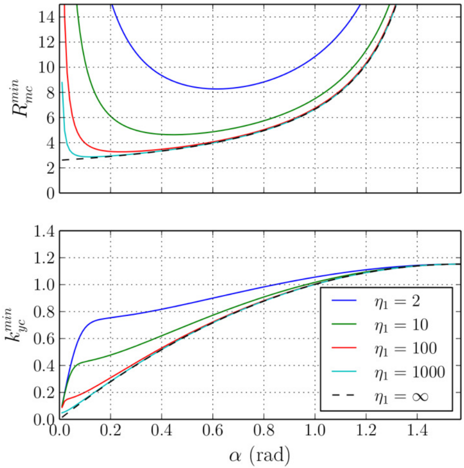
$$R_{mc} = \frac{k_y \frac{\frac{1}{\eta_1} + q_x^2}{q_x q_z} (\mathcal{N} - 1) \left[1 + e^{-2\sqrt{\mathcal{N}} k_y}\right]}{1 + e^{-2\sqrt{\mathcal{N}} k_y} \left[1 + \frac{1}{\sqrt{\mathcal{N}}}\right] - \frac{1}{\sqrt{\mathcal{N}}} - 2e^{-(1+\sqrt{\mathcal{N}}) k_y}}$$

$\alpha = 0.5$ rad
(28.65 °)



Solution in the limiting case $\eta_1 \to \infty$

$$\lim_{\eta_1 \to \infty} R_{mc} = \frac{q_z \frac{k_y}{q_x}\left[1 + e^{-2\frac{k_y}{q_x}}\right]}{1 + e^{-2\frac{k_y}{q_x}}\left[1 + q_x\right] - q_x - 2e^{-\left(1+\frac{1}{q_x}\right)k_y}}$$



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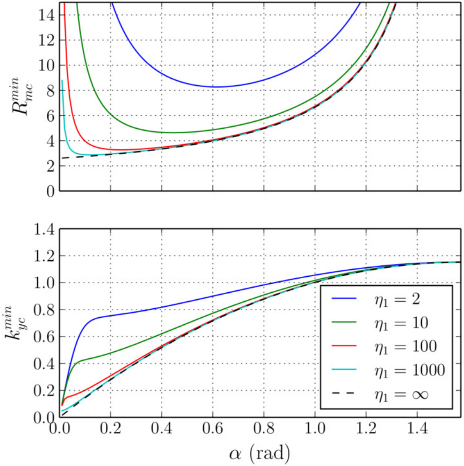
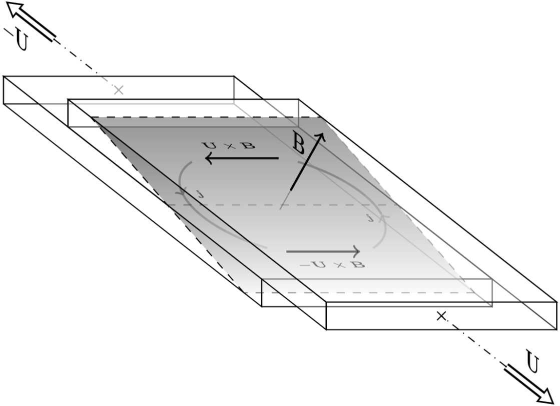
Minimum
$R_{mc} \simeq 2.609$



A bit of hand-waiving



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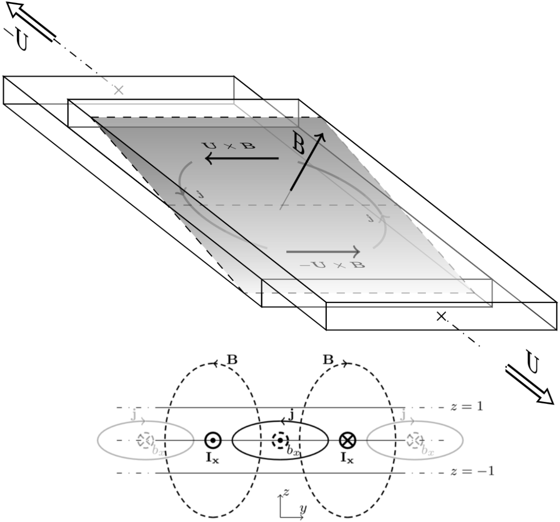
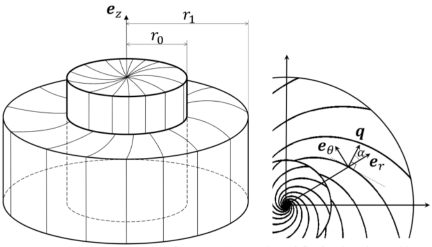
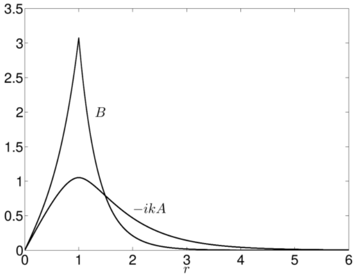


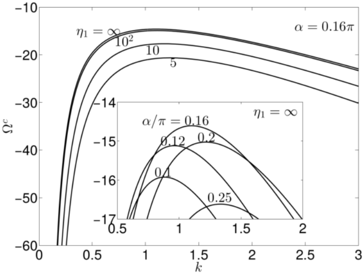
Axisymmetric version

A cylinder is rotating in an infinite domain. Electrical conductivity is anisotropic everywhere.

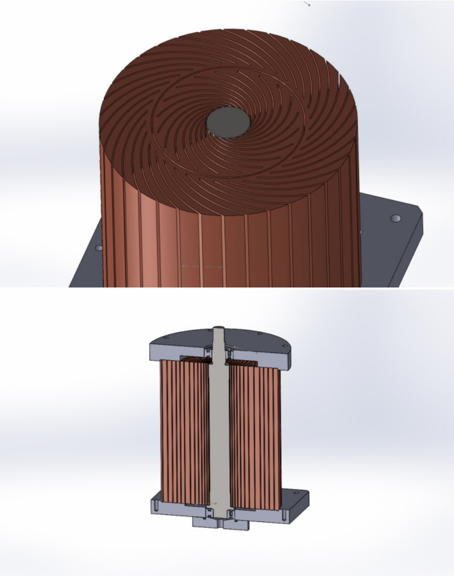


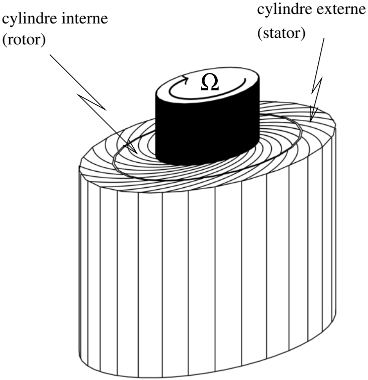
The eigenvalues and eigenvectors are obtained analytically, using Bessel functions





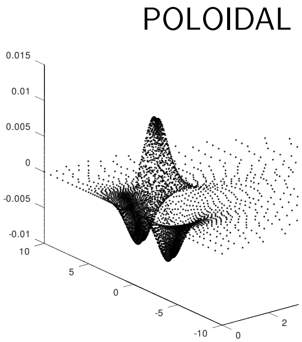
PLanning a cylindrical experiment

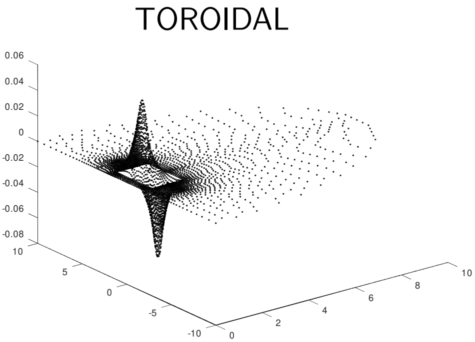


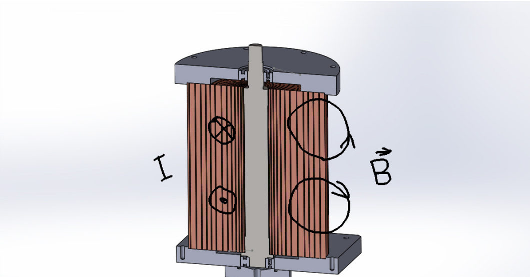


$\Phi = 170$ mm, $L = 205$ mm
100 Watt, 20 rpm

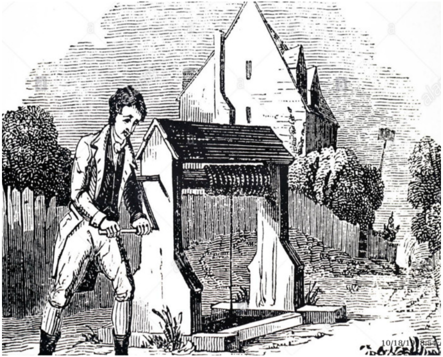
Eigenmode

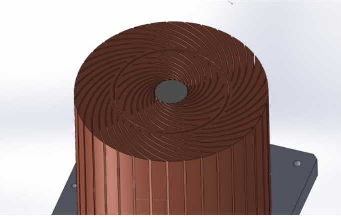

POLOIDAL
0.015
0.01
0.005
0
-0.005
-0.01
10
5
0
-5
-10
0
2


TOROIDAL
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
10
5
0
-5
-10
0
2
4
6
8
10


I
$\vec{B}$

A hand-powered dynamo





An axisymmetric magnetic field maintained by a dynamo

ANISODYNE

experiment

Building blocks

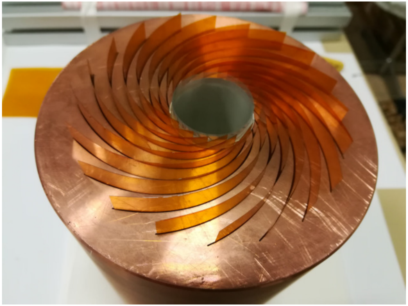
Stator diameter: 17 cm
Height: 20.5 cm

Copper CuA1, 100 % IACS, *i.e.* $5.8001 \times 10^{7}\,\Omega^{-1}\mathrm{m}^{-1}$

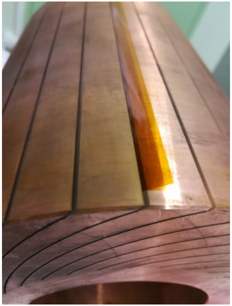
Cutting grooves

Electro-erosion
Width: 0.33 mm

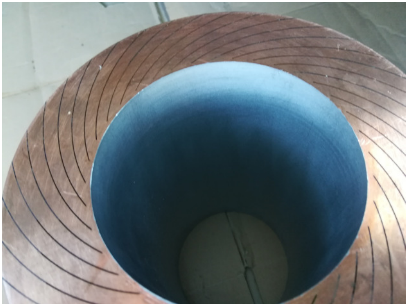
Insulating



film of kapton
and resin



Chrome plating

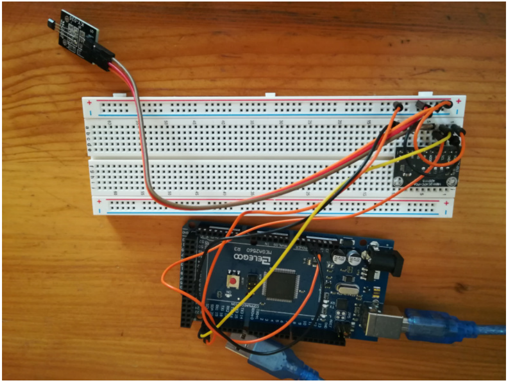


Thickness: 20 μm

Put a shaft

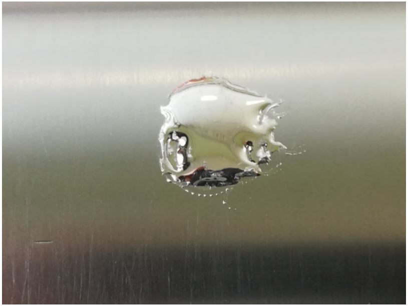
Connect to a crank

Install a few Hall probes

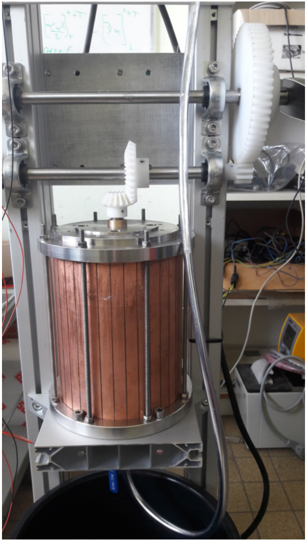


Arduino
16 bit A/D converter

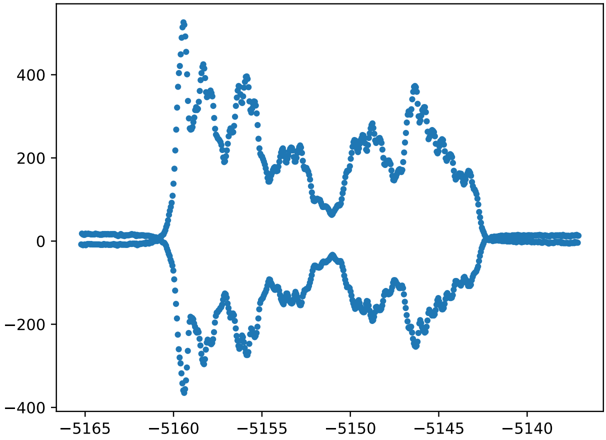
Fill the gap with Galinstan



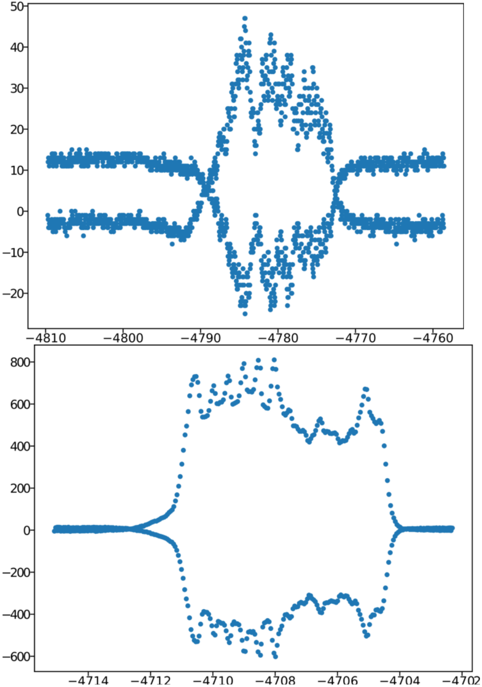
Gallium-Indium-Tin alloy
near eutectic
+ unknown stuff



The first run: March 23, 2021

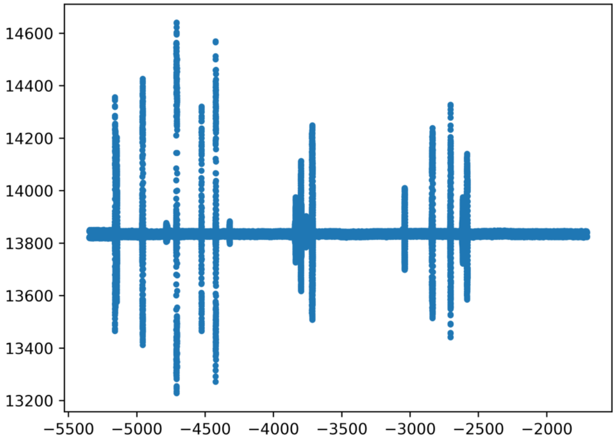


Run 3 / Run 4

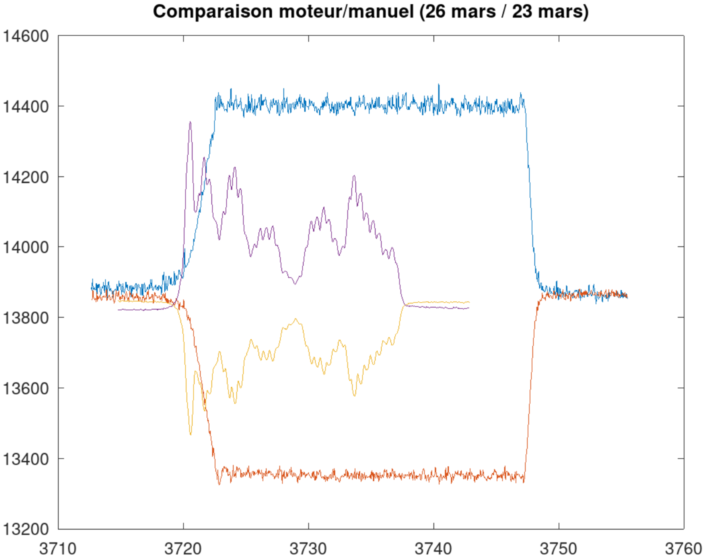

50
40
30
20
10
0
−10
−20
−4810
−4800
−4790
−4780
−4770
−4760
800
600
400
200
0
−200
−400
−600
−4714
−4712
−4710
−4708
−4706
−4704
−4702

All runs in the afternoon

JB, Renaud, Maëlis, Victor, Stéphanie, Yanick, Franck, myself...



A little help from an electric motor



Anisotropic or heterogeneous?

Cowling theorem applies in the heterogeneous case

Preliminary conclusions on anisotropy

- Dynamos generated by a plane flow are possible
- Axisymmetric dynamos exist
- Some have simple analytical solutions
- The critical magnetic Reynolds number is obtained analytically
- Anisotropic magnetic permeability also leads to dynamo
- 'Superfast" dynamo
- The experiment has been working for one week
- Version 2 is coming soon...