Multifractal Description of Lagrangian Velocity Statistics in Turbulent Flows : From Dissipative to Inertial Scales

L. Chevillard, S.G. Roux, E. Levêque, N. Mordant, J.-F. Pinton, and A. Arneodo
Laboratoire de Physique, École Normale Supérieure de Lyon
46 allée d'Italie, F-69007 Lyon, France
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We use the multifractal formalism to describe the effects of dissipation on Lagrangian velocity statistics in turbulent flows. We analyze high Reynolds number experiments and direct numerical simulation (DNS) data. We show that this approach reproduces the shape evolution of velocity increment probability density functions (PDF) from Gaussian to stretched exponentials as the time lag decreases from integral to dissipative time scales. We observe that numerical and experimental data are accurately described by a unique quadratic $D(h)$ spectrum which is found to extend from $h_{min} \approx 0.18$ to $h_{max} \approx 1$, as the signature of the highly intermittent nature of Lagrangian velocity fluctuations.

Statistical properties of homogeneous three dimensional turbulence have been studied for a long time in the Eulerian framework [1]. Recently a growing interest in studying intermittency from a dynamical point of view has been motivated by high precision Lagrangian experiments. Essentially two experimental groups have performed particle tracking in highly turbulent flows. The group at Cornell [2] reports measurements of Lagrangian acceleration in a turbulent water flow between two counter-rotating disks for Taylor-based Reynolds numbers $200 < R_\lambda < 900$. The experiment carried out at ENS-Lyon [3], in a similar von Kármán flow, is based on acoustic tracking. It provides Lagrangian velocity records covering the inertial range of turbulent motion, up to several integral time scales. In addition to these complementary experiments, DNS of the Navier-Stokes equations [3, 4] have produced comparative numerical results in the range $75 < R_\lambda < 380$. The aim of the present work is to provide a comprehensive *description* of the Lagrangian intermittency, using a formalism that describes both the inertial and dissipative range of time scales. The multifractal description [5], already widely used in Eulerian studies of turbulence, is a natural choice [6, 7]. In the present description, a first-order Lagrangian velocity increment over a time scale τ is written as :

$$\delta_\tau v(t) = v(t+\tau) - v(t) = \beta\,(\tau/T)\,\delta_T v\ , \qquad (1)$$

where all the time scale dependence is contained in the independent random function $\beta(\tau/T)$. The PDF of integral time scale increments $\delta_T v$ is thus assumed to be Gaussian ($\mathcal{G}$) — a result of a central limit argument, also in agreement with Eulerian observations. Once the distribution of $\mathcal{P}(\beta)$ is known, the PDF of increments at any time scale τ is computed as

$$\mathcal{P}(\delta_\tau v) = \int \frac{d\beta}{\beta}\,\mathcal{G}\left(\frac{\delta_\tau v}{\beta}\right)\mathcal{P}(\beta)\ . \qquad (2)$$

In the standard mutifractal formalism [5], β is assumed to have a power law scale dependence in the inertial range, $\beta \sim (\tau/T)^h$, with a spectrum $\mathcal{D}(h)$ (meaning that the PDF of observing an exponent h at scale τ is proportional to $(\tau/T)^{1-\mathcal{D}(h)}$). To describe the entire range of scales covered in experimental measurements and computer simulations, one must take into account the effects of viscosity (finite R_λ). In the dissipative range, velocity fluctuations are smoothed by viscous damping (or by measurement filtering) and the velocity increments become proportional to the time scale $\delta_\tau v(t) = \tau a(t)$,

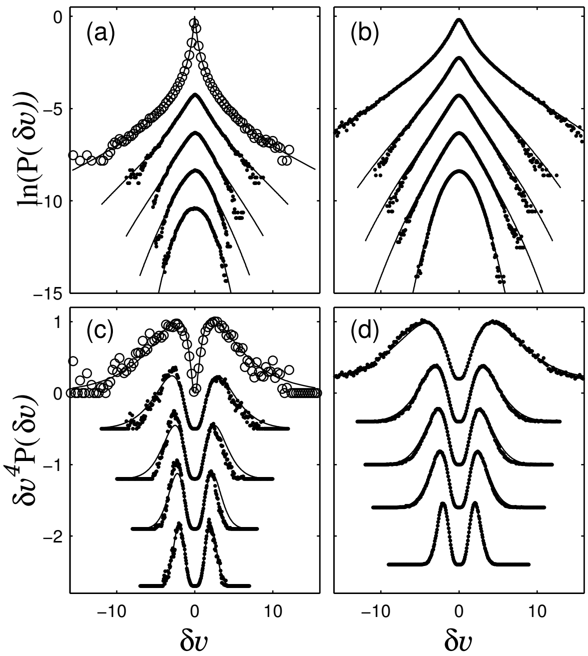

FIG. 1: Comparison of the experimental (a,c) and numerical (b,d) data for the normalized velocity increment PDF $\mathcal{P}(\overline{\delta_\tau v})$, where $\overline{\delta_\tau v} = \delta_\tau v / \langle(\delta_\tau v)^2\rangle^{\frac{1}{2}}$, with the predictions of the multifractal description. (•)(a,c): ENS-Lyon experimental data, for time lags $\tau/T = 1$, 0.35, 0.16 and 0.07, from bottom to top; the solid lines are the model fit with $c_2 = 0.075$.(•)(b,d) DNS data calculated for $\tau/T = 1$, 0.25, 0.17, 0.11 and 0.05, from bottom to top; the solid lines correspond to parameter value $c_2 = 0.086$.(∘)(a,c) Cornell acceleration data, the solid lines are the model predictions for $c_2 = 0.079$. The original δv-axis for the acceleration PDF (∘) has been shrunk by a factor of 4.

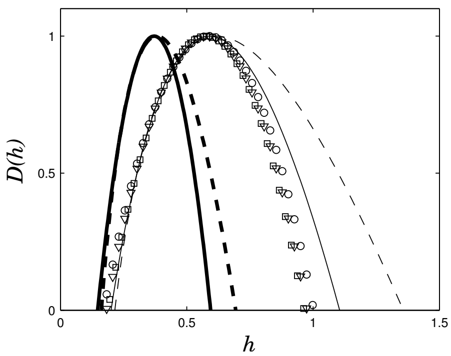

FIG. 2: $\mathcal{D}(h)$ curves extracted from: (□) ENS-Lyon velocity data, (∇) Cornell acceleration data and (∘) DNS numerical data. Are also represented for comparison the Eulerian log-normal (thick solide line) and log-Poisson (thick dashed line) spectra as well as their Lagrangian counter-parts.

where $a(t)$ is the Lagrangian acceleration. We shall consider that the cross-over between inertial and dissipative statistics occurs when the local Reynolds number is of order unity,

$$Re(\tau/T) = \frac{\tau}{T}\beta^2\left(\frac{\tau}{T}\right) Re = 1 \ , \qquad (3)$$

where Re is the integral scale Reynolds number. This defines a local Kolmogorov dissipative time $\tau_\eta(h) = T Re^{\frac{-1}{2h+1}}$ where the local velocity increments change from inertial scale invariance to dissipative scaling (this implies $h \geq -1/2$, for time scales to be shorter than T). Changing the integration variable from β to h in Eq.(2), the PDF of velocity increments at any scale τ/T can be written as the sum of two contributions,

$$\mathcal{P}(\delta_\tau v) = \int_{-1/2}^{h^*\left(\frac{\tau}{T},Re\right)} dh \, \frac{\mathcal{P}_i\left(h,\frac{\tau}{T},\mathcal{D}(h)\right)}{\beta_i\left(\frac{\tau}{T},h\right)} \, \mathcal{G}\left(\frac{\delta_\tau v}{\beta_i\left(\frac{\tau}{T},h\right)}\right) + \int_{h^*\left(\frac{\tau}{T},Re\right)}^{+\infty} dh \, \frac{\mathcal{P}_d\left(h,Re,\mathcal{D}(h)\right)}{\beta_d\left(\frac{\tau}{T},h,Re\right)} \, \mathcal{G}\left(\frac{\delta_\tau v}{\beta_d\left(\frac{\tau}{T},h,Re\right)}\right) \qquad (4)$$

where the functions $\beta_{i,d}$ and $\mathcal{P}_{i,d}$ have the proper inertial ($\beta_i \sim (\tau/T)^h$, $\mathcal{P}_i \sim (\tau/T)^{1-\mathcal{D}(h)}$) and dissipative ($\beta_d \sim \tau/T$, $\mathcal{P}_d \sim (\tau_\eta(h)/T)^{1-\mathcal{D}(h)}$) scalings. The change occurs at the critical value h^* for which the local Reynolds number is unity:

$$h^*\left(\frac{\tau}{T}, Re\right) = -\frac{1}{2}\left(1 + \frac{\ln Re}{\ln\frac{\tau}{T}}\right) \ . \qquad (5)$$

For $h < h^*(\tau/T, R_e)$, the increments $\delta_\tau v$ are in the inertial range, while they lie in the dissipative range for $h > h^*$. Finally, we impose that the function $\beta(\tau/T)$ be continuous and differentiable at the transition, following a strategy used in the Eulerian domain [7], and inspired from an elegant interpolation formula proposed by Batchelor [8] (see [10] for details). In this work, and as *a posteriori* justified, we assume a quadratic form $\mathcal{D}(h) = 1 - (h - c_1)^2/2c_2$. A first finding of our analysis is that almost identical functions $\mathcal{D}(h)$ are obtained for the three sets of data (Cornell, Lyon and DNS), although they cover a wide range of scales and of turbulent Reynolds numbers: the symbols in Fig.2 are undistiguishable, certainly within error bars.

To conclude, we return to our observation that a unique $\mathcal{D}(h)$ spectrum yields an accurate description of the Lagrangian velocity statistics at all scales. Such a spectrum $\mathcal{D}^E(h)$ has been extensively studied in the Eulerian domain [1, 9]. Two widely used forms (corresponding to log-normal and log-Poisson statistics) are shown in Fig.2. They can be mapped into the Lagrangian domain :

$$\mathcal{D}(h) = -h + (1+h)\mathcal{D}^E\left(h/(1+h)\right) \ , \qquad (6)$$

using a Kolmogorv Refined Similarity argument in the spirit of the work done by Borgas [11] . The resulting curves are shown in Fig.2.; we note that the agreement with the measured Lagrangian $\mathcal{D}(h)$ functions is excellent on the left-hand side of the curves, *i.e.* for values $h < c_1$ corresponding to intense velocity increments. On the right-hand side ($h > c_1$) there is a noticeable difference. Whether this difference is significant deserves more investigation. It may be of importance since the above relationship clearly shows that the Eulerien and Lagrangian singularity spectra cannot be both log-normal.

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