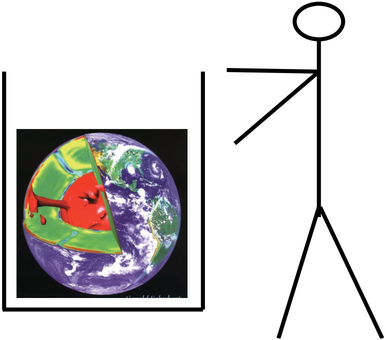
Dimensional analysis applied to the deep Earth



Maylis Landeau[1],

[1] Institut de Physique du Globe de Paris,

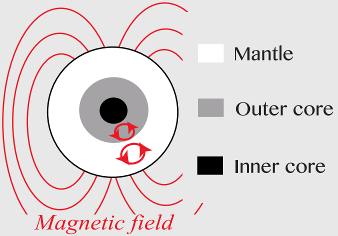
Les Houches, 2021

This session: practicals!

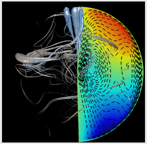
- A few introductory slides & **exercises**.
- These who are familiar with dimensionless numbers: help others!
- At least **one person familiar with dimensionless numbers** sitting next to these who do not use dimensional analysis on a regular basis.
- Gong sound at the end of each exercise.

Dimensional analysis: key to studying the dynamics of Earth’s interior

Earth's interior



Modelling



simulations

What ingredients to keep in simulations?

experiments

What fluids to use in experiments?

What is a dimensionless number?

- Dimensionless number = **ratio** of two physical parameters that have the same dimension (two **forces**, two **time scales**, two **length scales**, two **energy scales**).

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- Example : the **Reynolds number**,

$$Re = \frac{\text{inertia}}{\text{viscous forces}} = \frac{\text{viscous time}}{\text{advective time}} = \frac{UL}{\nu},$$

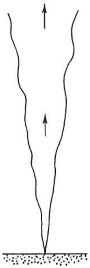
where U a typical velocity, L a typical length scale, ν the kinematic viscosity.

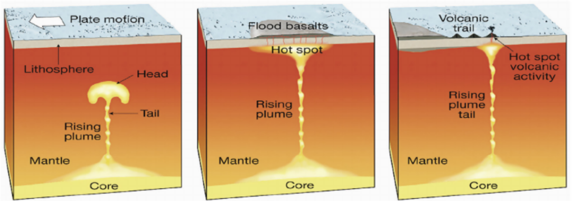
What is a dimensionless number?

Exercise 1: Reynolds number

- Compare inertia and viscous forces in the momentum conservation equation (see lecture by S. Labrosse) to show that $Re = U\,L/\nu$,
- Write the advective and viscous time scales to show that $Re = U\,L/\nu$.

The Reynolds number: Application to plumes





Mantle plume, from *Niu et al. 2017*

The Reynolds number: Application to plumes

$Re \gg 1$: turbulent plume when inertia dominates

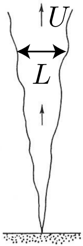
e.g. Kitamura & Sumita, 2011

The Reynolds number: Application to plumes

$Re \lesssim 1$: viscous plume when viscosity dominates

Davaille et al., 2010

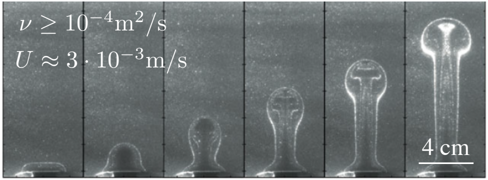
The Reynolds number: Application to plumes



Reynolds

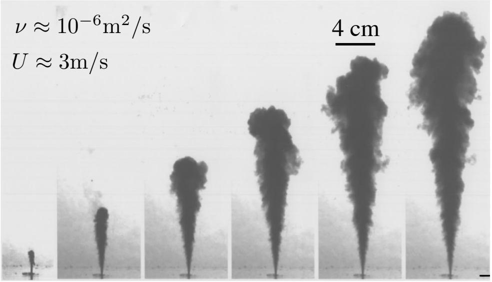
$$Re = \frac{UL}{\nu}$$

$Re \approx 1$ Laminar plume



Davaille et al., 2011

$Re \approx 9000$ Turbulent plume



Diez et al., 2003

Why are dimensionless numbers useful?

Why are dimensionless numbers useful?

1. To **reduce** the number of free **parameters**.

The Vaschy-Buckingham π-theorem:

If a physical system has n physical parameters, with k physical dimensions, the system can be entirely described by a set of $n - k$ dimensionless parameters.

Reduce the number of parameters

Exercise 3: Thermal convection
(see lectures of S. Labrosse, T. Alboussière)

Reduce the number of parameters

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- Under the Boussinesq approximation, the physical parameters that govern thermal convection are: the temperature difference ΔT, the layer depth H, the buoyancy force $\alpha \Delta T g$, the kinematic viscosity ν and the thermal diffusion k.
- Based on the π-theorem, how many dimensionless parameters govern the system? Write them.

Reduce the number of parameters

Exercise 3: Thermal convection
(see lectures of S. Labrosse, T. Alboussière)

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- Based on the π-theorem, how many dimensionless parameters govern the system? Write them.

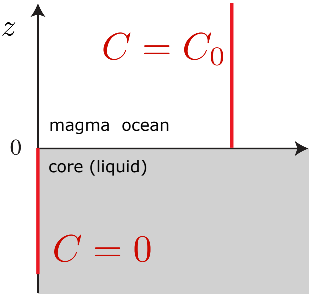
Answer:

- 2 dimensions (length, time),
- $4 - 2 = 2$ dimensionless numbers: $Pr = \dfrac{\nu}{k}$, $Ra = \dfrac{\alpha \Delta T g H^3}{\nu k}$

Reduce the number of parameters

Bonus exercise: Diffusion

In the early Earth, giant impacts might have entirely melted the mantle into a magma ocean. We consider the diffusion of light elements (Si, O, etc) from the mantle into the core (no convection).



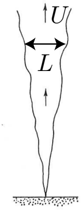
Based on the π-theorem, how many dimensionless parameters govern the system? What does that imply for the normalised concentration $C(z, t)/C_0$?

Why are dimensionless numbers important ?

2. They tell you what **forces dominate** $\rightarrow$ the **dynamical regime**.

Control the dynamical regime

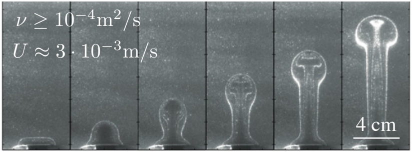
Ex. 4: Viscous or turbulent plumes in mantle & outer core?



Reynolds

$$Re = \frac{UL}{\nu}$$

$Re \approx 1$ Laminar plume



Davaille et al., 2011

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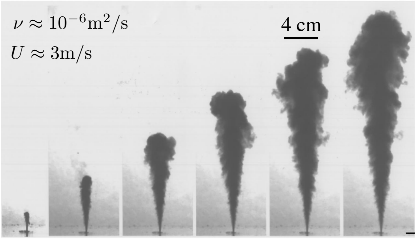
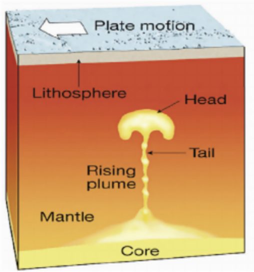


Diez et al., 2003

Control the dynamical regime

Mantle plume



$U \sim \mathrm{cm/yr}$

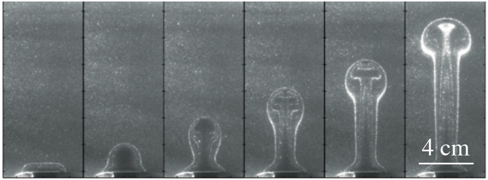
$\nu \sim 10^{18} \mathrm{m}^2/\mathrm{s}$

$L \sim 100 - 1000\mathrm{km}$

$Re \sim 10^{-22}$

→

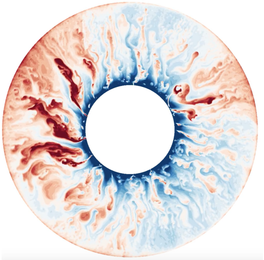
Laminar plume



Davaille et al., 2011

Control the dynamical regime

Outer core plume



Schaeffer et al. 2017

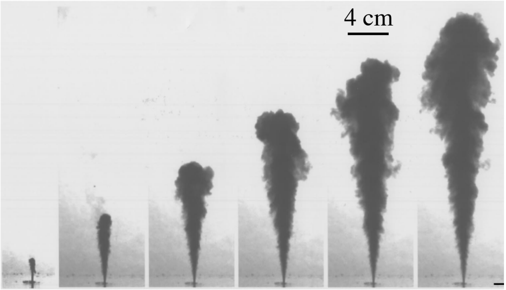
$\nu \sim 10^{-7} \mathrm{m}^2/\mathrm{s}$

$U \sim 5 \cdot 10^{-4} \mathrm{m/s}$

$L \sim 1000 \mathrm{km}$

$\longrightarrow$ $Re \sim 10^9$

Turbulent plume



Control the dynamical regime

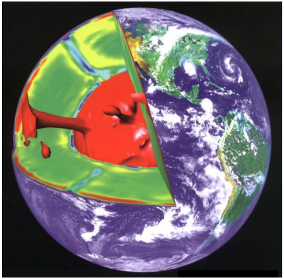
Exercise 5: Convection in outer core vs mantle

Based on what you learned in previous lectures (T. Alboussière, S. Labrosse, N. Schaeffer), estimate ratios of the different forces for convection in the outer core and in the mantle.

What are the important forces for both cases?

Control the dynamical regime

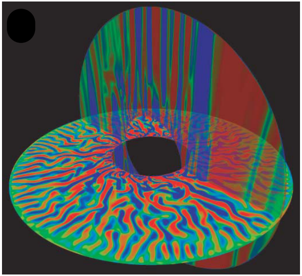
Mantle



Schubert, Turcotte, Olson, 2001

Viscous flow,
governed by
viscosity & buoyancy

Outer core



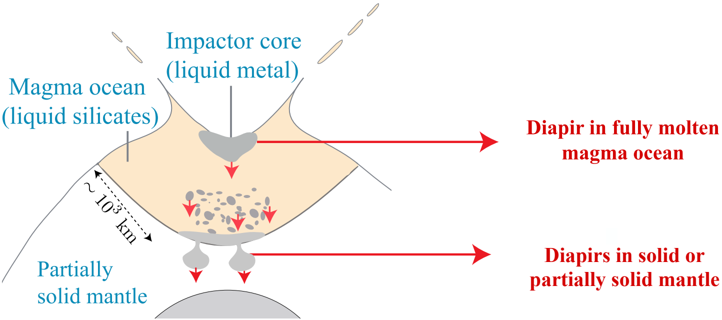
Kageyama et al. 2008

Turbulent flow, governed by
rotation, buoyancy, inertia
(& Lorentz force)

Control the dynamical regime

Exercise 6: Core-mantle differentiation

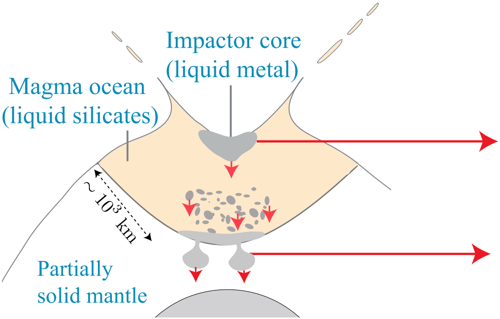
Core-mantle differentiation



Control the dynamical regime

Exercise 6: Core-mantle differentiation

Core-mantle differentiation



Questions:

Dimensionless numbers ?

Dynamical regime ?

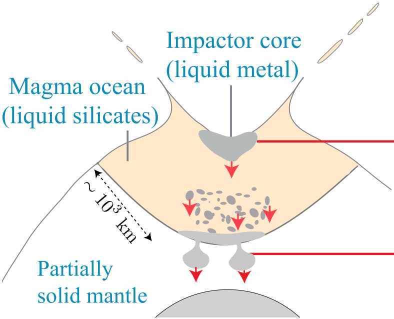
Diapir in fully molten magma ocean

Diapirs in solid or partially solid mantle

Control the dynamical regime

Exercise 6: Core-mantle differentiation

Core-mantle differentiation



Questions:

Dimensionless numbers ?

Dynamical regime ?

Diapir in fully molten magma ocean
Turbulent

Deguen et al. 2014
Landeau et al. 2014
Lherm & Deguen, 2018
Wacheul & Le Bars, 2018

Diapirs in solid or partially solid mantle
Viscous

Fleck et al 2018
Monteux et al., 2009

Why are dimensionless numbers important?

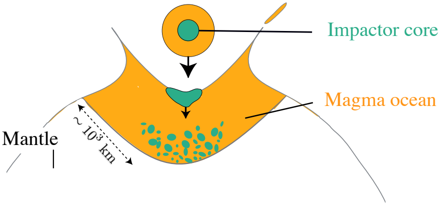
3. A necessary step to **scale down analogue experiments**.

Scaling-down experiments

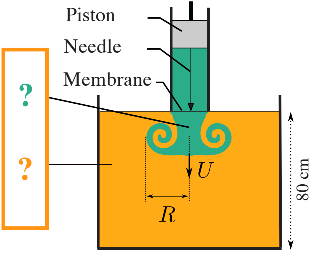
Exercise 7: Core-mantle differentiation

What fluids to match the dynamical regime of metal fragmentation in magma oceans?

Earth's differentiation



Experiment



Scaling-down experiments

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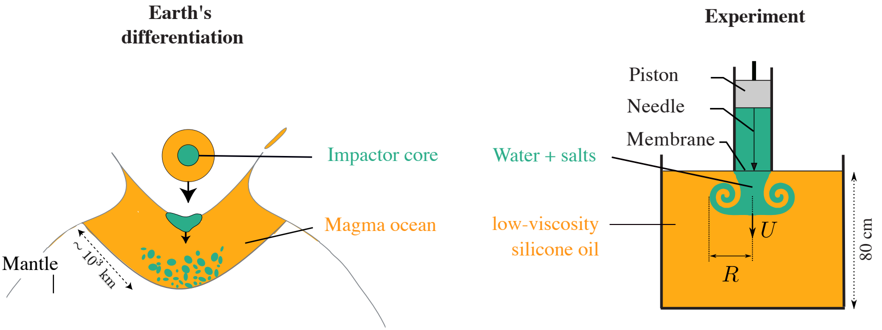


Scaling-down experiments

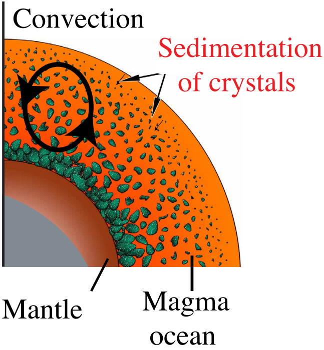
Landeau et al., AGU 2020.

Why are dimensionless numbers important?

4. A necessary step to obtain **scaling laws**.

Scaling laws: cooling of magma oceans

Cooling rate ?



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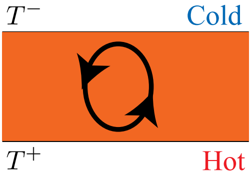
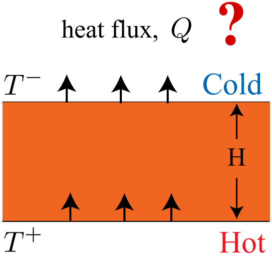
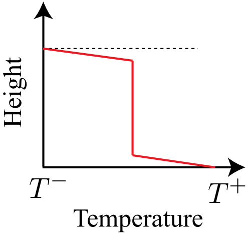
Analog model



Scaling laws: cooling of magma oceans



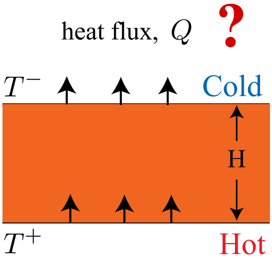


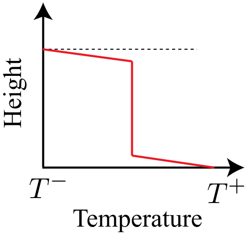
$$Ra = \frac{\alpha g \Delta T H^3}{\nu \kappa} \sim \frac{\text{Diffusion time}^2}{\text{Advection time}^2}$$

$$Nu = \frac{Q}{\rho C_p \kappa \Delta T / H} \sim \frac{\text{Convective flux}}{\text{Conductive flux}}$$

Scaling laws: cooling of magma oceans

Fully-liquid magma ocean:	$Ra \sim 10^{27}$	Not achievable in experiments or simulations !





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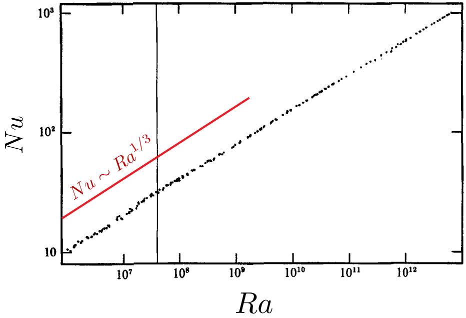
Scaling laws

Exercise 8: Cooling of a magma ocean

Use the scaling law to estimate the cooling time of a magma ocean.



Castaing et al. 1989

Theory & experiments: $Nu \approx 0.1 Ra^{1/3}$

Scaling laws

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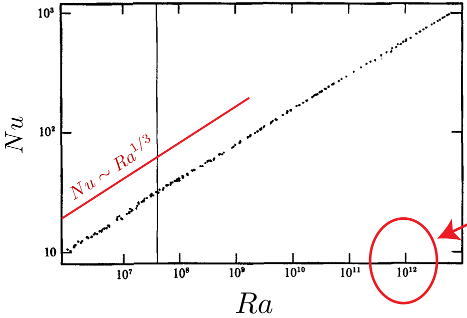


Castaing et al. 1989

Extrapolation of scaling to fully-liquid magma oceans

$Ra \sim 10^{27}$

$Nu \sim 10^8$

$Q \sim 10^5 \mathrm{W/m^2}$

Cooling time $\sim$ 1000 yr

Theory & experiments: $Nu \approx 0.1 Ra^{1/3}$

Scaling laws

Exercise 9: Maximum elevation on planetary objects

Why is the Earth (left) spherical while asteroids like Vesta (right) have a more irregular shape?

Scaling laws

Exercise 9: Maximum elevation on planetary objects

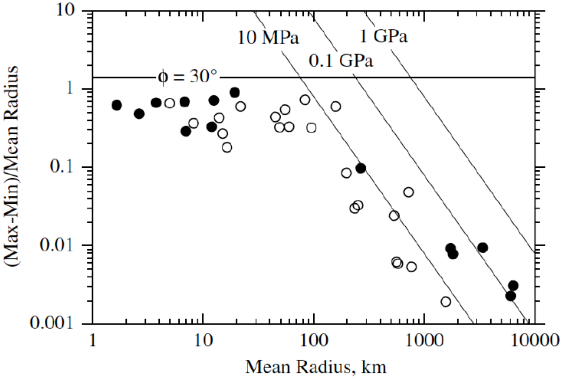
Why is the Earth (left) spherical while asteroids like Vesta (right) have a more irregular shape?

- Hint:
  Compare the pressure below a mountain to the yield strength Y.
  For rocks, $Y \sim 100$ MPa.
  The gavitational acceleration at the planet surface is $g = \dfrac{4\,\pi\,\rho\,G\,r}{3}$,
  where r is the planet radius.

Scaling laws

Exercise 9: Maximum elevation on planetary objects
Why is the Earth spherical while Vesta is irregular?

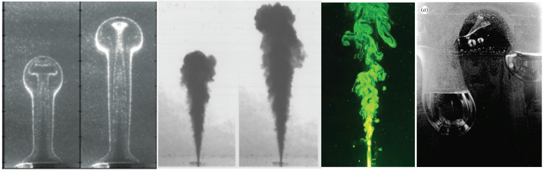
From Melosh, 2011:



Summary

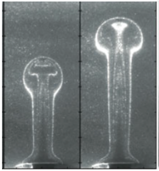
Dimensional analysis & dimensionless numbers are important to:

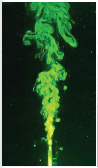
1. Reduce the **number of parameters!**
2. Predict the **dynamical regime** in the core and mantle: choose what **forces** to include in simulations and what **processes** in experiments.
3. **Scale down analogue lab experiments**,
4. Extrapolate quantitative **scalings** to the deep Earth.



References

- For beginners: *Les lois d'échelle*, Thomas Séon
- To go further: *Similarity, Self-Similarity, and Intermediate Asymptotics*, Barenblatt.
- *Laboratory experiments on the dynamics of the core*, P. Olson, 2011.
- *Dynamical similarity and density (non-) proportionality in experimental tectonics*, N. Ribe, A. Davaille, 2013.





Working groups

- Ideas of geophysical processes you would like to model?

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- Examples: convection in mantle or core, convection in icy satellites, planetary impacts & craters, core-mantle differentiation, dynamic topography, cooling of magma oceans, convection & stratification in the outer core, subduction, etc.

Working groups

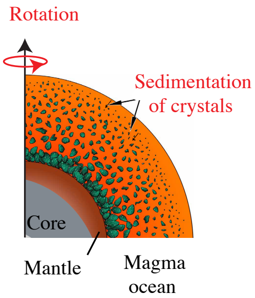
- Ideas of geophysical processes you would like to model?
- Examples: convection in mantle or core, convection in icy satellites, planetary impacts & craters, core-mantle differentiation, dynamic topography, cooling of magma oceans, convection & stratification in the outer core, subduction, etc.
- Choose 4 or 5 systems $\rightarrow$ 4 or 5 working groups.
- For each group:
  Find the dimensionless numbers,
  Estimate them.
  What forces dominate?
  Propose an analogue experiment or a numerical system.

Extra slides

Why are dimensionless numbers important ?

1. They control the **morphology of the flow** and the dynamical regime.

Convection in magma oceans



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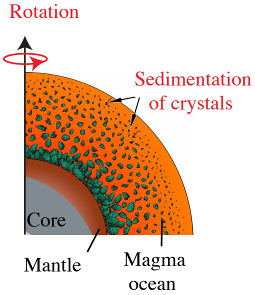
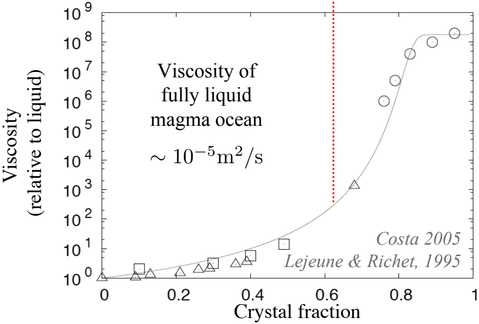


Low crystal fraction
↓
Turbulent flow
∼ outer core
↓
rotation, buoyancy, inertia

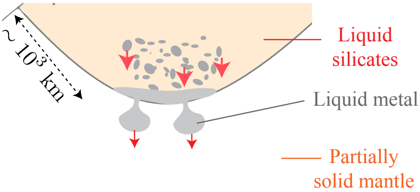
High crystal fraction
↓
Ecoulement visqueux
∼ solid mantle
↓
buoyancy, viscosity



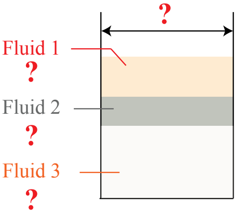
Scaling-down experiments

Exercise 7: Core-mantle differentiation

Core-mantle differentiation
in viscous magma ocean



Analog experiment



Olson & Weeraratne, 2008

What fluids can we use to match the dimensionless numbers of the geophysical system?

Scaling-down experiments

Exercise 7: Core-mantle differentiation

Core-mantle differentiation
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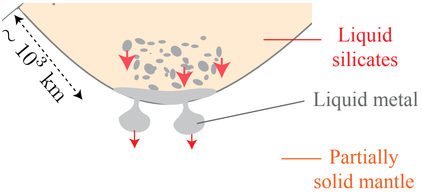
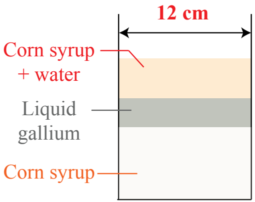


Analog experiment



Olson & Weeraratne, 2008

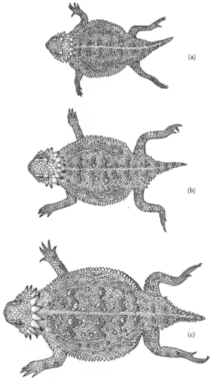
What fluids can we use to match the dimensionless numbers of the geophysical system?

Self-similarity

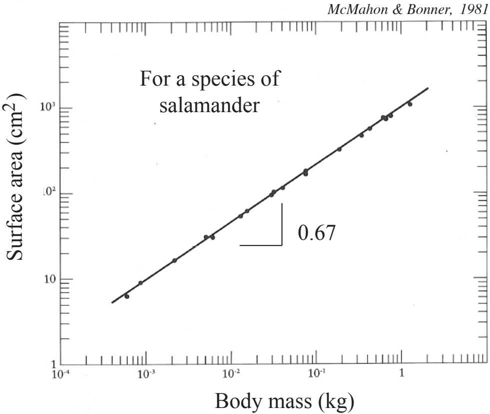
- **Self-similarity** : most often it refers to scale similarity in time.

$$f(x,t) = A(t)F\left(\frac{x}{\delta(t)}\right)$$

Example : Isometry within a species

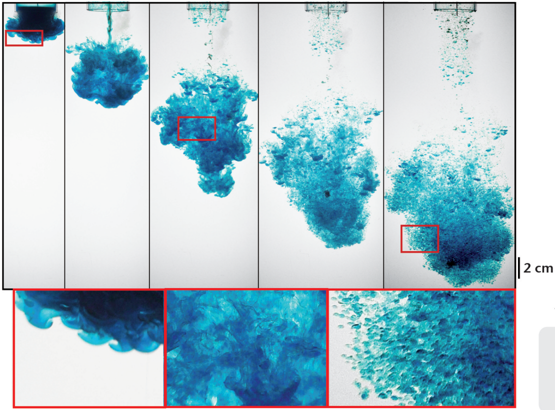


Lezards of different ages



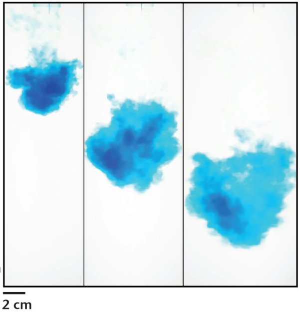
Self-similarity. Example: Turbulent thermal of metal in magma oceans

Immiscible fluids



Immiscible turbulent thermals

Miscible fluids



Morphologically similar to miscible experiments

Self-similarity. Example: Turbulent thermals

Model

Turbulent entrainment concept

Morton et al. (1956),
Taylor (1945, public since 1996)

$$\frac{d(\rho V)}{dt} = \alpha \rho_a u S$$

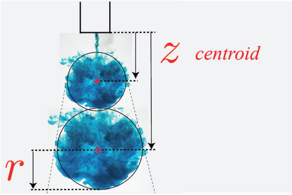
entrainment coefficient

+ **Self-similarity**

r: cloud radius

$$\boxed{\frac{dr}{dz} = \alpha}$$

Experiments



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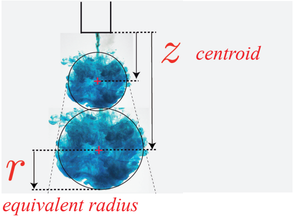

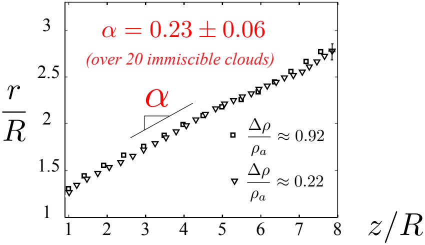

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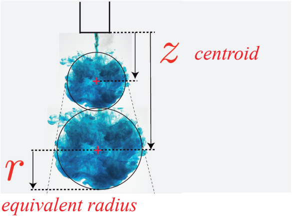
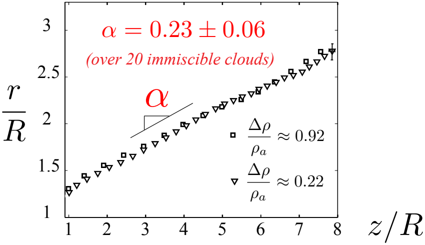
$$\frac{\text{Entrained silicate volume}}{\text{Impactor core volume}} = \left(1 + \alpha \frac{z}{R}\right)^3 - 1$$

Experiments





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And from dimensional analysis ?
(on the blackboard)

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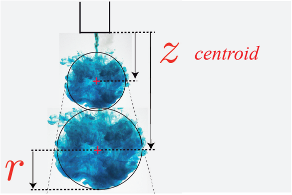
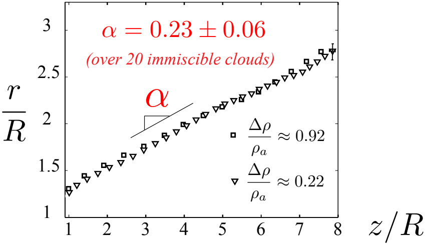
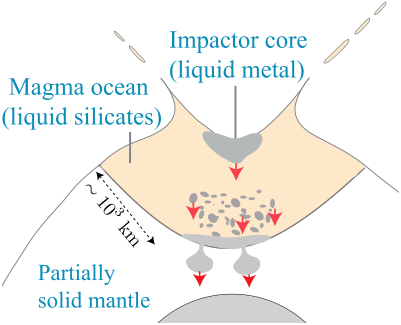




1-Reduce the number of parameters

Exercise 4: Metal diapir during Earth's differentiation

A spherical metal diapir of radius R and density ρ falls in a viscous mantle of density ρ_a and viscosity ν_a under gravity g.



Based on the π-theorem, how many dimensionless parameters govern the system? Write them.

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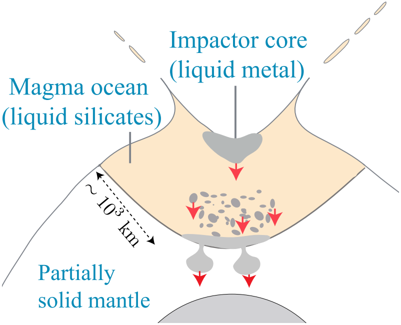


Based on the π-theorem, how many dimensionless parameters govern the system? Write them.

Answer:

5 physical parameters,

3 dimensions (length, mass, time),

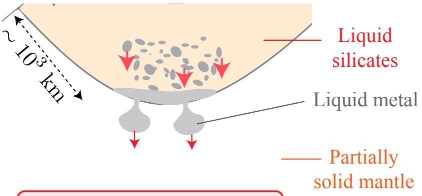
$5 - 3 = 2$ dimensionless numbers:

$$\frac{\rho}{\rho_a}, \quad \frac{(\rho - \rho_a) g R^3}{\rho_a \nu_a^2}$$

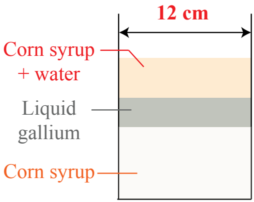
3-Scaling-down experiments

Exercise 7: Core-mantle differentiation

Core-mantle differentiation in viscous magma ocean



Analog experiment



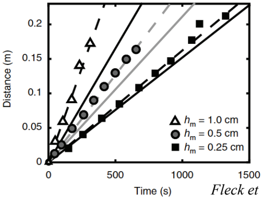
Olson & Weeraratne, 2008

$R \sim 100$ km

$U \sim$ mm/s

time ~ 100 yr

$$U \sim \frac{\Delta\rho g R^2}{\rho\nu} \longleftarrow$$



Fleck et al. 2018