Wheel-free graphs

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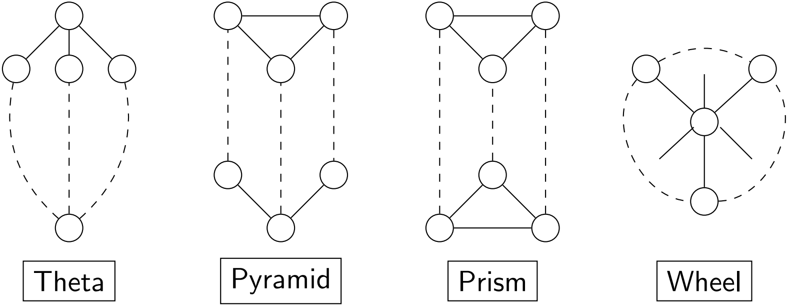
Truemper configurations

There are 4 Truemper configurations : theta, pyramid and prism are called **3-paths configurations** and the last one is called the **wheel**.

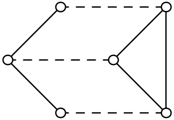


Truemper's Theorem

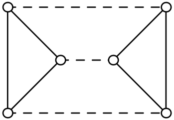
Theorem (Truemper, 1982)

Let β be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a subset F of the edge set of G such that $|F \cap C| \equiv \beta_C \pmod 2$ for all chordless cycles C of G, if and only if every induced subgraph G' of G that is a Truemper configuration there exists a subset F' of the edge set of G' such that $|F' \cap C| \equiv \beta_C \pmod 2$, for all chordless cycles C of G'.

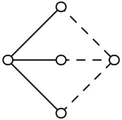
Detecting Truemper configurations

- 

  Polynomial, $O(n^9)$,
  Chudnovsky and Seymour, 2002

- 

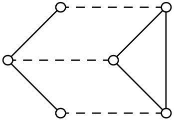
  NP-complete,
  Maffray, NT, 2003
  Follows from a construction
  of Bienstock

- 

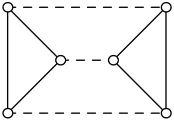
  Polynomial, $O(n^{11})$,
  Chudnovsky and Seymour, 2006

Coloring Truemper configurations

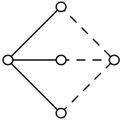
We say that a class of graph $\mathcal{C}$ is **χ-bounded** if there exists a function f such that for any $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$.

- 

  Not χ-bounded,
  Zykov, 1949

- 

  Not χ-bounded,
  Zykov, 1949

- 

  χ-bounded,
  D. Kühn and D. Osthus, 2004

Wheels : open question

There are three main open (and difficult) questions on wheel-free graphs are :

- Is there a polytime algorithm to detect a wheel (as an induced subgraph)?
- Is there a constant C such that any graph G with no induced wheel satisfy : $\chi(G) \leq C$?
- Do they have an interesting structure?

Subclasses of wheel-free graph

Two subclasses of wheel-free graphs have a complete structural description:

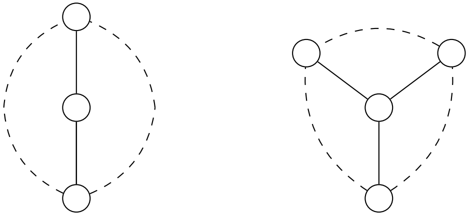
- (Unichord,K_4)-free graphs (Trotignon, Vušković 2008)
- Graphs that do not contain an induced subdivision of a wheel (Lévêque, Maffray, Trotignon 2009)

Remark : Let H be a graph. We say that a graph G is ***H*-free** if G does not contain H as an **induced** subgraph.

2-wheel (with Radovanović, Trotignon and Vušković)

A **k-wheel** ($k \geq 2$) is a cycle together with a vertex (called the center), not in the cycle, that has at least k neighbors in the cycle.

We are now going to study the class of **2-wheel-free graphs**.



It is clear that the class of 2-wheel-free graphs is a subclass of wheel-free graphs.

Decomposing graphs that do not contain a 2-wheel as a subgraph

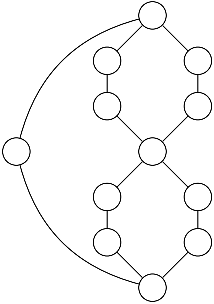
Theorem (PA, Radovanović, Trotignon, Vušković 2011)

If G does not contain a 2-wheel as a subgraph then either G is basic, or G admits a decomposition.

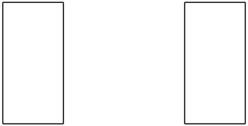
Basic class

A graph G is **sparse** if no vertices have at least 2 neighbors of degree at least 3.

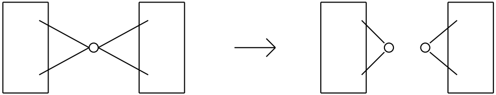
Obviously:

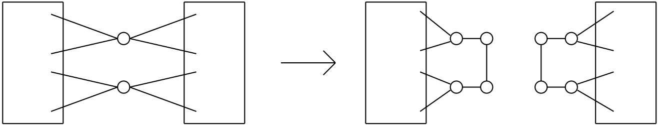
- If G is sparse, then it is 2-wheel-free.
- The converse is false : 
- It is a very simple class.

Decomposition

- **0-cutset**: 

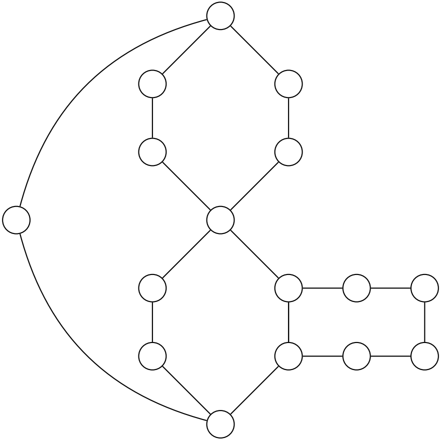
- **1-cutset**:



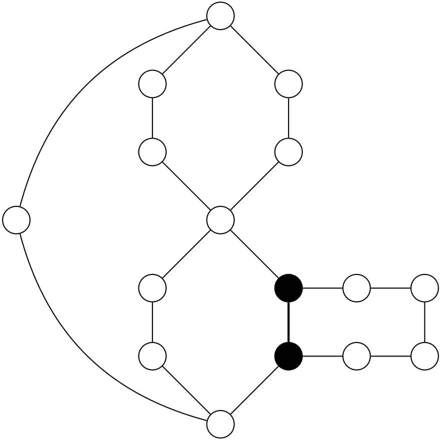
- **proper 2-cutset**: 

- **K_2-cutset**

Example of a decomposition



Example of a decomposition



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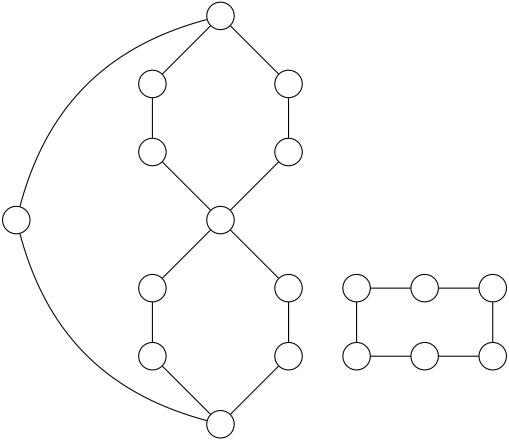

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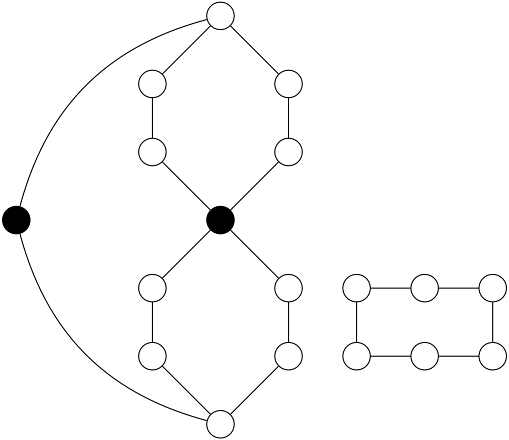
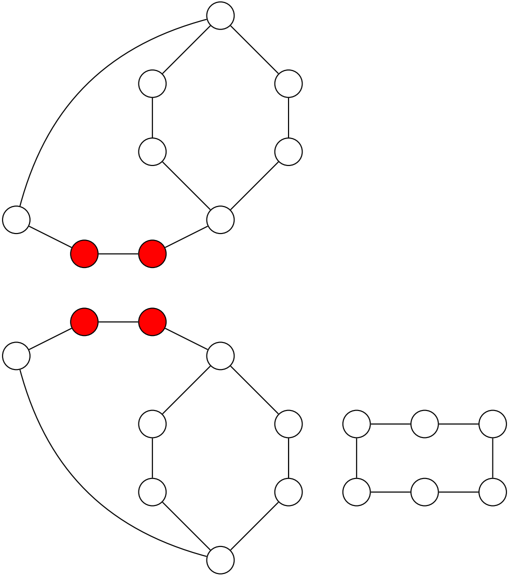


Example of a decomposition



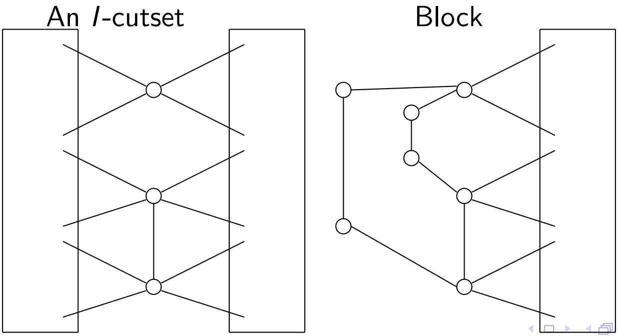
Decomposing graphs that do not contain a 2-wheel as an induced subgraph

Theorem (PA, Radovanović, Trotignon, Vušković 2011)

If G does not contain a 2-wheel as an induced subgraph then either G is basic, or G admits a decomposition.

Decomposing graphs with no 2-wheel as an induced subgraph

- To decompose graphs with no 2-wheel as an **induced subgraph**, one more decomposition is needed.
- An ***I*-cutset** in a graph G is a cutset S made of three vertices, with only one edge linking them, and such that $G \setminus I$ has at least two components containing neighbors of all vertices of S.



Application (1): detecting an induced 2-wheel in polytime

Theorem (PA, Radovanović, Trotignon, Vušković 2011)

Detecting a 2-wheel as a subgraph can be done in time $\mathcal{O}(mn)$.

Theorem (PA, Radovanović, Trotignon, Vušković 2011)

Detecting a 2-wheel as an induced subgraph can be done in time $\mathcal{O}(m^2n^2)$.

If it was easy to see that detecting a 2-wheel as a subraph was polynomial, detecting a 2-wheel as an induced subraph does not have an easy solution (at least to my knowledge).

Detecting k-wheel : state of play

Theorem (PA, Radovanović, Trotignon, Vušković 2011)

Detecting a 2-wheel can be done in polytime.

Open Question

Is it possible to detect a 3-wheel (= wheel) in polytime?

Theorem (PA, Radovanović, Trotignon, Vušković 2011)

Detecting a k-wheel with $k \geq 4$ is NP-complete.

Application (2): edge-colouring

If G is a graph, we note $\mathbf{\Delta(G)}$ the greatest degree of G.

Theorem (Vizing)

Any graph G is either $\Delta(G)$-edge colourable or $\Delta(G) + 1$-edge colourable

Theorem (PA, Radovanović, Trotignon, Vušković 2011)

A 2-connected graph with no induced 2-wheel contains an edge with both ends of degree at most 2.
Hence, every graph G with no induced 2-wheel, and which is not an odd cycle, is $\Delta(G)$-edge colourable.

Open questions

Question : what can we say about graphs that do not contain a k-wheel as a subgraph when $k \geq 3$?

Theorem (Turner III 2005)

Graphs that do not contain a k-wheel ($k \geq 3$) as a subraph ($k \geq 3$) admit a vertex of degre at most k and thus are $(k+1)$-colourable.

Open questions

Theorem (Thomassen and Toft 1981 $\setminus$ PA, F. Havet and N. Trotignon 2011)

Graphs that do not contain a 3-wheel as a subgraph either contains a pair of twin or contains a vertex of degree at most 2 and thus are 3-colourable.

Theorem (PA 2012)

Graph that does not contain a 4-wheel as a subgraph either contains a pair of twin or contains a vertex of degree at most 3 and thus are 4-colourable.

Conjecture

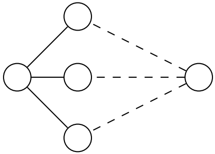
Graphs that do not contain a k-wheel ($k \geq 5$) as a subgraph are k-colourable.

Excuding wheels with exactely 2 spokes and colouring

Theorem (Radovanović, Vusković 2010)

If a graph contains no triangle, no cube and no theta (as induced subgraphs), then it has a vertex of degree at most 2.

Remind that a theta is:



Conjecture

Every graph with no triangle, no cube and no wheel with exactly 2 spokes (as induced subgraphs) admits a vertex of degree at most 2.

Detecting 2-wheels

Open Question

Is there a polytime algorithm to detect wheels with exactly 2 spokes (as induced subgraphs)?

Back to wheel-free graph

Let us now return to the study of the class of **wheel-free graph**.

In the rest of the talk, **K_4 won't be considered as a wheel** anymore.
Note that this enlarge the class of wheel-free graphs.

Some definitions

A **module** in a graph G is a set $A \subset V(G)$ such that each vertex of A share the same external neighborhoud.

A **minimal separator** in a graph G is a set $S \subset V(G)$ such that $G \setminus S$ is not connected and at least 2 connected components of $G \setminus S$ see every vertices of S.

A **moplex** of a graph G is a set $S \subseteq V(G)$ such that S is a clique, S is a module and $N(S)$ is a minimal separator.

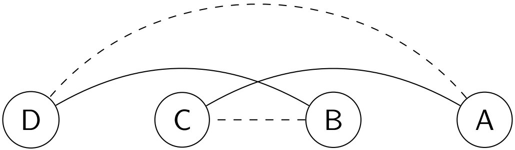
Theorem (A. Berry and J.-P. Bordat 1998)

Every graph contains a moplex.

LexBFS

Theorem (Brandstädt, Dragan and Nicolai 1997)

An order $\prec$ of the vertices of a graph G is a LexBFS order if and only if it satisfies the following property: for all vertices a, b, c of G such that $c \prec b \prec a$, $ca \in E(G)$ and $cb \notin E(G)$ there exists a vertex d in G such that $d \prec c$, $db \in E$ and $da \notin E$.
There exists an $O(n+m)$-time algorithm that, given an input graph G, outputs a LexBFS order of G.



Around LexBFS

Corollary

If $(v_1, v_2, \ldots, v_n)$ *is a LexBFS order of the graph* G*, then* $(v_1, v_2, \ldots v_i)$ *is a LexBFS order of the graph* $G[v_1, v_2 \ldots, v_i]$.

It permits to get **elimination ordering**.

Theorem (A. Berry and J.-P. Bordat 1998)

Let G *be a graph that is not a clique,* $\prec$ *a LexBFS order of* G*, and* z *the last vertex for* $\prec$*. Then* z *is contained in a moplex of* G.

Diamond-wheel-free graphs and multi-simplicial vertices

A wheel is a **diamond-wheel** if it center has three consecutive neighbours on the rim.

A vertex v is **multi-simplicial** if $N(v)$ is a union of disjoints cliques or, equivalently, if $N(v)$ is P_3-free (where P_3 is the path on 3 vertices).

A graph G admits a **multi-simplicial elimination order** if there exists an order $(v_1, \ldots, v_n)$ on the vertices of G such that, for $i = 1, \ldots, n$, v_i is multi-simplicial in $G[v_1, \ldots, v_i]$.

Theorem (PA, Charbit, Chudnovsky, Trotignon and Vušković 2011)

If G is a diamond-wheel-free graph then any LexBFS order of G is a multi-simplicial elimination order of G.

Theorem (PA, Charbit, Chudnovsky, Trotignon and Vušković 2011)

There is an algorithm with the following specifications.

Input: *A diamond-wheel-free graph G.*

Output: *A maximum clique of G is given.*

Running time: $O(n + m)$

A property that goes well with lexBFS and moplexes

A wheel is a **complete-wheel** if it center is adjacent to every vertex of the rim.

A **hole** is an induced cycle of length at least 4.

A graph is said to be **triangulated** if it does not contain any hole.

Definition (Property ($\star$))

A graph G satisfies Property ($\star$) if for any vertex x of G, for every hole H contained in $N(x)$ and for every connected component C of $G - N[x]$, there is a vertex $y \in H$ with $y \notin N(C)$.

A graph G admits a **triangulated elimination order** if there exists an order $(v_1, \ldots, v_n)$ on the vertices of G such that, for $i = 1, \ldots, n$, $N_{G[v_1,\ldots,v_i]}(v_i)$ is triangulated.

Property

If G satisfies Property $(\star)$, then any LexBFS order of G is a triangulated elimination order of G.

Observation

Complete-wheel-free graphs trivially satisfy Property $(\star)$.

Theorem (da Silva and Vušković 2007)

Even-hole-free graphs satisfy Property $(\star)$.

Theorem (Maffray, Trotignon and Vušković 2008)

Square-3PC(.,.)-free Berge graphs satisfy Property $(\star)$.

Algorithm

Theorem (PA, Charbit, Chudnovsky, Trotignon and Vušković 2011)

Let $\mathcal{C}$ be a class of graphs that satisfy Property $(\star)$ (such as even-hole-free graphs or square-3PC(.,.)-free Berge graphs or complete-wheel-free graphs). There is an algorithm with the following specifications.

Input: *A graph G.*

Output: *G is correctly identified as not belonging to $\mathcal{C}$, or a maximal clique of G is given.*

Running time: $O(nm)$

Algorithm

Step 1: Let $\mathcal{L} = \emptyset$.

Step 2: Run LexBFS on G.
Let $(v_1, \ldots, v_n)$ be the ordering given by LexBFS.
Define, for $i = 1, \ldots, n$, $N_i = N_{G[v_1,\ldots,v_i]}[v_i]$.

Step 3: For $i = 1, \ldots, n$, check if N_i is triangulated, if not return $G \notin \mathcal{C}$.

Step 4: For $i = 1, \ldots, n$, compute all maximal cliques of N_i and add them in $\mathcal{L}$. Then return a maximum (with respect to the cardinality) element of $\mathcal{L}$.

Merci pour votre attention