# DETECTING OSCILLATING SINGULARITIES IN MULTIFRACTAL ANALYSIS: APPLICATION TO HYDRODYNAMIC TURBULENCE

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## ABSTRACT

Multifractal analysis describes data as a collection of singularities. However, its classical formulation does not account for their possibly oscillating nature, while, in a number of applications, distinguishing between oscillating and non oscillating singularities may significantly enrich the analysis. This is notably the case in hydrodynamic turbulence, of interest here, where two different important heuristic models contradictorily lead to predict the existence or absence of oscillating singularities. This contribution proposes a wavelet Leader oscillation formalism enabling to evidence the presence of oscillating singularities in real data. It is first validated on synthetic data both with and without oscillating singularities and second applied to high quality 1D velocity turbulence data. This constitutes the first quantitative evidence against the presence of oscillating singularities in turbulence data.

***Index Terms***— Oscillating singularity, multifractal formalism, wavelet Leaders, hydrodynamic turbulence.

## 1. INTRODUCTION

**Multifractal analysis and oscillating singularities.** Multifractal analysis is now considered as a standard tool in Signal Processing, aiming at characterizing the fluctuations of local regularity in time and space, the roughness, of a given signal. This local regularity, at position $t_0$, is essentially measured by the Hölder (or singularity) exponent, $h(t_0) \geq 0$, obtained by comparing $X$ around $t_0$ to a locally singular behavior:

$$|X(t) - X(t_0)| \simeq C|t - t_0|^{h(t_0)}, |t - t_0| \to 0. \quad (1)$$

Multifractal analysis describes globally and geometrically the fluctuations of $h(t)$ via the multifractal (or singularity) spectrum $D(h)$, consisting of the Hausdorff dimension of the set of points $t_0$ having the same Hölder exponent $h$: $h(t_0) = h$. For thorough and detailed introductions to multifractal analysis, the reader is referred to e.g., [1, 2]. In essence, multifractal analysis hence describes the data $X(t)$ as a collection of singularities. Such singularities are potentially superimposed to smooth (polynomial-like) behaviors, disregarded by multifractal analysis. However, this heuristic reading of multifractal analysis, commonly underlying the intuition leading to its use on real-life data in applications, does not fully account for the variety of singular behaviors that can actually be encountered in data. Indeed, stating that $X$ has Hölder exponent $h$ at $t_0$ can actually correspond to any of the following collection of oscillating singularity behaviors:

$$X(t) - X(t_0) \simeq C|t - t_0|^h \sin\left(\frac{1}{|t - t_0|^\beta}\right), \beta \geq 0. \quad (2)$$

The case $\beta > 0$ corresponds to the so-called chirp-type singularity, while $\beta = 0$ actually only constitutes a particular case, referred to as a cusp-type singularity. Such behaviors are illustrated in Fig. 1. By definition, the multifractal spectrum $D(h)$ concentrates on the regularity exponents but misses the potentially oscillating nature of the singularities existing in data. Nevertheless, the detection and characterization of such chirp type singularities is a key-issue in a number of applications ranging from gravitational wave detection to hydrodynamic turbulence, of interest here.

**Oscillating singularities and hydrodynamic turbulence.** Hydrodynamic turbulence refers to the analysis of non laminar fluid flows [3]. It has long been accepted that velocity fluctuations in turbulent flows are well described by their multifractal spectra. The seminal work of Mandelbrot [4] initiated these analyses and formalized their relations to the heuristic Richardson energy cascade model (cf. e.g., [3]): Energy is injected by flow at a coarse (or integral) scale and dissipated at a fine (viscosity) scale, the non linear term of the Navier-Stokes equation ensuring the energy transfer from coarse to fine scales. This energy transfer is modeled by multiplicative cascades, known as the very (and for long the only) paradigm of stochastic processes with well controlled multifractal spectra. It has recently been conjectured [5] that for multiplicative cascades, used to model scaling properties in turbulence, singularities can only consist of cusps. However, independently from the Richardson energy cascade model and in parallel, qualitative analyses of the Navier-Stokes equation predict that the competition between the advective non linear term and the Laplacian dissipative term may induce a vortex stretching mechanism and hence oscillating singularity behavior (cf. [3, 6]). This generated a number of research works proposing to model and describe turbulence data by the superimposition in time and space of oscillating singularities (cf. [7] and references therein). These two heuristic models are to some extent hence in contradiction and the use of statistical signal processing tools that enable to detect the presence of oscillating singularities in turbulence velocity data could enable to shed light on this challenging issue.

**Goals, contributions and outline.** So far, the actual characterization of multifractal spectra issued from oscillating singularities on real data only received preliminary attempts [8, 9, 5]. In this context, the goal of this contribution is to propose an extension of the classical wavelet Leader multifractal formalism, enabling to measure $D(h)$, to an *oscillation formalism*, permitting to evidence the existence (or not) of oscillating singularities in data. This construction, relying on fractional integration, is introduced, detailed and analyzed (cf. Section 2), and validated on known synthetic data (cf. Section 3). It is then shown at work on a set of velocity data, collected from well controlled high Reynolds turbulence experiments, and considered as reference data in hydrodynamic turbulence studies (cf. Section 4).
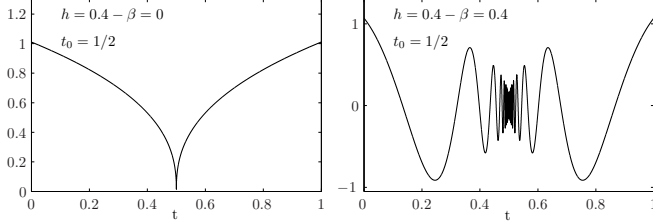
**Fig. 1**. Cusp vs. Chirp singularities.

## 2. WAVELET LEADER OSCILLATION SPECTRUM

### 2.1. Singularity and Oscillations Exponents and Spectra

**Regularity Hölder exponent.** The local regularity of a bounded function $X(t)$ is quantified by comparison against a local power law behavior: $|X(t) - P_{X,t_0}(t)| \leq C|t - t_0|^\alpha$, where $P_{X,t_0}$ is a polynomial of degree less than or equal to the integer part of $\alpha \geq 0$ and $C > 0$. Technically, the Hölder exponent of $X$ at $t_0$, $h(t_0)$, is defined as the supremum of the values of $\alpha$ such that the inequality above holds. Multifractal analysis then characterizes globally and geometrically the fluctuations of $h$ along $t$, through the multifractal spectrum $D(h)$, consisting of the Hausdorff dimension of the set of points $t_0$ having the same Hölder exponent $h$: $h(t_0) = h$. For chirp singularities such as (2), the Hölder exponent takes the same value $h$, whatever the value of $\beta > 0$. $D(h)$ hence conveys no information on the oscillating nature of the singularities.

**Fractional Integration.** Let $X$ denote a $L^2(\mathbb{R})$ function and $\tilde{X}$ its Fourier transform. The fractional integral of order $s > 0$ of $X$, $X^{(-s)}$, is defined via its Fourier transform $\tilde{X}^{(-s)}$ as:

$$\tilde{X}^{(-s)}(\nu) = (1 + |\nu|^2)^{-s/2} \tilde{X}(\nu). \tag{3}$$

**Oscillating exponents.** The fractional integration, of integer order $n$, of the function $X(t) = C|t|^h \sin(1/|t|^\beta)$ has a Hölder exponent, in $t = 0$, of the form $h' = h + (1+\beta)n$, where $h$ is the Hölder exponent of $X$ at $t = 0$. This particular chirp-type case suggests to propose a definition of the local oscillation exponent $\beta(t_0)$ as:

$$\beta(t_0) = \lim_{s \to 0} \frac{\partial}{\partial s}(h^{-s}(t_0)) - 1, \tag{4}$$

where $h^{-s}(t_0)$ denotes the Hölder exponent of $X^{(-s)}$ at $t_0$. As for the Hölder exponent, one can define the oscillation spectrum $\mathcal{D}(\beta)$, as the Hausdorff dimension of the set of points where $\beta(t_0) = \beta$.

It is crucial to note that the definition of the oscillation exponent is closely tied to that of the regularity exponent. The use of a regularity definition other than the Hölder exponent would hence yield a different oscillation measure.

### 2.2. Multifractal Formalisms

The so-called multifractal formalism consists of a practical procedure that enables to estimate, from real data $X$, their multifractal spectrum $D(h)$, that describes the range of $h$ actually observed. Recently, it has been shown that this formalism must be based on wavelet Leaders [2]. This section aims at extending this procedure to the practical measure of the oscillation spectrum $\mathcal{D}(\beta)$.

**Wavelet coefficients and Leaders.** Let $d_X(j,k) = \langle X, \psi_{j,k} \rangle$ denote the ($L^1$-normalized) discrete wavelet transform coefficients of $X$, where $j$ refers to the analysis scale ($a = 2^j$) and $k$ to time ($t = 2^j k$), and where $\psi$ denotes the oscillating reference pattern referred to as the mother wavelet, and $\psi_{j,k}(t) = 2^{-j}\psi(2^{-j}t - k)$ the dilated and translated wavelets [10]. The Wavelet Leaders $L_X(j,k)$ are defined as the local supremum of wavelet coefficients taken within a spatial neighborhood over all finer scales [2]: $L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_X(\lambda')|$, where $\lambda_{j,k} = [k2^j, (k+1)2^j)$ and $3\lambda_{j,k} = \bigcup_{m\{-1,0,1\}} \lambda_{j,k+m}$.

**Hölder Formalism.** The following procedure, referred to as the multifractal formalism (because originally based on thermodynamic formalism used in statistical physics [3]) can be used in order to obtain $D(h)$. The structure functions $S_L(2^j, q) = \frac{1}{n_j}\sum_k L_X(j,k)^q$ exhibit power-law behavior in the limit of fine scales $a = 2^j \to 0$,

$$S_L(2^j, q) \simeq S_0(q)2^{j\zeta_L(q)}. \tag{5}$$

The $\zeta_L(q)$ are called the scaling exponents and their Legendre transform $D_L(h) = \inf_q(1 + qh - \zeta_L(q))$ provides an upper bound of the multifractal spectrum, i.e., $D_L(h) \geq D(h)$. For further theoretical details on multifractal analysis and wavelet Leader formalism, the reader is referred to, e.g., [1] and [2], respectively.

**Wavelet Leaders and Oscillation Exponents.** The derivation of the oscillation formalism follows the following heuristic: Let $X$ be characterized at $t_0$ by an oscillating singularity with exponents $h(t_0), \beta(t_0)$. The wavelet leaders of $X$ at $t_0$ satisfies $L_X(j,k) \simeq C2^{jh}, 2^j \to 0$, for $j, k$, such that $2^{-j}k \simeq t_0$. The wavelet leaders of $X^{(-s)}$, at $t_0$ satisfies $L_{X^{(-s)}}(j,k) \simeq C2^{j(h+s(1+\beta))}, 2^j \to 0, s \to 0$, for $j, k$, such that $2^{-j}k \simeq t_0$. Therefore, the $\beta$-Leaders, defined as

$$B^{(-s)}(j,k) = 2^{-j}(L_{X^{(-s)}}(j,k)/L_X(j,k))^{1/s}, \tag{6}$$

behave as (for $(j,k)$, such that $2^{-j}k \simeq t_0$):

$$B_X^{(-s)}(j,k) = 2^{j\beta(t_0)}, \ 2^j \to 0, \ s \to 0. \tag{7}$$

**Oscillation Formalism.** Once multiresolution quantities such as the $B^{(-s)}(j,k)$ above are obtained, the key thermodynamic argument that leads to the obtention of the multifractal spectrum for Hölder exponents [3] can be rephrased mutatis mutandis. The new structure functions

$$S_\beta(2^j, q) = \frac{1}{n_j}\sum_k B_X^{(-s)}(j,k)^q \tag{8}$$

exhibit power-law behaviors in the limit of fine scales $a = 2^j \to 0$ and $s \to 0$,

$$S_\beta(2^j, q) \simeq S_0(q)2^{j\zeta_\beta(q)}. \tag{9}$$

The Legendre transform $D_\beta(\beta) = \inf_q(1 + q\beta - \zeta_\beta(q))$ of the exponents $\zeta_\beta(q)$ is expected to provide an upper bound of the oscillation spectrum: $D_\beta(\beta) \geq \mathcal{D}(\beta)$. The study of this conjecture is beyond the scope of the present contribution, that focuses on its practical use in turbulence. It will be detailed in [11]. However, its formulation follows from the combined use of the thermodynamic formalism and of fractional integration. In Section 3, it is validated by application of the oscillation formalism to a variety of synthetic processes whose oscillation spectrum is known theoretically.

**Pseudo-fractional Integration.** The numerical computation of the fractional integration $X^{(-s)}$ of $X$, that can be complicated in practice can actually be avoided [12]: For $X$ characterized at $t_0$ by an oscillating singularity with exponents $(h(t_0), \beta(t_0))$, the pseudo-wavelet Leaders $L_X^{(-s)}(j,k)$, computed as leaders of the modified wavelet coefficients $2^{js}d_X(j,k)$, exhibit a power law behavior, $L_X^{(-s)}(j,k) \simeq C2^{j(h+s(1+\beta))}, 2^j \to 0, s \to 0$, that reproduces that of the wavelet Leaders $L_{X^{(-s)}}(j,k)$ computed from the true fractional integration $X^{(-s)}$ of $X$, and hence exactly yields the same multifractal spectrum as that of $X^{(-s)}$. The $L_X^{(-s)}(j,k)$ are therefore referred to as the pseudo-fractional integration wavelet

Leaders. In the practical implementation of the oscillation formalism, the $L_X^{(-s)}(j,k)$, that are straightforward to compute, naturally replace the $L_{X^{(-s)}}(j,k)$ in the definition of $B^{(-s)}(j,k)$.

**Practical choice of $s$.** Practically the limit $s \to 0$ cannot be taken. The actually chosen value results from the following practical trade-off: a too small $s$ yields practically unstable and higly variable results, as can be anticipated from the use of the power $1/s$ entering the definition of the $B^{(-s)}(j,k)$ ; for too large $s$, the linear behavior in $s$ founding the definition of $\beta$ (cf. Eq. (4)) may no longer be valid hence potentially inducing a bias. Practically and for this contribution the numerical experimentations conducted on the synthetic processes described below indicate that when $s$ is varied in the narrow range $s \in [0.2, 0.5]$, results and hence conclusions are not significantly varied. All figures are shown here for $s = 0.25$.

## 3. SYNTHETIC PROCESSES

To assess the validity of the oscillation formalism, it is applied to a set of synthetic process whose multifractal properties are well known a priori. Results and Figures are obtained from average over 100 independent copies of the same process.

**Fractional Brownian motion (FBM).** FBM, with parameter $0 < H < 1$, is defined as the only Gaussian exactly self-similar process with stationary increments [13]. It is well known to be a mono-Hölder process (i.e., a process whose local regularity takes the only value $h = H$) so that its multifractal spectrum is degenerate: $D(H) = 1$ and $D(h) = -\infty$ else. Also, FBM contains only and everywhere cusp-type ($\beta = 0$) singularities, hence $\mathcal{D}(\beta) = \delta(\beta)$. Fig. 2 (a) compares the results of the oscillation formalism procedure applied to FBM to its theoretical oscillation spectrum. As commonly observed for multifractal formalisms for Hölder exponents, the estimated spectrum does not collapse onto a single point, yet remains concentrated around the theoretical single point constituting the spectrum, a very satisfactory result.

**Lacunary Wavelet Series (LWS).** LWS are defined as wavelet series: $X_{\alpha,\gamma}(t) = \sum_{j,k} d_{\alpha,\gamma}(j,k)\psi_{j,k}(t)$, $\alpha > 0, 0 < \gamma < 1$, where a random fraction of $2^{-j\gamma}$ of $d_{\alpha,\gamma}(j,k)$ takes the single value $2^{j\alpha}$, while the others are set to 0. Such processes possess an oscillation spectrum of the form $\mathcal{D}(\beta) = \gamma(\beta + 1)$, $\beta \in [0, 1/\gamma - 1]$ [11]. Fig. 2 (b) compares the results of the oscillation formalism procedure applied to LWS to its theoretical oscillation spectrum. Despite a clear discrepancy between the theoretical and estimated spectra, the two satisfactory outcomes of the current procedure stem from the fact that the maximum of the estimated $\mathcal{D}(\beta)$ falls far to the right of $\beta = 0$ and that the estimated $\mathcal{D}(\beta)$ is found significantly above 0 for a range of $\beta$ that matches the theoretical range $\beta \in [0, 1/\gamma - 1]$. By comparing the top plots of Fig. 2, the oscillation formalism clearly and unambiguously leads to conclude that FBM is a cusp only process, while LWS contains chirp singularities over a wide range of $\beta$s. The extent to which the observed discrepancy is caused by numerical instabilities inherent to the proposed procedure or by difficulties in the synthesis of LWS remains to be understood [11].

**Random Wavelet Cascades (RWC) versus Random Wavelet Series (RWS).** As mentioned in Section 1, turbulence data are commonly modeled as multiplicative cascade random processes to account for the Richardson energy cascade and to model their well-accepted multifractal properties [4, 3]. It is hence natural to apply the oscillation formalism to classes of processes that mimic these properties. RWC [14] are defined as an expansion on an orthonormal wavelet basis $\sum_{j,k} d_X(j,k)\psi_{j,k}(t)$, where the wavelet coefficients
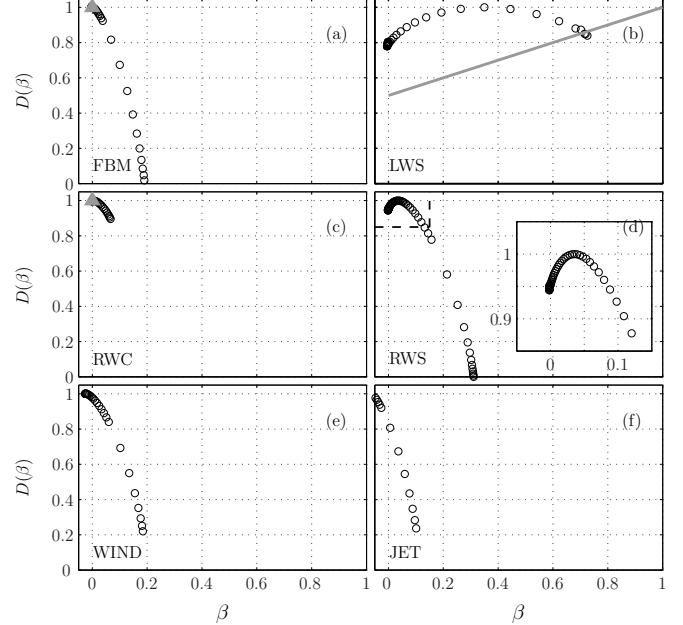


**Fig. 2**. **Wavelet Leader Oscillating Singularity Formalism.** applied to FBM (a), LWS (b), RWC (c), RWS (d), *Wind-Tunnel* (e) and *Jet* (f) experiments turbulence 1D Velocity Data.

are obtained as products of i.i.d. positive random variables (called the multipliers):

$$d_X(j,k) = (-1)^{\epsilon_{j,k}} \prod_{\{(j',k')\in I_{j,k}\}} W_{j,k}, \qquad (10)$$

where $I_{j,k}$ denote the dyadic intervals $\{[k2^{-j}, (k+1)2^{-j}), j = 1, \ldots, J, k = 1, \ldots, 2^j\}$ and where the $\epsilon_{j,k}$ consist of random variable taking values 1 or 2 with probability $1/2$. It has been proven that for certain classes of multipliers $W$, RWC contain only cusp type singularities, this is conjectured to be valid for all multiplicative cascades [11]. RWS [15] are also defined as wavelet series:$\sum_{j,k} d_X(j,k)\psi_{j,k}(t)$, with $d_X(j,k)$ defined to have exactly the same marginal distribution as those of RWC at each scale. This is practically easily achieved by synthesizing first a RWC, whose $d_X(j,k)$ are then shuffled randomly scale by scale. By construction, RWS contain chirp type singularities almost everywhere [11]. Here, the multipliers are chosen log-normal, in agreement with the most commonly accepted model of multiplicative cascades in turbulence, the parameters of the log normal distribution are chosen to match well-aggreed measures in turbulence [4, 3]. Results reported in Fig. 2 (c, d) are obtained from average over a large number of independent copies of the process, the number of copies and the sample size being chosen to match those of the turbulence data analyzed in Section 4. For RWC, the estimated $\mathcal{D}(\beta)$ concentrates around the point $\beta = 0$ and $\mathcal{D} = 1$ as for FBM, hence comforting the cusp type only nature of the singularities. For RWS, the estimated $\mathcal{D}(\beta)$ has a maximum that falls clearly to the left of $\beta = 0$ and spans over a much wider range of values of $\beta$, hence validating the chirp type nature of the singularities. Various mother wavelets were used to simulate synthetic data and are unrelated to the mother wavelet involved in the application of the oscillation formalism, so that the fact that the synthetic data are constructed from wavelet expansion cannot cause any bias in the results obtained here.

## 4. HYDRODYNAMIC TURBULENCE

The wavelet Leader Oscillation formalism is now applied to turbulence velocity data with the aim of characterizing the presence (or not) of oscillating singularities.

**1D Eulerian Velocity Data.** Two data sets are analyzed, collected over different experiments, both consisting of 1D Eulerian velocity signals, obtained from hot-wire anemometry measurement techniques: *Wind-Tunnel* refers to data measured at the ONERA Modane wind tunnel (1995 campaign), at a very high Reynolds numbers $R_\lambda \simeq 2000$ and further described in [16] ; *Jet* refers to measurements collected on a Jet Turbulence performed in low temperature helium at moderately high Reynolds number, $R_\lambda \simeq 1000$ (as described in [17]). Results reported below are obtained from 192 and 492 independent time series collected in the *Wind-Tunnel* and *Jet* experiments respectively, each of sample size $n = 2^{17}$, with an estimated integral scale equivalent to $10^5$ samples. Both data sets hence consists of very large database and high quality turbulence experiment data, considered as reference in the field of turbulence. these data sets were made available to us by Y. Gagne, B. Castaing and C. Baudet, who are gratefully acknowledged.

**Oscillating singularities.** Fig. 2 (e, f) reports the results obtained from the *Wind-Tunnel* and *Jet* experiments, respectively, and clearly reveals for both cases that the $\mathcal{D}(\beta)$ essentially concentrate around the point $\beta = 0, \mathcal{D} = 1$: The ranges of $\beta$ where $\mathcal{D}(\beta)$ remains close to 1 is as narrow as in the FBM and RWC cases, and anyway significantly narrower than those observed for LWS and RWS, when RWS were synthesized to have the same statistical properties, sample size and number of independent copies as the experimental data ; For the *Jet* experiment, the maximum $\mathcal{D}(\beta)$ is observed to be located for $\beta < 0$, when by definition, $\beta \geq 0$, this biased estimation is in favor of the maximum of $\mathcal{D}(\beta)$ being actually located at $\beta = 0$. These experimental evidences clearly lead to conclude against the existence of oscillating singularities in 1D component of the velocity fluctuations collected on high Reynolds turbulence experiments, of various types.

## 5. CONCLUSIONS AND PERSPECTIVES

Elaborating on the now standard wavelet Leader based multifractal formalism aiming at characterizing the fluctuations along time of the Hölder exponents encountered in data, an oscillation formalism has been proposed and analyzed: It describes, with identical tools based on Hausdorff dimension, structure functions and Legendre transforms, the fluctuations along time of the oscillation exponents in data. This therefore enriches the classical multifractal analysis framework by investigating whether the singularities existing in data, and accounted for by the classical multifractal spectrum, are of cusp (non oscillating) or chirp (oscillating) nature. Proofs of the results obtained and used here will be detailed in [11]. This can be further extended by considering a spectrum, often referred to *Grand-Canonical*, measuring the Hausdorf dimension of the set of points having jointly Hölder and oscillation exponents $(h, \beta)$ and devising the corresponding formalism, in the spirit of [8, 5]. Also, further examples of stochastic processes possessing oscillating singularities with a known spectrum $\mathcal{D}(\beta)$ and that can be efficiently synthesized are being investigated.

For hydrodynamic turbulence, this is, to the best of our knowledge, and despite the huge corpus of literature dedicated to the analysis of turbulence data, the first attempt to perform a quantitative statistically grounded analysis centered on the existence of oscillating singularities. It unambiguously leads to conclude that 1D velocity fluctuations do not possess oscillating singularity, an important conclusion with respect to the heuristic understanding and modeling of turbulence. It leaves open the question of whether oscillating singularities could be detected on full 3D turbulence data, that are far more difficult to collect and whose analysis is much more involved.

## 6. REFERENCES

[1] R.H. Riedi, "Multifractal processes," in *Theory and applications of long range dependence*, P. Doukhan, G. Oppenheim, and M.S. Taqqu, Eds. 2003, pp. 625–717, Birkhäuser.

[2] H. Wendt, P. Abry, and S.Jaffard, "Bootstrap for empirical multifractal analysis," *IEEE Signal Processing Mag.*, vol. 24, no. 4, pp. 38–48, 2007.

[3] U. Frisch, *Turbulence, the Legacy of A.N. Kolmogorov*, Cambridge University Press, 1995.

[4] B.B. Mandelbrot, "Intermittent turbulence in self-similar cascades; divergence of high moments and dimension of the carrier," *J. Fluid Mech.*, vol. 62, pp. 331–358, 1974.

[5] S. Jaffard, P. Abry, and S. Roux, "Singularités oscillantes et coefficients d'ondelettes dominants," in *Proceedings of the 22nd GRETSI conference*, 2009.

[6] H. Moffat, *Turbulence and chaotic phenomena in fluids*, chapter Simple topological aspects of turbulent velocity dynamics, pp. 223–230, Elsevier, 1984.

[7] N. Kevlahan and J.C. Vassilicos, "The space and scale dependencies of the self-similar structure of turbulence," *Proc. R. Soc. Lond. A*, vol. 447, 1994.

[8] A. Arneodo, E. Bacry, S. Jaffard, and J.-F. Muzy, "Oscillating singularities on Cantor sets: A grand-canonical multifractal formalism," *J. Stat. Phys.*, vol. 87, no. 1-2, pp. 179–209, Apr. 1997.

[9] A. Arneodo, E. Bacry, S. Jaffard, and J.F. Muzy, "Singularity spectrum of multifractal functions involving oscillating singularities," *Journal of Fourier Analysis and Application*, vol. 4, no. 2, pp. 159–174, 1998.

[10] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, San Diego, CA, 1998.

[11] S. Jaffard, S. Roux, and P. Abry, "Oscillating singularities in multifractal analysis," *in preparation*, 2010.

[12] H. Wendt, S. G. Roux, P. Abry, and S. Jaffard, "Wavelet leaders and bootstrap for multifractal analysis of images," *Signal Proces.*, vol. 89, pp. 1100–1114, 2009.

[13] G. Samorodnitsky and M. Taqqu, *Stable non-Gaussian random processes*, Chapman and Hall, New York, 1994.

[14] A. Arneodo, E. Bacry, and J.-F. Muzy, "Random cascades on wavelet dyadic trees," *Journal of Mathematical Physics*, vol. 39, 1998.

[15] J.-M. Aubry and S. Jaffard, "Random wavelet series," *Commun. Math. Phys.*, vol. 227, pp. 483–514, 2002.

[16] B. Castaing, Y. Gagne, and M. Marchand, "Log-similarity for turbulent flows," *Physica D*, vol. 68, no. 3-4, pp. 387–400, Oct. 15 1993.

[17] O. Chanal, B. Chabaud, B. Castaing, and B. Hébral, "Intermittency in a turbulent low temperature gaseous helium jet," *Eur. Phys. J. B*, vol. 17, pp. 309, 2000.