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The motion of an electrically conducting fluid in the presence of a steady magnetic field is analyzed. For any non-uniform magnetic field and any non-electromagnetic driving force, a high Hartmann number asymptotic analysis is developed using curvilinear coordinates based on the magnetic field. This analysis yields the structure of the electric current density and velocity fields. In a second step, orthogonal planar symmetries lead to a significant simplification of the asymptotic structure, depending on the nature of the symmetry. The asymptotic solution is applied to some configurations, some of them corresponding to crystal growth from a melt. In the case of electrically insulating boundaries, the nature of the symmetry is found to govern the magnitude and structure of the damped velocity. [S1070-6631(96)02207-6]

I. INTRODUCTION

In many problems, especially in crystal growth from a melt, the knowledge of the velocity field is needed, e.g., to investigate the segregation of species in the resulting solid phase. For instance, Hurle[1] showed how an unsteady convection could be stabilized when a magnetic field is applied: this is of great interest for the elimination of striations.[2,3] But even a steady convection produces radial and longitudinal segregations.[4] Some of these steady movements were studied with or without a magnetic field in crystal growth configurations.[5–7] Outside the field of crystal growth, many other problems also require a good understanding of MHD convection. Convection in the tokamak blanket[8] is such a problem. The fusion energy must be carried away by a lithium or lithium-lead flow. The amount of energy required for the pumping and the efficiency of the heat transfer are still open questions. Another potential application consists in the determination of diffusion coefficients (thermal, solute or crossed) in electrically conducting fluids. In general, these measurements can easily be disturbed by unavoidable fluid movements. The use of a magnetic field, with its damping effect on convection, could help to get precise diffusion data.

Our purpose in this paper is to carry out an original asymptotic analysis adapted to flows of electrically conducting fluids under a strong, possibly non-uniform, steady magnetic field. An arbitrary known body force is included in our formulation, the only requirement being that it should not depend on the fluid velocity. This force need not be specified until applications are considered: for instance, it may be thermal buoyancy in the case of crystal growth configurations.

In section II, the governing equations are written under two main simplifying assumptions. One is quite classic for MHD at the laboratory scale and consists in assuming that the magnetic Reynolds number is small enough to neglect the perturbation of the applied magnetic field due to the fluid flow. The other, which assumes that inertia is negligible, limits the scope of this work either to weak driving forces, large interaction parameter or fully-established rectilinear flows. In addition, in section III, the magnetic field is assumed to be strong enough to justify an asymptotic analysis, valid for large Hartmann numbers; the flow is considered as the sum of a core flow (inviscid) and some boundary layer flows (viscid), each obeying a particular structure. The structure of the core flow, which is derived from an integration along the magnetic field lines, is expressed in a curvilinear system of coordinates such that one of them (x^1) is constant on the iso-potential surfaces of the curl-free magnetic field and the others (x^2 and x^3) are constant on the magnetic flux lines. This asymptotic study essentially differs from Kulikovskii's approach[9] by the fact that the flow may be driven by any kind of body force.

After the asymptotic analysis is completed, the initial three-dimensional problem is changed to a two-dimensional one. Nevertheless, it may still be very difficult to get the asymptotic solution, and we show in section IV that certain symmetries can significantly simplify the problem. Two kinds of orthogonal planar symmetries are examined. In both cases, the symmetry leads to simplifications in the structure of the asymptotic solution obtained in section III. The simplification is more important for one symmetry which we call ''singular''. In section V, the asymptotic solution is derived for some flows in crystal growth configurations and for some other convective flows with magnetic fields. Asymptotic solutions are also derived in the case of a ''regular'' symmetry. Finally, in section VI the role of the symmetries is highlighted and the scope of the asymptotic theory is discussed.

II. MATHEMATICAL MODEL

A cavity Ω containing an electrically conducting fluid is placed in a region of a steady magnetic field. The magnetic field, which is curl-free and divergence-free in the cavity, is generated by an external device (a coil or a permanent magnet) and may be non-uniform. The fluid motion results from boundary conditions and non-electromagnetic body forces. The electromagnetic body force is separated from other body forces. In addition to the classical MHD assumptions, the following hypotheses are assumed to hold:

- the magnetic field is not disturbed by the motion of the fluid;
- inertia is negligible;
- non-electromagnetic body forces $\tilde{\mathbf{f}}$ are known in the whole cavity and are independent of the velocity field;
- the physical properties of the fluid are uniform.

The motion of the fluid is modelled with four equations: the conservation of mass, electric charge and momentum, along with Ohm's law. Using a typical length scale H of the cavity, a typical magnitude B_0 of the magnetic flux density, the kinematic viscosity ν, the electric conductivity σ and the mass density ρ, non-dimensional variables are defined: $\mathbf{x}=\tilde{\mathbf{x}}/\mathrm{H}$, $\mathbf{B}=\tilde{\mathbf{B}}/B_0$, $\mathbf{f}=\tilde{\mathbf{f}}H^3/\nu^2$, $\mathbf{u}=\tilde{\mathbf{u}}H/\nu$, $\mathbf{j}=\tilde{\mathbf{j}}H/\nu\sigma B_0$, $p=\tilde{p}H^2/\rho\nu^2$ and $\varphi=\tilde{\varphi}/\nu B_0$. The symbols with a tilde, $\tilde{\mathbf{x}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{f}}$, $\tilde{\mathbf{u}}$, $\tilde{\mathbf{j}}$, $\tilde{p}$ and $\tilde{\varphi}$ denote the dimensional spatial coordinates, magnetic field, non-electromagnetic body force, velocity field, electric current density field, pressure field and electric potential field. The governing equations take the non-dimensional form

$$\operatorname{div} \mathbf{u}=0, \tag{1}$$

$$\operatorname{div} \mathbf{j}=0, \tag{2}$$

$$-\nabla p+\mathbf{f}+Ha^2\mathbf{j}\wedge\mathbf{B}+\Delta\mathbf{u}=0, \tag{3}$$

$$\mathbf{j}+\nabla\varphi-\mathbf{u}\wedge\mathbf{B}=0, \tag{4}$$

where the Hartmann number, $Ha=\sqrt{\sigma/\rho\nu}B_0H$, is the single non-dimensional parameter in the problem. The required boundary conditions are associated with each particular problem. A weak formulation and the Lax-Milgram theorem then prove the existence and uniqueness of the solution for such a linear problem.

Equations (1), (2), (3) and (4) are expressed in a vectorial framework. The expression of velocity, electric current density and magnetic field in terms of vector fields is not the only one possible. They may equivalently be considered as differential forms (the differential forms are the antisymmetric tensors) in the framework of differential geometry (see Westenholtz[10] for the use of differential forms). This approach will be adopted here for two reasons. First, from a theoretical point of view, differential geometry provides a clearer and more general tool than vectorial analysis. Differential properties (mainly the exterior derivation d) are clearly distinguished from the metric properties (the metric tensor g and the Hodge star operator $\star$). This is not the case in classical notations where the operators div, curl, ∇ and Δ all involve both differentiability and distance. The other reason is a practical one: we shall make use, in this paper, of non-orthogonal coordinate systems for which expressions of the classical operators (div, curl, Δ and Δ) are not easily available, whereas in the field of differential geometry, intrinsic definitions allow one to express the operators in any system.

In the following, a field as a vector or as a differential form, will be denoted by the same symbol, except for the component indices that will take their conventional position. If $\mathbf{u}$, $\mathbf{j}$, $\mathbf{B}$ and $\mathbf{f}$ are considered as differential forms of degree one, equations (1), (2), (3) and (4) now take the form

$$d\star\mathbf{u}=0, \tag{5}$$

$$d\star\mathbf{j}=0, \tag{6}$$

$$-dp+\mathbf{f}+Ha^2\star(\mathbf{j}\wedge\mathbf{B})+(\star d\star d)\mathbf{u}=\mathbf{0}, \tag{7}$$

$$\mathbf{j}+d\varphi-\star(\mathbf{u}\wedge\mathbf{B})=\mathbf{0}. \tag{8}$$

III. ASYMPTOTIC ANALYSIS

In this section, we concentrate on the asymptotic limit of high Hartmann numbers. When $Ha\gg1$, equation (3) or (7) indicates that the electromagnetic body force is large compared to the viscous forces, which can be neglected in the main part of the cavity, called the core region. Later, we shall consider thin boundary layers or free shear layers for which viscous terms may be comparable to the electromagnetic body force.

A. The core solution structure

Before treating the general case of a non-uniform magnetic field, we consider a uniform field because the algebra is easier while the basic ideas are the same. The curl of equation (3) (without the viscous term) is

$$\operatorname{curl}\,\mathbf{f}+Ha^2(\mathbf{B}.\nabla)\mathbf{j}=\mathbf{0}, \tag{9}$$

because $\operatorname{curl}(\mathbf{j}\wedge\mathbf{B})=(\mathbf{B}.\nabla)\mathbf{j}-(\mathbf{j}.\nabla)\mathbf{B}-(\operatorname{div}\mathbf{j})\mathbf{B}+(\operatorname{div}\mathbf{B})\mathbf{j}$ reduces to $(\mathbf{B}.\nabla)\mathbf{j}$ considering (2) and the uniformity of $\mathbf{B}$. Choosing the dimensional scale of reference B_0 as equal to the magnitude of the uniform magnetic field, the non-dimensional $\mathbf{B}$ is simply a unit vector field. We choose a Cartesian coordinate system (x^1,x^2,x^3) such that $\mathbf{B}=\nabla x^1$. Now, equation (9) gives the variation of $\mathbf{j}$ with the x^1 coordinate in terms of the known curl $\mathbf{f}$. By integrating (9) $\mathbf{j}$ has the form

$$\mathbf{j}=\mathbf{j}_0-Ha^{-2}\int_0^{x^1}\operatorname{curl}\,\mathbf{f}\;\,dx'^1, \tag{10}$$

where $\mathbf{j}_0$ is a vector field which is independent of x^1. The curl of equation (4) has exactly the same form as equation (9),

$$\operatorname{curl}\,\mathbf{j}-(\mathbf{B}.\nabla)\mathbf{u}=0.$$

It leads to an expression for $\mathbf{u}$, similar to (10),

$$\begin{aligned}\mathbf{u}&=\mathbf{u}_0+\int_0^{x^1}\operatorname{curl}\,\mathbf{j}\;\,dx'^1,\\&=\mathbf{u}_0+x^1\operatorname{curl}\,\mathbf{j}_0\\&\quad-Ha^{-2}\int_0^{x^1}\operatorname{curl}\left(\int_0^{x'^1}\operatorname{curl}\,\mathbf{f}\;\,dx''^1\right)dx'^1,\end{aligned} \tag{11}$$

where $\mathbf{u}_0$ is a vector field which is independent of x^1.

Conservation of mass (1) and electric charge (2) give equations governing $\mathbf{j}_0$ and $\mathbf{u}_0$. With the solutions (10) and (11), these constraints are

$$\text{div }\mathbf{j}_0 = Ha^{-2}(\text{curl }\mathbf{f})^1_{(x^1=0)}, \tag{12}$$

$$\text{div }\mathbf{u}_0 = -(\text{curl }\mathbf{j}_0)^1. \tag{13}$$

For a uniform magnetic field, the asymptotic core structure is thus defined by equations (10) and (11) with the conditions (12) and (13).

An analogous analysis can be performed for any arbitrary non-uniform magnetic field. It is useful to consider a curvilinear system of coordinates based on the magnetic field lines. In the cavity, the curl-free magnetic field may be written as the gradient of a scalar function x^1 (see Kulikovskii[11]). In an equivalent formulation, we write the differential form $\mathbf{B}$ as the differential of x^1: $\mathbf{B}=dx^1$. Moreover, it is possible to find two other scalar functions x^2 and x^3 such that the metric tensor, g, in the (x^1,x^2,x^3) system of curvilinear coordinates, can be expressed as

$$g=\begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{bmatrix}. \tag{14}$$

This is the simplest metric that can be chosen in general.[11] We denote $\eta=\sqrt{\det g}dx^1\wedge dx^2\wedge dx^3$, the volume Riemannian 3-form of the physical space. Furthermore, the magnetic field must satisfy the Maxwellian condition of zero divergence. In the context of differential geometry, the divergence is defined by the relation $d\star\mathbf{B}=(\text{div }\mathbf{B})\eta$. In the coordinates (x^1,x^2,x^3),

$$d\star\mathbf{B}=\frac{\partial}{\partial x^1}\left(\frac{\sqrt{\det g}}{g_{11}}\right)dx^1\wedge dx^2\wedge dx^3. \tag{15}$$

The quantity $\sqrt{\det g}/g_{11}$ is thus invariant along magnetic field lines. For a non-uniform magnetic field, the curl of equation (3) does not lead to (9) since $(\mathbf{j}.\boldsymbol{\nabla})\mathbf{B}$ is not zero. Nevertheless, the solution is very similar with a few added terms due to the non-uniform field. Recalling that $\mathbf{f}$, $\mathbf{j}$ and $\mathbf{B}$ are differential forms of degree one, we take the exterior derivative of (7), without the viscous term,

$$d\mathbf{f}+Ha^2 d\star(\mathbf{j}\wedge\mathbf{B})=0, \tag{16}$$

and express its $dx^1\wedge dx^2$ and $dx^3\wedge dx^1$ components relative to (x^1,x^2,x^3),

$$\frac{\partial}{\partial x^1}\left(\frac{\sqrt{\det g}}{g_{11}}\,j^3\right)=-Ha^{-2}(df)_{12},$$

$$\frac{\partial}{\partial x^1}\left(\frac{\sqrt{\det g}}{g_{11}}\,j^2\right)=-Ha^{-2}(df)_{31}.$$

[The components of a vector field (or a differential form) are denoted by the same symbol as the vector field itself, in italics instead of boldface.] Integration of these equations and the invariance of $\sqrt{\det g}/g_{11}$ along magnetic lines lead to

$$j^3=j_0^3-Ha^{-2}\,\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}(df)_{12}dx'^1, \tag{17a}$$

$$j^2=j_0^2-Ha^{-2}\,\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}(df)_{31}dx'^1, \tag{17b}$$

where j_0^3 and j_0^2 are components of $\mathbf{j}$ at $x^1=0$. The last component of $\mathbf{j}$ is derived from equation (6),

$$d\star\mathbf{j}=\left[\frac{\partial(\sqrt{\det g}j^1)}{\partial x^1}+\frac{\partial(\sqrt{\det g}j^2)}{\partial x^2}+\frac{\partial(\sqrt{\det g}j^3)}{\partial x^3}\right]dx^1\wedge dx^2\wedge dx^3=0,$$

so that

$$j_1=j_{01}-\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}\frac{\partial}{\partial x^2}(\sqrt{\det g}j^2)+\frac{\partial}{\partial x^3}(\sqrt{\det g}j^3)dx'^1. \tag{17c}$$

The exterior derivative of (8) is similar to (16), and $\mathbf{u}$ is also divergence-free. Therefore, the structure of $\mathbf{u}$ parallels that of $\mathbf{j}$:

$$u^3=u_0^3+\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}(dj)_{12}dx'^1, \tag{18a}$$

$$u^2=u_0^2+\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}(dj)_{31}dx'^1, \tag{18b}$$

$$u_1=u_{01}-\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}\frac{\partial}{\partial x^2}(\sqrt{\det g}u^2)+\frac{\partial}{\partial x^3}(\sqrt{\det g}u^3)dx'^1. \tag{18c}$$

The fields u_0^2, u_0^3, u_{01}, j_0^2, j_0^3 and j_{01} are independent of x^1 and are also submitted to constraints, namely the $dx^2\wedge dx^3$ component of (16) and of the exterior derivative of (8):

$$\frac{\partial}{\partial x^3}\left(\frac{\sqrt{\det g}}{g_{11}}\,j_0^3\right)+\frac{\partial}{\partial x^2}\left(\frac{\sqrt{\det g}}{g_{11}}\,j_0^2\right)=Ha^{-2}(df)_{23_{(x^1=0)}}, \tag{19}$$

$$\frac{\partial}{\partial x^3}\left(\frac{\sqrt{\det g}}{g_{11}}\,u_0^3\right)+\frac{\partial}{\partial x^2}\left(\frac{\sqrt{\det g}}{g_{11}}\,u_0^2\right)=-(dj)_{23_{(x^1=0)}}. \tag{20}$$

The core structure for any non-uniform magnetic field is defined, in the (x^1,x^2,x^3) coordinates, by equations (17) and (18) under the constraints (19) and (20). The formulas reduce to the previous ones for a uniform magnetic field. For either a uniform or a non-uniform magnetic field, the boundary conditions can not yet be considered with the hope to determine the unknown fields $\mathbf{j}_0$ and $\mathbf{u}_0$. The reason is that these boundary conditions are changed through thin boundary layers (see sections III B and III C). The general method is the following. An equivalent structure to that for the core regions has to be found for the boundary layers. Afterwards, the true boundary conditions have to be applied to the global structure of the solution, consisting in the matched structures of the cores and that of the boundary layers. A simultaneous solution has finally to be found for $\mathbf{j}_0$, $\mathbf{u}_0$ (in each core region) and the unknown fields related to the boundary layers. This nice program will be completed only in the presence of some symmetry or in particular cases with the help of the characteristic surfaces (see section IV).

B. Hartmann layers

Near each boundary, the velocity gradients increase so that the viscous term in equation (3) becomes comparable to the electromagnetic body force term. In each boundary layer, the solution can be considered as the sum of the neighbouring core solution and a solution which is governed by the homogeneous ($\mathbf{f}=\mathbf{0}$) version of equations (1), (2), (3) and (4) and which vanishes far from the boundary. The matching is simply a sum, thanks to the linear behavior of the solution with respect to **f**. The magnetic field being almost uniform at the layer thickness scale, the curl of the homogeneous momentum equation and of Ohm's law can be combined to get the following equations:

$$-Ha^2(\mathbf{B}.\nabla)^2\mathbf{j}+\Delta^2\mathbf{j}=\mathbf{0}, \tag{21}$$

$$-Ha^2(\mathbf{B}.\nabla)^2\mathbf{u}+\Delta^2\mathbf{u}=\mathbf{0}. \tag{22}$$

When the normal unit vector **n** (directed toward the fluid) of the boundary and the magnetic field are not parallel, an analytical solution can be derived for the system (21) and (22), the so-called Hartmann layer solution. The thickness of these Hartmann layers is found to scale as Ha^{-1}, in the asymptotic limit of high magnetic field: the magnetic field can indeed be assumed uniform at this scale. In the analysis, spatial derivatives along **n** are considered as very important compared to that along other directions. Let us limit ourselves to the projection of (21) and (22) on the plane tangent to the Hartmann boundary. The derivative operator $(\mathbf{B}.\nabla)$ is estimated as $(\mathbf{B}.\mathbf{n})(\mathbf{n}.\nabla)$ and the projections of equations (21) and (22) admit the simplified form

$$-Ha^2|\mathbf{B}.\mathbf{n}|^2\ \frac{\partial^2\mathbf{j_t}}{\partial n^2}+\frac{\partial^4\mathbf{j_t}}{\partial n^4}=\mathbf{0},$$

$$-Ha^2|\mathbf{B}.\mathbf{n}|^2\ \frac{\partial^2\mathbf{u_t}}{\partial n^2}+\frac{\partial^4\mathbf{u_t}}{\partial n^4}=\mathbf{0}.$$

The tangential part of the solution is therefore

$$\mathbf{j_t}=\mathbf{j_h}e^{-Ha|\mathbf{B}.\mathbf{n}|n}, \tag{23}$$

$$\mathbf{u_t}=\mathbf{u_h}e^{-Ha|\mathbf{B}.\mathbf{n}|n}, \tag{24}$$

where $\mathbf{j_h}$ and $\mathbf{u_h}$ are two tangential vector fields defined on the Hartmann boundary: they are vector fields on the $\partial\Omega$ manifold. It is well known (see for instance Moreau[12]) that Ohm's law implies a characteristic relation linking $\mathbf{j_h}$ and $\mathbf{u_h}$. Since the normal components u^n and j^n are very small, the curl of equation (4) leads to:

$$\mathbf{n}\wedge\mathbf{j_h}=(\mathbf{B}.\mathbf{n})\mathbf{u_h}. \tag{25}$$

One should not derive similar exponential-like functions for j^n and u^n, the normal components of **j** and **u**, which paradoxically seem to obey the same equations (21) and (22). The equations for the normal components involve non-negligible derivatives of the tangential components, and these derivatives lead to terms involving the gradient of the magnetic field along the surface and involving the boundary curvature. The normal components are obtained by direct integration of (1) and (2) using the expressions (23) and (24). The divergence of the electric current density and that of the velocity field are

$$\mathrm{div}\ \mathbf{j}=\frac{\partial j^n}{\partial n}+\mathrm{div}[\mathbf{j_h}e^{-Ha|\mathbf{B}.\mathbf{n}|n}],$$

$$\mathrm{div}\ \mathbf{u}=\frac{\partial u^n}{\partial n}+\mathrm{div}[\mathbf{u_h}e^{-Ha|\mathbf{B}.\mathbf{n}|n}].$$

The integration of these equations between n and $+\infty$ yields the expression for j^n and u^n, taking into account the condition that the deviation fields vanish far from the boundary:

$$j^n=\left(Ha^{-1}\ \mathrm{div}_{\partial\Omega}\left(\frac{\mathbf{j_h}}{|\mathbf{B}.\mathbf{n}|}\right)-n\ \frac{\mathbf{j_h}.\nabla_{\partial\Omega}|\mathbf{B}.\mathbf{n}|}{|\mathbf{B}.\mathbf{n}|}\right)e^{-Ha|\mathbf{B}.\mathbf{n}|n}, \tag{26}$$

$$u^n=\left(Ha^{-1}\ \mathrm{div}_{\partial\Omega}\left(\frac{\mathbf{u_h}}{|\mathbf{B}.\mathbf{n}|}\right)-n\ \frac{\mathbf{u_h}.\nabla_{\partial\Omega}|\mathbf{B}.\mathbf{n}|}{|\mathbf{B}.\mathbf{n}|}\right)e^{-Ha|\mathbf{B}.\mathbf{n}|n}. \tag{27}$$

The symbols $\mathrm{div}_{\partial\Omega}$ and $\nabla_{\partial\Omega}$ denote the divergence and gradient operators defined on the $\partial\Omega$ submanifold. The value of (26), at $n=0$, is the same as the conservation of electricity given by Holroyd *et al.*[13] The term $\nabla_{\partial\Omega}|\mathbf{B}.\mathbf{n}|$ is not zero for non-uniform magnetic fields or for curved boundaries, so that equations (26) and (27) each involve a combination of a constant and of a linear term times the basic exponential function.

C. Parallel layers

When the normal to the layer is orthogonal to the magnetic field, the boundary layer equations involve derivatives along both **n** and **B**. These parallel layers are much more difficult to handle. The literature contains solutions for several particular cases (Hunt,[14] Alty,[15] Petrykowski and Walker[16]), but a general solution is not available to our knowledge. In this paper, we do not treat any parallel layer. These layers have a typical thickness of $Ha^{-1/2}$, and either develop along walls which are parallel to the magnetic field or develop as free-shear layers between two core regions: any discontinuity on the boundary may generate an interior layer through the cavity. While these layers may carry a significant fraction of the total flow or electric current, they are sometimes passive (e.g., in section IV A) and produce no perturbation of the core flow, whereas the core solution can not be determined in many cases without matching the Hartmann layer solutions, exception made of the very special symmetry described in section IV A.

D. Characteristic surfaces

It has been stated in the past that flow streamlines and electrical current lines must often lie on certain characteristic surfaces.[13,17,18] In a cavity Ω, these characteristic surfaces are defined as the union of the magnetic lines $\mathscr{A}$ having the same value for $\int_{\mathscr{A}} ds/\|\mathbf{B}\|$, where s is the distance along each magnetic field line and the integration is performed only on the connected part of the field line inside Ω (see Fig. 1). The

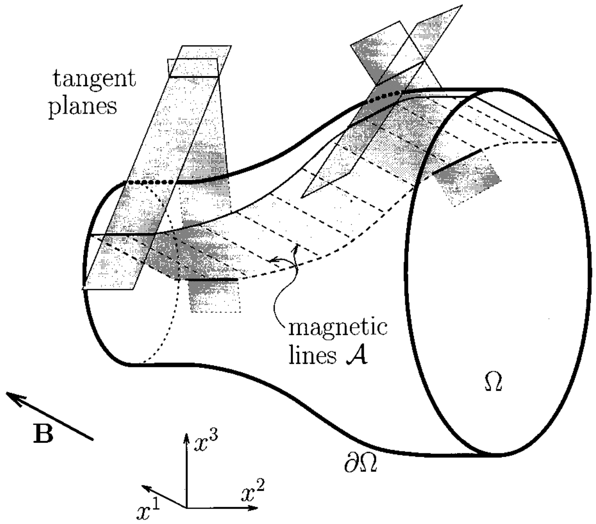

FIG. 1. Characteristic surface under a uniform magnetic field.

assumptions and proofs in the previous work are based on an analogy with the Kelvin-Helmholtz theorem and the induction equation while our purpose is to reexamine this concept with more physical arguments, somehow related to Kulikovskii's proof.[11] First we prove the following general theorem, which defines the exact conditions for the relevance of the characteristic surfaces, and then we discuss the application to MHD problems.

Theorem 1: *Let us consider a fluid physical domain* Ω *such that the length of each part of each magnetic field lines contained in* Ω *remains finite. We denote* $\mathbf{a}$ *a divergence-free vector field in* Ω*, such that*

- *curl* $(\mathbf{a}\wedge\mathbf{B})=\mathbf{0}$ *in* Ω;
- $\mathbf{a}.\mathbf{n}=0$ *on* $\partial\Omega$;

then, everywhere in Ω, $\mathbf{a}$ *is tangent to the characteristic surfaces*.

Proof: The field $\mathbf{a}$ is seen to satisfy the same equation as $\mathbf{j}$ in the core flow (16) when $d\mathbf{f}=0$. So its structure can be deduced from (17):

$$a^3=a_0^3, \tag{28a}$$

$$a^2=a_0^2, \tag{28b}$$

$$a_1=a_{01}-\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^1}\frac{\partial}{\partial x^2}\left(\sqrt{\det g}\,a_0^2\right)+\frac{\partial}{\partial x^3}\left(\sqrt{\det g}\,a_0^3\right)dx'^1, \tag{28c}$$

where $\mathbf{a}_0$ does not depend on the x^1 coordinate in the same metric (14) as previously defined. On each connected part of a magnetic field line in Ω, the two ends can be defined by the relations $x^1=x^+(x^2,x^3)$ and $x^1=x^-(x^2,x^3)$, with $x^-<x^+$. On the boundary defined by x^+, the condition of vanishing normal component, $d(x^1-x^+).\mathbf{a}=0$, can be rewritten using (28),

$$\frac{\sqrt{\det g}}{g_{11}}\,a_{01}-\frac{\partial}{\partial x^2}\int_0^{x^+}\sqrt{\det g}\,a_0^2dx^1-\frac{\partial}{\partial x^3}\int_0^{x^+}\sqrt{\det g}\,a_0^3dx^1=0. \tag{29}$$

The same relation holds for x^- and equation (15) states that $\sqrt{\det g}/g_{11}$ does not vary along a magnetic line. Subtracting the relation for x^- to that for x^+, (29) yields

$$\frac{\partial}{\partial x^2}\int_{x^-}^{x^+}\sqrt{\det g}\,a_0^2dx^1+\frac{\partial}{\partial x^3}\int_{x^-}^{x^+}\sqrt{\det g}\,a_0^3dx^1=0. \tag{30}$$

By analogy with (19), a satisfies

$$\frac{\partial}{\partial x^2}\left(\frac{\sqrt{\det g}}{g_{11}}\,a_0^2\right)+\frac{\partial}{\partial x^3}\left(\frac{\sqrt{\det g}}{g_{11}}\,a_0^3\right)=0. \tag{31}$$

Since $\sqrt{\det g}/g_{11}$ is independent of x^1, equations (30) and (31) give

$$d\left(\int_{x^-}^{x^+}g_{11}dx^1\right).\mathbf{a}=0. \tag{32}$$

Since the norm of $\mathbf{B}$ is $1/\sqrt{g_{11}}$, $s = x^1\sqrt{g_{11}}$ is the distance along a magnetic field line. The change of variable yields $\int_{x^-}^{x^+}g_{11}dx^1=\int_{s^-}^{s^+}ds/\|\mathbf{B}\|$. Equation (32) exactly means that $\mathbf{a}$, if considered as a vector field, is tangent to the surfaces composed of magnetic lines with the same value for $\int_{s^-}^{s^+}ds/\|\mathbf{B}\|$. □

Second proof: For a uniform magnetic field, there is a simpler and more visual proof. Under our assumptions, curl$(\mathbf{a}\wedge\mathbf{B})$ is equal to $(\mathbf{B}.\boldsymbol{\nabla})\mathbf{a}$. The vector field $\mathbf{a}$ is thus invariant along a magnetic line and must also be tangent to both boundaries crossed by the magnetic line. So, the direction of $\mathbf{a}$ is given as the intersection of both tangent plane surfaces (see figure 1). In this direction, the length of the part of the magnetic line intercepted by the fluid domain obviously does not vary, which proves the theorem for uniform magnetic fields. □

In MHD, the characteristic surfaces are important when either $\mathbf{u}$ or $\mathbf{j}$ in the core satisfies the conditions for $\mathbf{a}$, so that $\mathbf{u}$ or $\mathbf{j}$ must be tangent to the surfaces, i.e., the flow or electric current must follow these surfaces. For a curl-free non-electromagnetic body force, the condition curl$(\mathbf{j}\wedge\mathbf{B})=0$ is fulfilled, but the assumption of electrically insulating walls is not enough to state that the condition $\mathbf{j}.\mathbf{n}=0$ holds on $\partial\Omega$ for the core solution. A significant electric current may develop within the Hartmann layer so that there may be a significant electric current between the core and the Hartmann layer. For the velocity field, curl$(\mathbf{u}\wedge\mathbf{B})$ is only $\mathbf{0}$ when the electric current density is very small compared to the velocity. If the no-slip condition is available, then the non-permeable wall condition is still correct for the core since the normal velocity in the Hartmann layers is very small [see equation (27)].

Therefore $\mathbf{u}$ or $\mathbf{j}$ are only tangent to the characteristic surfaces under certain special conditions, which will be explored further in section IV.

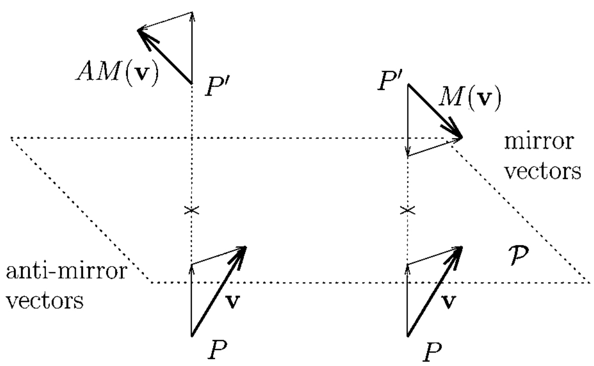

FIG. 2. Anti-mirror and mirror properties.

IV. SYMMETRY

It is in principle always fruitful to identify a symmetry in a given problem, since this allows one to consider only half a cavity. In addition, a symmetry can provide much valuable simplification when associated to the asymptotic structure of MHD flows. Here, we concentrate on orthogonal symmetries with respect to a plane surface.

A. Singular symmetry

Before describing the symmetry that we call singular, some definitions are needed. Let us consider a plane surface $\mathscr{P}$ and two points P and P' symmetric with respect to $\mathscr{P}$. The anti-mirror operator AM is defined for a scalar function f and a vector field $\mathbf{v}$ as

• $AM(f)(P)=-f(P')$;

• $AM(\mathbf{v})_{\parallel}(P)=-\mathbf{v}_{\parallel}(P')$ and $AM(\mathbf{v})_{\perp}(P)=+\mathbf{v}_{\perp}(P')$.

The subscripts $_{\parallel}$ and $_{\perp}$, respectively, denote the parallel and orthogonal parts of a vector relative to $\mathscr{P}$. In the following we say that an anti-mirror scalar function f satisfies $AM(f)=f$, and that an anti-mirror vector field $\mathbf{v}$ satisfies $AM(\mathbf{v})=\mathbf{v}$ (see Fig. 2). For the class of problems governed by equations (1) to (4); the occurrence of an anti-mirror solution is related to the following theorem.

Theorem 2: *If* $(\mathbf{u}, \mathbf{j}, p, \varphi)$ *is the unique solution of the equations (1), (2), (3) and (4) in a cavity* Ω, *and if*

• *there exists a plane surface* $\mathscr{P}$ *such that* Ω *is orthogonally symmetric with respect to* $\mathscr{P}$;

• $\mathbf{B}$, $\mathbf{f}$ *and the boundary conditions are invariant under the* AM *operator relative to* $\mathscr{P}$;

then, $\mathbf{u}$, $\mathbf{j}$, p *and* φ *are all anti-mirror in the whole cavity.*

Proof: The uniqueness of the $(\mathbf{u}, \mathbf{j}, p, \varphi)$ solution is proved by the Lax-Milgram theorem for linear differential equations with adequate boundary conditions. However, the set $[AM(\mathbf{u}), AM(\mathbf{j}), AM(p), AM(\varphi)]$ also satisfies equations (1) to (4). This is due to the commutative property of the AM operator with the gradient, divergence, curl and Laplacian operators, as well as its distributivity with respect to the cross product: indeed, for any given scalar function f and vector fields $\mathbf{v}$ and $\mathbf{w}$, it can be simply proved that $AM(\boldsymbol{\nabla} f)=\boldsymbol{\nabla}(AM(f))$, $AM(\text{div } \mathbf{v})=\text{div}(AM(\mathbf{v}))$, $AM(\text{curl } \mathbf{v})=\text{curl}(AM(\mathbf{v}))$, $AM(\boldsymbol{\Delta}\mathbf{v})=\boldsymbol{\Delta}(AM(\mathbf{v}))$ and $AM(\mathbf{v}\wedge\mathbf{w})=AM(\mathbf{v})\wedge AM(\mathbf{w})$. As the boundary conditions are assumed AM invariant, the transformation of the solution by AM is a solution and must by uniqueness be equal to the original solution. It is interesting to notice that the inertial term ''$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}$'', neglected in our analysis, does not satisfy this anti-mirror property. □

When this theorem can be applied, we propose to call the problem ''singularly symmetric''. This result is true for any value of Ha. Nevertheless, its application becomes especially fruitful when the large Hartmann-number asymptotic analysis can be applied, since the core solution must have the same symmetry. An important consequence of theorem 2 is that both $\mathbf{j}$ and $\mathbf{u}$ have zero tangential components in the symmetry plane $\mathscr{P}$, where $\mathbf{B}$ is perpendicular. This greatly simplifies the core structure (17) and (18) since, if $x^1=0$ was chosen on $\mathscr{P}$, the four components j_0^2, j_0^3, u_0^2 and u_0^3 vanish. The core solution is only defined in terms of j_{01} and u_{01} which depend on x^2 and x^3. Now, the complete asymptotic solution for the flow which is the sum of the core flow and Hartmann layers only depends on j_{01}, u_{01}, and both components of $\mathbf{u_h}$ on the boundary defined by x^+ since the other boundary is symmetric, and since $\mathbf{j_h}$ is related to $\mathbf{u_h}$.

A very interesting characteristic of the present singularly symmetric problems is that, in the derivation of the core solution, the Hartmann layers play a passive role. Let us assume that, at the wall, the velocity field vanishes. The condition of zero tangential velocity on the walls implies that $\mathbf{u_h}$ is of the same order of magnitude as the core velocity and thus the normal component in the Hartmann solution (27) is negligible. Therefore, the non-permeable boundary condition also applies for the core solution. By equation (18c), u_{01} has the same order of magnitude as the two other velocity components and it can be deduced, using (18a) and (18b) with $u_0^2=u_0^3=0$, that the core velocity is of the same order of magnitude as the core electric current density. The surface fields $\mathbf{j_h}$ and $\mathbf{u_h}$ are also of this same magnitude order, $\mathbf{u_h}$ because it cancels the core velocity at the wall and $\mathbf{j_h}$ because it is linked to $\mathbf{u_h}$ by the relation (25). In this special case of singular symmetry, because there is no significant mass or electric current flux within the Hartmann layers, the core solution can be derived separately.

The core solution is not yet complete, since j_{01} and u_{01} are undetermined. The information necessary to complete the solution comes from the nature of the electrical connection between the fluid and the exterior. In the following, we limit ourselves to the two simplest (and opposite) electrical boundary conditions: perfectly insulating walls or perfectly conducting walls (more precisely, with a uniform electric potential on the boundary).

1. Insulating walls

Using the same argument as that for the velocity, it is straightforward to deduce from equation (26) that the normal electric current component within the Hartmann layer is negligible compared to the core electric current density. Therefore, the normal component of the core current density must vanish at the wall. After some transformation, this condition for the core current density, $d(x^1-x^+).\mathbf{j}=0$ at the walls, takes the form

$$-\frac{\sqrt{\det g}}{g_{11}}\, j_{01}-\frac{\partial}{\partial x^2}\left[\int_0^{x^+} g_{11}\int_0^{x^1}\frac{(df)_{31}}{Ha^2}\,dx'^1 dx^1\right]$$

$$-\frac{\partial}{\partial x^3}\left[\int_0^{x^+} g_{11}\int_0^{x^1}\frac{(df)_{12}}{Ha^2}\,dx'^1 dx^1\right]=0. \tag{33}$$

This equation determines j_{01} since the non-electromagnetic force field **f** is known. The core electric current density is now completely determined. We have pointed out that the condition $d(x^1-x^+).\mathbf{u}=0$ also holds for the core velocity, so that

$$-\frac{\sqrt{\det g}}{g_{11}}\, u_{01}+\frac{\partial}{\partial x^2}\left[\int_0^{x^+} g_{11}\int_0^{x^1}(dj)_{31}dx'^1 dx^1\right]$$

$$+\frac{\partial}{\partial x^3}\left[\int_0^{x^+} g_{11}\int_0^{x^1}(dj)_{12}dx'^1 dx^1\right]=0. \tag{34}$$

This equation determines u_{01}. Both components of $\mathbf{u_h}$ may now be determined using the no-slip condition.

2. Conducting walls

Let us now assume a uniform electric potential at the wall, corresponding to a perfectly conducting (connected) wall: by singular symmetry, this wall potential must be zero. There is no electric potential jump through the Hartmann layer (because there is no significant mass flux) and the core potential is the same on $\mathscr{P}$ and at the walls. The integration of the core current **j** along a magnetic line between $\mathscr{P}$ and the wall $(x^1=x^+)$ can be written as

$$\int_{\mathscr{A}}\mathbf{j}=\int_0^{x^+} j_1 dx^1=\int_{\mathscr{A}}-d\varphi+\star(\mathbf{u}\wedge\mathbf{B})=0.$$

Using the structure of j_1 (17c) for a singular symmetry, j_{01} is given by

$$j_{01}x^+=\frac{g_{11}}{\sqrt{\det g}}\int_0^{x^+}\int_0^{x^1}\frac{\partial}{\partial x^2}\,(\sqrt{\det g}\,j^2)$$

$$+\frac{\partial}{\partial x^3}\,(\sqrt{\det g}\,j^3)dx'^1 dx^1. \tag{35}$$

Then, as for the previous case, the core velocity is determined by the non-permeable condition at the boundary (34).

For singular symmetry, for either insulated or perfectly conducting walls, the electric current density and the velocity are of the same magnitude order, namely $\mathbf{f}/Ha^2$. The relation $\mathrm{curl}(\mathbf{u}\wedge\mathbf{B})=\mathbf{0}$ is far from being satisfied and thus, the characteristic surfaces are not relevant. Two applications will be studied in section V which have a singular symmetry. The solution along each magnetic field line is determined by conditions at the symmetry plane and wall, and is entirely independent of the solution along other magentic field lines. Thus, the local flow is independent of the flow at a distance, provided that the singular symmetry holds everywhere. As a consequence, the parallel layers do not affect the core flow.

B. Regular symmetry

Some flows have a different symmetry, with respect to a plane $\mathscr{P}$, which we call a regular symmetry. The mirror operator M with respect to the planar surface $\mathscr{P}$ is needed to define this symmetry. With the same convention we define

- $M(f)(P)=+f(P')$;
- $M(\mathbf{v})_{\parallel}(P)=+\mathbf{v}_{\parallel}(P')$ and $M(\mathbf{v})_{\perp}(P)=-\mathbf{v}_{\perp}(P')$.

A mirror scalar function f or a mirror vector field **v** is invariant under the mirror M operator (see figure 2). An analogous result as theorem 2 is derived for a regular symmetry.

Theorem 3: *If* $(\mathbf{u},\mathbf{j},p,\varphi)$ *is the unique solution for equations (1), (2), (3) and (4) in a cavity* Ω *and if*

- *there exists a plane surface* $\mathscr{P}$ *such that* Ω *is orthogonally symmetric with respect to* $\mathscr{P}$;
- **B** *is invariant under the AM operator*;
- **f** *and the boundary conditions are invariant under the mirror M operator;*

then, **u**, **j**, p *and* φ *are all mirror in the whole cavity*.

The only difference between the proof and that for theorem 2 is that $M(\mathbf{v}\wedge\mathbf{w})=M(\mathbf{v})\wedge AM(\mathbf{w})$ for vector fields **v** and **w**. Contrary to the case of singular symmetry, it is interesting to notice that the inertial term ''$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}$'' would satisfy this mirror property. When the conditions of theorem 3 are satisfied, we have a regular symmetry. A consequence of the theorem is that the normal component of **j** and **u** are zero on $\mathscr{P}$. When combined with an asymptotic core structure (17) and (18), this condition yields $j_{01}=0$ and $u_{01}=0$. There is less simplification than for a singular symmetry since four unknown core functions j_0^2, j_0^3, u_0^2 and u_0^3, and both components of $\mathbf{u_h}$ still have to be determined with equations (19) and (20), and with the boundary conditions.

We shall concentrate on electrically insulating boundaries for a bounded cavity Ω. Thus, the characteristic surfaces close on themselves: they form rings.[11] We shall later see that the velocity streamlines lie on these characteristic surfaces. In this special case, the asymptotic solution can be derived by general formulas. By integrating the velocity and electric current density along magnetic lines, we first derive expressions for the mass and electrical transfers (**Q** and **I**) as functions of x^2 and x^3:

$$\star\mathbf{Q}=\int_{\mathscr{A}}\star\mathbf{u}, \tag{36}$$

$$\star\mathbf{I}=\int_{\mathscr{A}}\star\mathbf{j}+\frac{\star\mathbf{j_{h+}}}{Ha|\mathbf{B}.\mathbf{n}|}+\frac{\star\mathbf{j_{h-}}}{Ha|\mathbf{B}.\mathbf{n}|}, \tag{37}$$

where $\mathscr{A}$ is the portion of a magnetic line contained in Ω. The last two terms in (37) are the electrical flux in the upper and lower Hartmann layer: the $(Ha|\mathbf{B}.\mathbf{n}|)^{-1}$ factor is the thickness deduced from (23). The corresponding terms for (36) are negligible due to the no-slip condition of the velocity field. Thanks to the regular symmetry, the last two terms of (37) are equal and the relation (25) can also be written as

$\star\mathbf{j_h}=|\mathbf{B}.\mathbf{n}|\mathbf{u_h}$, while $\mathbf{u_h}$ is related to the core velocity with the no-slip condition:

$$u_{h2}+u_2+u_1\,\frac{\partial x^+}{\partial x^2}=0, \tag{38a}$$

$$u_{h3}+u_3+u_1\,\frac{\partial x^+}{\partial x^3}=0. \tag{38b}$$

The relations (36) and (37) can be expressed using the core structure of the velocity and electric current density fields:

$$(\star Q)_2dx^2+(\star Q)_3dx^3=\left[-u_0^3\,\frac{\sqrt{\det g}}{g_{11}}\int_{x^-}^{x^+}g_{11}dx^1-\int_{x^-}^{x^+}g_{11}\int_0^{x^1}(dj)_{12}dx'^1dx^1\right]dx^2+\left[u_0^2\,\frac{\sqrt{\det g}}{g_{11}}\int_{x^-}^{x^+}g_{11}dx^1+\int_{x^-}^{x^+}g_{11}\int_0^{x^1}(dj)_{31}dx'^1dx^1\right]dx^3, \tag{39}$$

$$(\star I)_2dx^2+(\star I)_3dx^3=\left[-j_0^3\,\frac{\sqrt{\det g}}{g_{11}}\int_{x^-}^{x^+}g_{11}dx^1+\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{12}}{Ha^2}\,dx'^1dx^1+2\,\frac{u_{h2}}{Ha}\right]dx^2+\left[j_0^2\,\frac{\sqrt{\det g}}{g_{11}}\int_{x^-}^{x^+}g_{11}dx^1-\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{31}}{Ha^2}\,dx'^1dx^1+2\,\frac{u_{h3}}{Ha}\right]dx^3. \tag{40}$$

For electrically insulating walls, **Q** and **I** have a zero component through the line boundary of the 2D surface obtained by ''compressing'' Ω along magnetic lines. Moreover **Q** and **I** are divergence-free. In order to simplify the notations, we perform a change of variable by choosing $x^2=\int_{x^-}^{x^+}g_{11}dx^1$ (this new coordinate x^2 is independent of the initial choice of x^2 and x^3 and its isovalues are the characteristic surfaces; furthermore, it is orthogonal to x^1: $g_{12}=0$). The divergence-free condition for **Q** and **I** can be written, using (19) and (20), as

$$x^2(-dj_{23|x^1=0})+u_0^2\,\frac{\sqrt{\det g}}{g_{11}}+\frac{\partial}{\partial x^3}\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}(dj)_{12}dx'^1dx^1\right]+\frac{\partial}{\partial x^2}\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}(dj)_{31}dx'^1dx^1\right]=0, \tag{41}$$

$$x^2\left(\frac{df_{23|x^1=0}}{Ha^2}\right)+j_0^2\,\frac{\sqrt{\det g}}{g_{11}}-\frac{\partial}{\partial x^3}\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{12}}{Ha^2}\,dx'^1dx^1\right]-\frac{\partial}{\partial x^2}\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{31}}{Ha^2}\,dx'^1dx^1\right]+2Ha^{-1}du_{h23}=0. \tag{42}$$

It is useful to consider the total amount of the electrical flux across a characteristic surface ($x^2=c^{st}$) which is zero since it forms a ring. We integrate (40) along Γ, a closed line in (x^2,x^3) corresponding to a characteristic surface:

$$x^2\oint_\Gamma\left[j_0^2\,\frac{\sqrt{\det g}}{g_{11}}\right]-\oint_\Gamma\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{31}}{Ha^2}\,dx'^1dx^1\right]+2Ha^{-1}\oint_\Gamma u_{h3}=0. \tag{43}$$

Equation (42) gives j_0^2 in terms of $d\mathbf{u}_h$. After substitution, (43) becomes an ordinary differential equation governing $\oint_\Gamma u_{h3}$:

$$-(x^2)^2\oint_\Gamma\left[\frac{df_{23|x^1=0}}{Ha^2}\right]+x^2\,\frac{d}{dx^2}\oint_\Gamma\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{31}}{Ha^2}\,dx'^1dx^1\right]-2\,\frac{x^2}{Ha}\,\frac{d}{dx^2}\left[\oint_\Gamma u_{h3}\right]-\oint_\Gamma\left[\int_{x^-}^{x^+}g_{11}\int_0^{x^1}\frac{(df)_{31}}{Ha^2}\,dx'^1dx^1\right]+\frac{2}{Ha}\oint_\Gamma u_{h3}=0. \tag{44}$$

The solution takes the form

$$\oint_\Gamma u_{h3}=-\frac{1}{2Ha}\oint_\Gamma\left[\int_x^{x^+}-g_{11}f_3dx^1\right]. \tag{45}$$

It is obvious that $\oint_\Gamma u_{h3}$ is comparable to $d\mathbf{f}/Ha$ while equation (42) implies that $j_0^2\sim d\mathbf{f}/Ha^2$. Then (41) leads to $u_0^2\sim j_0^2$ (as the characteristic surface is closed, $j_0^3\sim u_0^2$ because there is no global electric potential difference over one turn). So u_0^3 must be of order $d\mathbf{f}/Ha$ and $\mathbf{u_h}$, given by (38), takes the form

$$u_{h2}=\left[g_{23}+\left(\frac{\partial x^+}{\partial x^2}\right)\left(\frac{\partial x^+}{\partial x^3}\right)g_{11}\right]u_0^3,$$

$$u_{h3}=\left[g_{33}+\left(\frac{\partial x^+}{\partial x^3}\right)^2g_{11}\right]u_0^3.$$

Moreover, equation (20) shows that $\sqrt{\det g}/g_{11}u_0^3$ is independent of x^3 (to the Ha^{-1} order). So, the main component of the velocity, u_0^3, is given by the relation

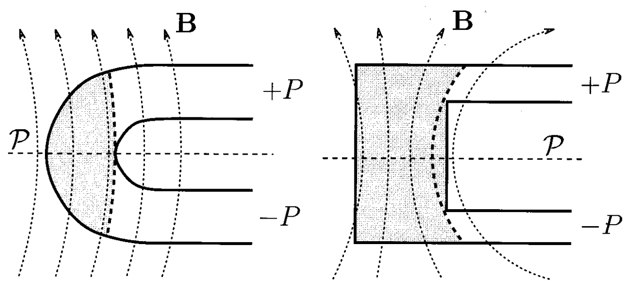

FIG. 3. Stagnant zone in a bend.

$$u_0^3 = \frac{g_{11}\oint_\Gamma u_{h3}}{\sqrt{\det g}\oint_\Gamma[[g_{33}+(\partial x^+/\partial x^3)^2 g_{11}]g_{11}/\sqrt{\det g}]}. \quad (46)$$

In applications, the velocity field, following the characteristic surfaces, is determined by the key solution (45) and by substituting the result for $\oint_\Gamma u_{h3}$ in equation (46).

Unlike the streamlines, the electric current lines do not lie on the characteristic surfaces: the topological constraint of a closed cavity (of finite extend) leads to closed characteristic surfaces forming rings, thus precluding large electric potential differences on each surface. This flow pattern is quite different from that for duct flows with varying cross-section or magnetic field intensity.[13]

The order of magnitude for the velocity is found to be $d\mathbf{f}/Ha$ in this case of regular symmetry. A similar (yet more complicated) solution as (45) could be written for general configurations, which are neither singularly nor regularly symmetric: the functions x^+ and x^- would be different (not just their sign) as well as the tangent fields $\mathbf{j}_{\mathbf{h}+}$ and $\mathbf{j}_{\mathbf{h}-}$. It appears that the $d\mathbf{f}/Ha^2$ order of magnitude for the velocity can only be expected for singular configurations.

V. APPLICATIONS

In this section, we present some practical applications of our analysis. First, the case of a pressure-driven flow in a bend is considered; it is singularly symmetric and admits a simple solution. Then, asymptotic solutions for three buoyancy-driven flows are given: two are singularly symmetric, the other regularly.

A. A singular pressure-driven flow in a bend

A pipe is bended in a region of magnetic field as illustrated in the Fig. 3. The pipe as well as the magnetic lines are symmetric with respect to the plane $\mathscr{P}$. The flow is pressure-driven: a pressure $+P$ is assumed at the entrance and $-P$ at the outlet. First, let us show that this configuration is singularly symmetric, i.e. that the requirements of theorem 2 are fulfilled. The volume force is zero (gravity here only leads to hydrostatic pressure) and can thus be considered anti-mirror invariant. Concerning the boundary conditions $\mathbf{u}=0$ and $\mathbf{j}.\mathbf{n}=0$ for electrically insulating walls (or $\mathbf{j}\wedge\mathbf{n}=0$ for perfectly conducting walls), they also are anti-mirror invariant. The pressure boundary condition is also invariant under the anti-mirror operator. As the configuration is singular, the results derived in section IV A for insulating or perfectly conducting walls apply in a region of connected magnetic lines in the fluid (the shaded regions in figure 3). As $\mathbf{f}$ is zero, the solution of equation (33) or (35) is $j_{01}=0$. The wall being either insulating or perfectly conducting, there is no electric current density in the bend region. Equation (34) then leads to $u_{01}=0$. There is a zero velocity field in the bend. The whole mass flux must flow within the interior layer (bold dashed lines in figure 3). A region of stagnant fluid exists, which should be taken into account when heat or species transfer are to be considered. A poor heat or species transfer, only due to diffusion, is expected in such a configuration. This result is true for any bend geometry and for any magnetic lines, provided the symmetry is maintained. Figure 3 illustrates two such configurations.

It is interesting to notice that Molokov and Bühler[19] have found a non-zero velocity field in such regions under a uniform magnetic field, in the case of a finite electrical wall conductivity: the electric current flow in the wall creates potential gradients and precludes the derivation of a local solution. On the contrary, insulating walls prevent any current flow and perfectly conducting walls, by impressing a uniform electric potential at the walls, prevent any electromotive force within the fluid. Molokov and Bühler used another asymptotic approach based on pressure and electrical potential, combined with numerics. In the limiting case of large wall conductivity they also predict that the velocity vanishes in the bend, the mass flux being confined to the parallel layers.

B. Two singular buoyancy-driven flows

We first consider the case of a buoyancy-driven flow with a uniform magnetic field, since the algebra is simpler than that for a non-uniform field. In the Boussinesq approximation, there is no mathematical contradiction between the existing buoyant force due to mass density variations and our assumption of uniform physical properties; indeed, the mass density is assumed uniform and the buoyant force proportional to the temperature. The dimensional force density is expressed in the Boussinesq approximation,

$$\tilde{\mathbf{f}} = \rho\mathbf{g}[1-\beta(T-T_0)],$$

where β is the volumetric expansion coefficient of the fluid, T the temperature field and T_0 some temperature reference. Since only the curl of $\tilde{\mathbf{f}}$ produces convection, we take equivalently $\tilde{\mathbf{f}}=-\rho\beta T\mathbf{g}$. In our dimensional scales defined in section II, with a uniform thermal gradient $T=Gx^2$ (this coordinate x^2 is independent of the characteristic surfaces as seen in Fig. 4), this force density and its curl take the form

$$\mathbf{f}=Grx^2dx^1,$$
$$d\mathbf{f}=-Grdx^1\wedge dx^2, \quad (47)$$

where $Gr=\beta gGH^4/\nu^2$ is the Grashoff number. The reader can check that this configuration (figure 4) is singularly sym-

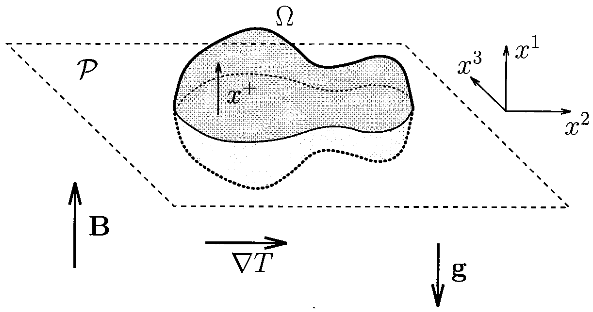

FIG. 4. A singular buoyancy-driven configuration.

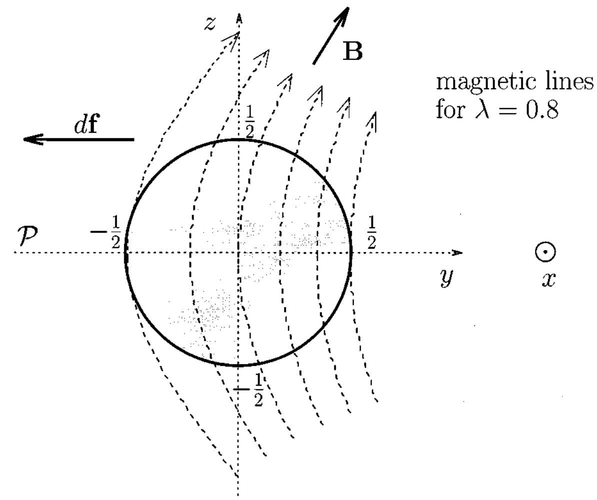

FIG. 5. Non-uniform magnetic field and buoyancy.

metric. The analysis of section IV A can be applied; using expression (47) for $d\mathbf{f}$, equations (33) and (34) become for the case of insulating walls

$$j^2=0,$$

$$j^3=+\frac{Gr}{Ha^2}\,x^1,$$

$$j^1=+\frac{Gr}{2Ha^2}\,\frac{\partial x^{+2}}{\partial x^3},$$

$$u^2=\frac{Gr}{Ha^2}\left(-x^1+\frac{x^1}{2}\,\frac{\partial^2 x^{+2}}{\partial (x^3)^2}\right),$$

$$u^3=-\frac{Gr}{Ha^2}\,\frac{x^1}{2}\,\frac{\partial^2 x^{+2}}{\partial x^3 \partial x^2}, \tag{48}$$

$$u^1=\frac{Gr}{Ha^2}\left(-\frac{1}{2}\,\frac{\partial x^{+2}}{\partial x^2}-\frac{1}{4}\,\frac{\partial x^{+2}}{\partial x^3}\,\frac{\partial^2 x^{+2}}{\partial x^3\partial x^2}+\frac{1}{4}\,\frac{\partial x^{+2}}{\partial x^2}\,\frac{\partial^2 x^{+2}}{\partial (x^3)^2}\right).$$

The singular symmetry allows us to derive easily this three-dimensional asymptotic solution. The velocity depends on the shape of the cavity, defined by the function x^+, involving its variations up to its second derivatives. The choice of this shape may lead to very different flows, always of magnitude order $d\mathbf{f}/Ha^2$, provided the singular symmetry holds.

For perfectly conducting walls of uniform electric potential, equations (35) and (34) lead to

$$j^2=0,$$

$$j^3=+\frac{Gr}{Ha^2}\,x^1,$$

$$j^1=0,$$

$$u^2=-\frac{Gr}{Ha^2}\,x^1,$$

$$u^3=0,$$

$$u^1=-\frac{Gr}{2Ha^2}\,\frac{\partial x^{+2}}{\partial x^2}. \tag{49}$$

In this case, the electric current density is seen to be independent of the shape of the cavity and the velocity depends only slightly on it, just in order to satisfy the non-permeability condition.

Such results as (48) and (49) can be applied in configurations of Bridgman crystal growth: some particular cases of these results can be found in Ref. 7. The velocity field can then be used to study the species transfer governing solute segregation in crystals.[20]

Let us now consider a quite similar case of buoyancy-driven flow in the presence of a non-uniform magnetic field. For the sake of algebraic simplicity, we assume here a cylindrical shape (Fig. 5) with an axial thermal gradient and magnetic lines lying in the cross-section plane: $\mathbf{B}=(1+\lambda y)\mathbf{e}_z+\lambda z\mathbf{e}_y$, defined in the (x,y,z) Cartesian coordinate system. This magnetic field is a superposition of a uniform vertical component and a linear field scaled by a parameter λ (when $\lambda=0$, the magnetic field is purely uniform). This field is clearly divergence-free and curl-free. A look at figure 5 will convince the reader that the singular symmetry holds. We first derive a coordinate system (x^1,x^2,x^3) adapted to the magnetic field such that $\mathbf{B}=dx^1$ and the metric tensor in this orthogonal coordinate system,

$$\begin{cases} x^1=z+\lambda yz, \\ x^2=-y+\frac{\lambda}{2}\,z^2-\frac{\lambda}{2}\,y^2, \\ x^3=x, \end{cases}$$

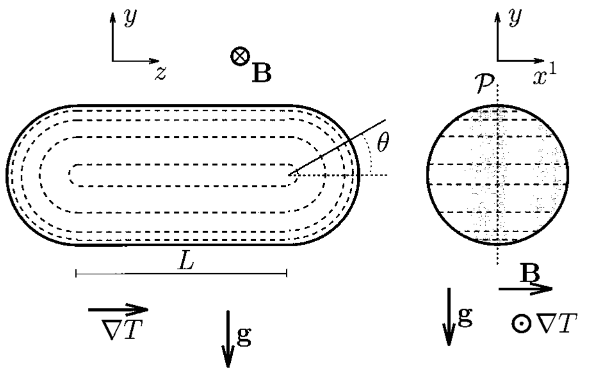

FIG. 6. Buoyancy in a regular configuration.

$$\begin{cases} g_{11}=\frac{1}{2}\left(\left(\frac{1}{2}-\lambda x^2\right)^2+(\lambda x^1)^2\right)^{-1/2}, \\ g_{22}=g_{11}, \\ g_{33}=1. \end{cases}$$

Then $d\mathbf{f}$ must be expressed in the (x^1,x^2,x^3) coordinate system as well as the shape function $(y^2+z^2=0.25)$. For insulating boundaries, equation (33) must be solved, while equation (34) leads to $u_{01}=0$. Then, u^3, the single non-zero velocity component, is obtained using equation (18a). After some tedious algebra (calculations of integrals), we find

$$u^3=-2\frac{Gr}{Ha^2}x^1\left[1+\frac{\lambda^2}{2}-4\lambda x^2\right]^{-3/2}.$$

Expressed in the (x,y,z) system, the velocity can be written as

$$u_x=-2\frac{Gr}{Ha^2}(z+\lambda yz)\left[1+4\lambda y+\frac{\lambda^2}{2}-2\lambda^2z^2+ +2\lambda^2y^2\right]^{-3/2}. \tag{50}$$

For small values of λ, a first order term takes the form $u_x \simeq -2(Gr/Ha^2)(z-5\lambda yz)$, previously found using an analysis suitable for weakly non-uniform magnetic fields.[21] When λ is zero, we find: $u_x=-2(Gr/Ha^2)z$, which is also the result of equation (48) when the shape is given by $x^+ = \sqrt{1/4-y^2}$ for a cylindrical circular cavity.

C. A regular buoyancy-driven flow

We consider a cavity with insulating boundaries: it is a cylinder of length L, capped by two hemispheres (see Fig. 6). An axial thermal gradient normal to the gravity is imposed. The uniform magnetic field is assumed to be orthogonal to both gravity and thermal gradient. This configuration is regularly symmetric with respect to the vertical plane of symmetry orthogonal to the magnetic field. Characteristic surfaces form closed rings (dashed lines in figure 6) and the analysis developed in section IV B can be applied. In the cylindrical part of the cavity, the only ''free'' coordinate x^3, is chosen as the axis linear coordinate $(x^3=z)$, while an angular one is of practical use at the ends $(x^3=\theta)$. The functions defining the shape of the cylinder and the hemispheres are the same, $y(x^2)=r(x^2)$, and for a circular cross-section of diameter unity: $x^2 = \sqrt{1-4y^2}$. The general solution (45) allows us to derive the explicit solution:

$$\oint_\Gamma u_{h3}=\frac{Gr}{Ha}\left(Ly+\pi\frac{y^2}{2}\right)x^2.$$

From equation (46), the velocity magnitude can be expressed,

$$\sqrt{g_{33}u_0^3}=\frac{Gr}{Ha}\frac{L+\pi(y/2)}{2L+2\pi y}x^2y. \tag{51}$$

In the two limits of large or small length L, this solution is written as

$$L\to\infty\ \sqrt{g_{33}u_0^3}=\frac{Gr}{Ha}\frac{y}{2}x^2,$$

$$L\to 0\ \sqrt{g_{33}u_0^3}=\frac{Gr}{Ha}\frac{y}{4}x^2.$$

In the analysis, the symmetric shape in the plane (x^1,x^2) has not yet been specified. One can substitute the circular equation, $x^2 = \sqrt{1-4y^2}$, or any other, to fully describe the solution.

VI. CONCLUDING REMARKS

We have presented an original asymptotic analysis for the flows of electrically conducting fluids under a strong and possibly non-uniform magnetic field (see section III). This analysis is very general since it can handle any given driving body force, denoted by $\tilde{\mathbf{f}}$. When $\tilde{\mathbf{f}}=\mathbf{0}$, the classical case of pressure-driven flows can be analyzed. For example, the classical Shercliff solution[22] for the fully-established flow in a circular duct in the presence of a strong uniform magnetic field can be derived within this general theory. In natural convection problems, the buoyant force can also be taken into account for a small thermal Péclet number. Indeed, the force density $\tilde{\mathbf{f}}$ must be known, *a priori*, which is the case if the temperature field is not affected by convection (otherwise, the force density depends on the solution for the velocity). Other external forces than buoyancy can be treated within our theoretical frame, e.g., the magnetic force with $\tilde{\mathbf{f}}=(\chi/2\mu_0)\nabla B^2$ where χ denotes the magnetic susceptibility of the fluid and μ_0 the vacuum permeability: it is independent of the velocity solution under our assumption of small magnetic Reynolds number. As shown in the analysis, only the curl of the body force can produce motion in a closed cavity. So, the magnetic force leads to convection in the fluid only if χ is non-uniform. This occurs in a paramagnetic fluid in a thermal gradient, since its magnetic susceptibility is proportional to the inverse of the thermodynamic temperature.[23]

Our general asymptotic analysis is simplified for two symmetries. It is found that the nature of symmetry, denoted here by regular or singular, has as much effect as the electrical conductivity of the walls on the characteristics of the flow and on the magnitude of the velocity.

In singularly symmetric configurations, the simplification leads to a local (i.e., in the vicinity of a magnetic line) asymptotic solution. The non-dimensional velocity is proportional to Ha^{-2}, for any wall conductivity, and there is no significant electric current flowing within the Hartmann layers.

In regularly symmetric configurations, the velocity is proportional to Ha^{-2} for conducting walls and to Ha^{-1} for insulating walls. Most of the classical pressure-driven duct flows pertain to the regular symmetry, such as the Hartmann flow. The asymptotic solution is generally difficult to derive; a two-dimensional partial differential equation must be solved. With insulating walls, there is a significant electric current inside the Hartmann layers and the characteristic surfaces provide insights in the flow solution.

The terms ''singular'' and ''regular'' have been chosen because general configurations (neither singular nor regular) have the same characteristics as regular ones, since the regular contribution may be Ha times larger than the singular one. Besides, it can be inferred from Murphy's law that the combination of an easy singular problem and a difficult regular problem remains difficult. Nevertheless, singular configurations are of great interest, because the mass transport may be very desirable or undesirable.

Our analysis shows that the concepts of differential geometry can be a powerful mathematical tool in fluid mechanics. In the asymptotic analysis, the choice of x^1 such that $\mathbf{B}=dx^1$ and, in section IV B, the choice of $x^2=\int ds/\|\mathbf{B}\|$ are both of the utmost importance. These curvilinear coordinates, based on the magnetic field and on the geometry, have been proved to be very useful. Differential geometry provides a very general framework.

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