Fluid Dynamics Research

Note on braking and stabilization laws for buoyant flows under a weak magnetic field

To cite this article: T Alboussière *et al* 2003 *Fluid Dyn. Res.* **33** 287

FLUID DYNAMICS RESEARCH

Fluid Dynamics Research 33 (2003) 287–297

Note on braking and stabilization laws for buoyant flows under a weak magnetic field

T. Alboussière[a], D. Henry[b,*], S. Kaddeche[c]

[a] *Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom*
[b] *Laboratoire de Mécanique des Fluides et d'Acoustique, UMR-CNRS 5509, Ecole Centrale de Lyon/Université Claude Bernard - Lyon 1, ECL-BP 163, 69131 Ecully Cedex, France*
[c] *Institut National des Sciences Appliquées et de Technologie, BP 676, 1080 Tunis Cedex, Tunisia*

Received 24 April 2003; received in revised form 28 May 2003; accepted 16 June 2003
Communicated by T. Mullin

Abstract

We consider the effect of a constant magnetic field on buoyant flows generated by temperature gradients. We focus on the domain of weak magnetic fields, i.e., small values of the Hartmann number Ha, for which general scaling laws can be derived. Concerning the braking of these buoyant flows, it was found to scale at small Ha as even powers of Ha. Concerning the damping of the oscillations, it can be shown that the instability characteristics, critical threshold expressed through the critical Grashof number Gr_c, critical eigenvector, and critical pulsation also scale as even powers of Ha. In particular, this gives an initial MHD stabilization effect at small Ha of the form $Gr_c - Gr_{c_0} \sim Ha^2$ where Gr_{c_0} is the critical Grashof number at $Ha = 0$. These findings have been illustrated by results obtained in the case of the flow in an infinite layer.

1. Introduction

Material processing technologies often involve a liquid phase where convective motions and thus heat and mass transport must be controlled in order to improve the material quality (Hurle, 1966; Pimputkar and Ostrach, 1981). When metallic materials are concerned, a magnetic field can be used as it allows, through the Lorentz forces, both the braking of the flow and the damping of the instabilities. One of these material processing techniques is the Bridgman crystal growth where the melt sample, pulled out of the furnace, progressively solidifies, and where the convective motions correspond to long convection loops. The action of the magnetic field on such convective flows has

* Corresponding author. Tel.: +33-4-72-18-61-70; fax: +33-4-78-64-71-45.
E-mail address: henry@mecaflu.ec-lyon.fr (D. Henry).

0169-5983/$30.00
doi:10.1016/S0169-5983(03)00081-9

been the subject of many studies. Among them, for buoyant flows, we can cite experimental works (Hurle et al., 1974; Okada and Ozoe, 1992; Juel et al., 1999; Davoust et al., 1999; Hof, 2001; Hof et al., 2003), theoretical works (Alboussière et al., 1993) and numerical works (Oreper and Szekely, 1983, 1984; Kim et al., 1988; Ozoe and Okada, 1989; BenHadid and Henry, 1996, 1997; BenHadid et al., 1997; Touihri et al., 1999; Gelfgat and Bar-Yoseph, 2001; Kaddeche et al., 2003) which concern different configurations, either close to industrial processes or more academic. These studies have generally shown the relative efficiency of the magnetic field on the flow braking or on the flow stabilization. Very often, scaling laws were sought for these effects, principally for large magnetic field, i.e., large values of the Hartmann number Ha. Classical asymptotic laws of decrease of the flow velocity $\boldsymbol{V}$ for large magnetic field have been obtained, typically, $\boldsymbol{V} \sim Ha^{-2}$ and $\boldsymbol{V} \sim Ha^{-1}$ laws, depending on the configuration of the cavity and on the part of the flow considered, i.e., the flow in the core, in the Hartmann layers or in the parallel layers. Concerning the stabilization of flows, the laws principally give the increase of the thresholds where the instabilities are triggered. Asymptotic laws as $Gr_c \sim Ha$ have been found in a simplified infinite layer model under horizontal magnetic field (Kaddeche et al., 2003), but generally strong increase of the thresholds are observed over a moderate range of Ha with fitted laws for $Gr_c - Gr_{c_0}$ (where Gr_{c_0} is the critical Grashof number at $Ha=0$) which scale as Ha^2 or even as $\exp(Ha^2/K)$ where K is a constant (Hof, 2001; Gelfgat and Bar-Yoseph, 2001; Kaddeche et al., 2003). From all these studies, the obtention of universal laws for the action of the magnetic field, particularly for the stabilization of the flow, seems difficult, even at high Ha, as it depends on many factors as the configuration of the cavity and the orientation of the field. In this study, we show that for a weak magnetic field universal laws exist for the decrease of the convective flow as well as for the stabilization of the flow, which scale as Ha^2. Despite its validity for small values of Ha, this result, which has never been clearly stated in the literature, is interesting as most stability experiments on convective flows have been performed with moderate (even weak) magnetic fields because of the good efficiency of the magnetic field to damp oscillations and the difficulty to experimentally generate the high Grashof numbers which would be necessary to destabilize flows at high magnetic fields (Hof, 2001). Moreover, this result sheds new light on the Ha^2 law found as initial stabilization under magnetic field in many experiments (Hof, 2001). These findings will be illustrated by the results obtained in the case of an infinite layer submitted to a horizontal temperature gradient and subject to a magnetic field with different orientations.

2. Governing equations and scaling laws for the magnetic damping of the basic flow

We consider a viscous electrically conducting fluid with a constant electric conductivity σ_e contained in a cavity subject to a temperature gradient and submitted to an external constant magnetic field $\boldsymbol{B}_0$. We assume the existence of a reference length L and of a reference temperature gradient $\nabla\tilde{T}$. The orthogonal reference axes are x, y, and z, z being in the vertical direction, with respective unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$. The fluid is assumed to be Newtonian with constant kinematic viscosity v and thermal diffusivity κ. According to the Boussinesq approximation, the fluid density is considered as constant except in the buoyancy term where it is taken as temperature dependent according to the law $\rho = \rho_0(1 - \beta(\tilde{T} - \tilde{T}_0))$ where β is the thermal expansion coefficient and $\tilde{T}_0$ a reference temperature.

The magnetic field $\boldsymbol{B}$ is the sum of the applied magnetic field $\boldsymbol{B}_0$ and the induced field $\boldsymbol{b}$ such that $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$. Since in most laboratory experiments the magnetic Reynolds number $Re_{\mathrm{m}} = \mu\sigma_{\mathrm{e}}\tilde{V}_0 H$, where μ is the magnetic permeability and $\tilde{V}_0$ a characteristic velocity, is very small, the induced magnetic field $\boldsymbol{b}$ is negligible. Considering L, L^2/v, v/L, $\nabla\tilde{T}L$ and $v|\boldsymbol{B}_0|$ as scale quantities for length, time, velocity, temperature and induced electric potential, respectively, the dimensionless equations are then:

$$\nabla \cdot \boldsymbol{V} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\nabla P + \nabla^2 \boldsymbol{V} + Gr\, T \boldsymbol{e}_z + Ha^2\, \boldsymbol{J} \times \boldsymbol{e}_{\boldsymbol{B}_0}, \tag{2}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{V} \cdot \nabla T) = \frac{1}{Pr}\nabla^2 T, \tag{3}$$

where the dimensionless variables are the velocity vector $\boldsymbol{V} = U\boldsymbol{e}_x + V\boldsymbol{e}_y + W\boldsymbol{e}_z$, the pressure P and the temperature T, and $\boldsymbol{e}_{\boldsymbol{B}_0}$ is the unit vector in the direction of $\boldsymbol{B}_0$. The dimensionless parameters are the Grashof number $Gr = g\beta\nabla\tilde{T}L^4/v^2$, the Prandtl number $Pr = v/\kappa$, and the Hartmann number $Ha = |\boldsymbol{B}_0|L\sqrt{\sigma_{\mathrm{e}}/\rho_0 v}$.

In the equation of motion (2), the body force $Ha^2\, \boldsymbol{J} \times \boldsymbol{e}_{\boldsymbol{B}_0}$ is the Lorentz force due to the interaction between the induced electric current density $\boldsymbol{J}$ and the applied magnetic field $\boldsymbol{B}_0$. The dimensionless electric current density $\boldsymbol{J}$ is given by Ohm's law for a moving fluid:

$$\boldsymbol{J} = -\nabla\Phi + \boldsymbol{V} \times \boldsymbol{e}_{\boldsymbol{B}_0}, \tag{4}$$

where Φ is the dimensionless electric potential. This electric potential can be determined through the continuity equation for the electric current density, $\nabla \cdot \boldsymbol{J} = 0$. Appropriate mechanical, thermal and electric boundary conditions in agreement with the chosen configuration have to be considered. In such a situation, for not too large values of Gr, a steady flow can be obtained. This steady flow is a solution of Eqs. (2) and (3) at steady state coupled with Eq. (4) and with the continuity constraints for the two vector fields $\boldsymbol{V}$ and $\boldsymbol{J}$, i.e., $\nabla \cdot \boldsymbol{V} = 0$ and $\nabla \cdot \boldsymbol{J} = 0$.

In the limit of small Ha, the solution for $\boldsymbol{V}$, $\boldsymbol{J}$, P, Φ, and T can be expressed as a Taylor expansion with respect to Ha. The expression for $\boldsymbol{V}$ is

$$\boldsymbol{V} = \boldsymbol{V}_0 + Ha\, \boldsymbol{V}_1 + Ha^2\, \boldsymbol{V}_2 + Ha^3\, \boldsymbol{V}_3 + O(Ha^4), \tag{5}$$

where $\boldsymbol{V}_0$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, and $\boldsymbol{V}_3$ depend on Gr and Pr. Similar expansions are defined for $\boldsymbol{J}$, P, Φ, and T. Using such expansions, we can develop the steady system deduced from (2) to (3) at the different orders in Ha. At order 0, we have

$$(\boldsymbol{V}_0 \cdot \nabla)\boldsymbol{V}_0 = -\nabla P_0 + \nabla^2 \boldsymbol{V}_0 + Gr\, T_0 \boldsymbol{e}_z,$$

$$(\boldsymbol{V}_0 \cdot \nabla T_0) = \frac{1}{Pr}\nabla^2 T_0.$$

This non-linear system together with Eq. (4), the two continuity equations and the boundary conditions leads to the steady solution at $Ha = 0$ expressed by $\boldsymbol{V}_0$, $\boldsymbol{J}_0$, P_0, Φ_0, and T_0. At higher orders, linear systems are obtained. At order 1, we get a homogeneous linear system for $\boldsymbol{V}_1$, $\boldsymbol{J}_1$, P_1, Φ_1, and

T_1 (no Lorentz forces, and homogeneous boundary conditions since the initially non-homogeneous boundary conditions are accounted for by the zero order solution). As this system is not singular, the only solution is the zero solution ($\boldsymbol{V}_1 = \mathbf{0}$, $\boldsymbol{J}_1 = \mathbf{0}$, $P_1 = 0$, $\Phi_1 = 0$, $T_1 = 0$). Using this result in the expansion at order 2, we obtain

$$(\boldsymbol{V}_0 \cdot \nabla)\boldsymbol{V}_2 + (\boldsymbol{V}_2 \cdot \nabla)\boldsymbol{V}_0 = -\nabla P_2 + \nabla^2 \boldsymbol{V}_2 + Gr\, T_2 \boldsymbol{e}_z + \boldsymbol{J}_0 \times \boldsymbol{e}_{\boldsymbol{B}_0},$$

$$(\boldsymbol{V}_0 \cdot \nabla T_2) + (\boldsymbol{V}_2 \cdot \nabla T_0) = \frac{1}{Pr} \nabla^2 T_2,$$

together with Eq. (4), the two continuity equations and the boundary conditions. The Lorentz term $\boldsymbol{J}_0 \times \boldsymbol{e}_{\boldsymbol{B}_0}$ makes this linear system non-homogeneous. A generally non-zero solution for $\boldsymbol{V}_2$, $\boldsymbol{J}_2$, P_2, Φ_2, and T_2 will then be obtained. The expansion at order 3 will use the nullity of the solution at order 1, which in particular induces a zero Lorentz force $\boldsymbol{J}_1 \times \boldsymbol{e}_{\boldsymbol{B}_0}$. At order 3, similarly to order 1, it is found that $\boldsymbol{V}_3$, $\boldsymbol{J}_3$, P_3, Φ_3, and T_3 are all zero, because the linear system is still homogeneous. As a conclusion, the steady-state solution (denoted by $F = (\boldsymbol{V}, \boldsymbol{J}, P, \Phi, T)$ with an index s) can be expressed by the following expansion at small Ha:

$$F_s = F_0 + Ha^2 F_2 + O(Ha^4), \tag{6}$$

indicating an evolution of the steady solution with respect to the even powers of Ha. Again, F_0 and F_2 depend on Gr and Pr.

Conclusion (6) can be reached by another method. If the initial set of steady governing equations derived from (2) to (4) is considered and if, for a given direction of the imposed magnetic field, its orientation is changed, the solution is unchanged. This can be seen easily because changing Ha to $-Ha$ does not affect the equations. Hence, all solutions are even functions of Ha. Provided the solution is analytic at $Ha = 0$, a condition also required to write the previous Taylor expansion (5), it follows that all solution derivatives of odd order with respect to Ha must vanish. Explicit Taylor expansions give the possibility of computing directly the effect of a weak magnetic field by solving the second-order problem. Another advantage is that one can identify which term in the equations is responsible for a change in the steady solution when Ha is increased from zero. This term is $\boldsymbol{J}_0 \times \boldsymbol{e}_{\boldsymbol{B}_0}$ in the second-order expansion. It can be seen from the equation at order zero that a non-zero electric current density $\boldsymbol{J}_0$ will be generated if $\boldsymbol{V}_0 \times \boldsymbol{e}_{\boldsymbol{B}_0}$ is not a gradient. Equivalently, if $\boldsymbol{V}$ is not uniform along magnetic lines, there will be some current $\boldsymbol{J}_0$. Each case has to be analyzed independently to assess the magnitude of $\boldsymbol{J}_0 \times \boldsymbol{e}_{\boldsymbol{B}_0}$, which will ultimately be responsible for the magnitude of F_2.

3. Scaling laws for the magnetic stabilization of the flow

The stability of the basic flow solution obtained in the previous section is investigated here in a general way by the linear analysis of infinitesimal perturbations. The solution of the non-dimensional problem is written as

$$(\boldsymbol{V}, \boldsymbol{J}, P, \Phi, T) = (\boldsymbol{V}_s, \boldsymbol{J}_s, P_s, \Phi_s, T_s) + (\boldsymbol{v}, \boldsymbol{j}, p, \phi, \theta)\mathrm{e}^{qt}, \tag{7}$$

i.e., the sum of the basic flow quantities with perturbations seen as normal modes in time. Substitution into Eqs. (2)–(4) and linearization with respect to the perturbations yield to the following system written at the marginal state (q purely imaginary, i.e., $q = \mathrm{i}\omega$ where ω is the pulsation):

$$\mathrm{i}\omega \boldsymbol{v} + (\boldsymbol{V}_s \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{V}_s = -\nabla p + \nabla^2 \boldsymbol{v} + Gr\,\theta \boldsymbol{e}_z + Ha^2 \boldsymbol{j} \times \boldsymbol{e}_{\boldsymbol{B}_0}, \tag{8}$$

$$\boldsymbol{j} = -\nabla\phi + \boldsymbol{v} \times \boldsymbol{e}_{\boldsymbol{B}_0}, \tag{9}$$

$$\mathrm{i}\omega\theta + \boldsymbol{V}_s \cdot \nabla\theta + \boldsymbol{v} \cdot \nabla T_s = \frac{1}{Pr}\nabla^2\theta \tag{10}$$

with the continuity constraints for the two vector fields $\boldsymbol{v}$ and $\boldsymbol{j}$, $\nabla \cdot \boldsymbol{v} = 0$ and $\nabla \cdot \boldsymbol{j} = 0$. We use for $\boldsymbol{V}_s$ and T_s the general form given by (6). System (8)–(10) can be written as

$$f(Gr, Pr, Ha, \omega, \Gamma, \boldsymbol{V}_s, T_s) = 0, \tag{11}$$

where $\Gamma = (\boldsymbol{v}, \boldsymbol{j}, p, \phi, \theta)$ is an eigenvector and ω the associated pulsation. For given values of Ha and Pr, the critical threshold, expressed by the Grashof number Gr, is defined as

$$Gr_\mathrm{c} = \min\{Gr/\exists\Gamma_\mathrm{c} = (\boldsymbol{v}_\mathrm{c}, \boldsymbol{j}_\mathrm{c}, p_\mathrm{c}, \phi_\mathrm{c}, \theta_\mathrm{c}) \text{ non-zero associated to } \omega_\mathrm{c}, \text{ so that } f = 0\}. \tag{12}$$

At criticality, Gr, the eigenvector Γ, and the pulsation ω can be expressed around $Ha = 0$ as Taylor expansions with respect to Ha. We obtain for Gr_c:

$$Gr_\mathrm{c} = Gr_{\mathrm{c}_0} + Ha\,Gr_{\mathrm{c}_1} + Ha^2\,Gr_{\mathrm{c}_2} + Ha^3\,Gr_{\mathrm{c}_3} + O(Ha^4)$$

and similar expansions for Γ_c and ω_c. The coefficients at different orders will depend on Pr. The equation $f = 0$ at criticality can then be written as the sum of linear terms (denoted $\mathscr{L}$) and bilinear terms (denoted $\mathscr{B}$) with respect to the unknowns, under the following form:

$$\omega_\mathrm{c}\mathscr{L}_a(\Gamma_\mathrm{c}) + \mathscr{L}_b(\Gamma_\mathrm{c}) + Gr_\mathrm{c}\mathscr{L}_\mathrm{c}(\Gamma_\mathrm{c}) + \mathscr{B}(\Gamma_\mathrm{c}, F_s(Gr_\mathrm{c})) + Ha^2\,\mathscr{L}_d(\Gamma_\mathrm{c}) = 0. \tag{13}$$

This expression can be developed at different orders in Ha. At order 0, we have

$$\omega_{\mathrm{c}0}\mathscr{L}_a(\Gamma_{\mathrm{c}0}) + \mathscr{L}_b(\Gamma_{\mathrm{c}0}) + Gr_{\mathrm{c}0}\mathscr{L}_\mathrm{c}(\Gamma_{\mathrm{c}0}) + \mathscr{B}(\Gamma_{\mathrm{c}0}, F_0(Gr_{\mathrm{c}_0})) = 0, \tag{14}$$

expression which gives the critical characteristics without magnetic field. At order 1, we can write

$$\omega_{\mathrm{c}_1}\mathscr{L}_a(\Gamma_{\mathrm{c}0}) + \omega_{\mathrm{c}0}\mathscr{L}_a(\Gamma_{\mathrm{c}1}) + \mathscr{L}_b(\Gamma_{\mathrm{c}1}) + Gr_{\mathrm{c}_1}\mathscr{L}_\mathrm{c}(\Gamma_{\mathrm{c}0}) + Gr_{\mathrm{c}0}\mathscr{L}_\mathrm{c}(\Gamma_{\mathrm{c}1})$$
$$+\mathscr{B}(\Gamma_{\mathrm{c}_1}, F_0(Gr_{\mathrm{c}_0})) + \mathscr{B}\left(\Gamma_{\mathrm{c}_0}, Gr_{\mathrm{c}_1}\frac{\partial F_0}{\partial Gr}(Gr_{\mathrm{c}_0})\right) = 0. \tag{15}$$

This expression can be combined with (14). If we add ε times (15) to (14), we obtain an expression which can be written as

$$(\omega_{\mathrm{c}_0} + \varepsilon\omega_{\mathrm{c}_1})\mathscr{L}_a(\Gamma_{\mathrm{c}_0} + \varepsilon\Gamma_{\mathrm{c}_1}) + \mathscr{L}_b(\Gamma_{\mathrm{c}_0} + \varepsilon\Gamma_{\mathrm{c}_1}) + (Gr_{\mathrm{c}_0} + \varepsilon Gr_{\mathrm{c}_1})\mathscr{L}_\mathrm{c}(\Gamma_{\mathrm{c}_0} + \varepsilon\Gamma_{\mathrm{c}_1})$$
$$+\mathscr{B}\left(\Gamma_{\mathrm{c}_0} + \varepsilon\Gamma_{\mathrm{c}_1}, F_0(Gr_{\mathrm{c}_0}) + \varepsilon Gr_{\mathrm{c}_1}\frac{\partial F_0}{\partial Gr}(Gr_{\mathrm{c}_0})\right) + O(\varepsilon^2) = 0. \tag{16}$$

If ε is chosen small, we can neglect the last term of (16) which becomes similar to (14) with Γ_{c_0} replaced by $\Gamma_{\mathrm{c}_0} + \varepsilon\Gamma_{\mathrm{c}_1}$, ω_{c_0} by $\omega_{\mathrm{c}_0} + \varepsilon\omega_{\mathrm{c}_1}$, Gr_{c_0} by $Gr_{\mathrm{c}_0} + \varepsilon Gr_{\mathrm{c}_1}$, and with the steady regime taken at this new Gr. This means that if the unknowns at order 1, Γ_{c_1}, ω_{c_1}, and Gr_{c_1}, were non-zero, we could find solutions to the eigenvalue problem at order 0 which could be written $\Gamma_\mathrm{c} = \Gamma_{\mathrm{c}_0} + \varepsilon\Gamma_{\mathrm{c}_1}$ with $\omega_\mathrm{c} = \omega_{\mathrm{c}_0} + \varepsilon\omega_{\mathrm{c}_1}$ and would correspond to $Gr_\mathrm{c} = Gr_{\mathrm{c}_0} + \varepsilon Gr_{\mathrm{c}_1}$. With ε chosen so that $\varepsilon Gr_{\mathrm{c}_1} < 0$, we could find a critical Grashof number at $Ha=0$ less than Gr_{c_0}. This is impossible, which implies that Γ_{c_1}, ω_{c_1}, and Gr_{c_1} are zero. Taking this result into account, (13) gives at order 2:

$$\omega_{c_2}\mathscr{L}_a(\Gamma_{c_0})+\underline{\omega_{c_0}\mathscr{L}_a(\Gamma_{c_2})}+\underline{\mathscr{L}_b(\Gamma_{c_2})}+Gr_{c_2}\mathscr{L}_c(\Gamma_{c_0})+\underline{Gr_{c_0}\mathscr{L}_c(\Gamma_{c_2})}$$
$$+\underline{\mathscr{B}(\Gamma_{c_2},F_0(Gr_{c_0}))}+\mathscr{B}(\Gamma_{c_0},F_2(Gr_{c_0}))+Gr_{c_2}\mathscr{B}\left(\Gamma_{c_0},\frac{\partial F_0}{\partial Gr}(Gr_{c_0})\right)+\mathscr{L}_d(\Gamma_{c_0})=0. \qquad (17)$$

This equation is not an eigenvalue problem. The four underlined terms correspond to the linear operator of the zero order eigenvalue problem applied to Γ_{c_2}: the range of this operator is of codimension unity. The other five terms form an operator of range of dimension 1 (when the real numbers Gr_{c_2} and ω_{c_2} are varying independently, this amounts to the choice of an arbitrary complex number). There exists a unique combination of ω_{c_2}, and Gr_{c_2} such that the sum of these five terms lies on the image of the zeroth order stability operator. Then, Γ_{c_2} is the function whose image represents that intersection: in fact, the eigenvector being defined to within a multiplicative constant, Γ_{c_2} is unique once the multiplicative constant has been chosen for Γ_{c_0}. At last, at order 3, the same remarks as for the order 1 allow to show that Γ_{c_3}, ω_{c_3}, and Gr_{c_3} are zero. As a final result, we get the following expansion at small Ha:

$$\Gamma_c=\Gamma_{c_0}+Ha^2\,\Gamma_{c_2}+O(Ha^4),$$

$$\omega_c=\omega_{c_0}+Ha^2\,\omega_{c_2}+O(Ha^4),$$

$$Gr_c=Gr_{c_0}+Ha^2\,Gr_{c_2}+O(Ha^4).$$

This indicates that the initial increase with Ha of the main characteristics of the instabilities (critical threshold, pulsation and eigenvector) occurs through a Ha^2 law.

Again, this result can be derived directly from the fact that all solutions—including bifurcations—are even with respect to Ha. The advantage of writing explicitly the Taylor expansion is that it gives an idea of the origin and magnitude of the change in critical Grashof number when Ha grows from zero. In Eq. (17) the two constant terms, $\mathscr{B}(\Gamma_{c_0},F_2)$ and $\mathscr{L}_d(\Gamma_{c_0})$, contribute to the magnitude of Gr_{c_2}. The first term represents the change of the steady solution when changing Ha: a simple change in steady velocity profile or temperature distribution can of course affect the critical Grashof number. The second term represents the Lorentz damping force on the disturbances themselves. This last term can be very small or zero if the critical disturbance appears mainly in the form of rolls aligned with the magnetic field direction. It is also possible to solve Eq. (17) to obtain Gr_{c_2}.

4. Illustration in the case of the flow in an infinite layer

To illustrate what has been shown in the two previous sections, we consider the specific situation of an infinite layer of conducting liquid material confined vertically by two horizontal perfectly conducting rigid boundaries distant by L (z between -0.5 and 0.5). In such a situation, a horizontal temperature gradient (along x) drives a flow corresponding to an infinitely long convective cell and known as the Hadley circulation. This basic flow can be obtained analytically (Kaddeche et al., 2003). The expressions giving the horizontal velocity U_s and the temperature T_s in the case without magnetic field are:

$$U_s(z) = Gr\left(\frac{z^3}{6} - \frac{z}{24}\right), \tag{18}$$

$$T_s(x,z) = x + Gr\,Pr\left(\frac{z^5}{120} - \frac{z^3}{144} + \frac{7z}{5760}\right), \tag{19}$$

expressions which are also valid if the magnetic field is horizontal, along x or along y (Kaddeche et al., 2003). Under vertical magnetic field (Kaddeche et al., 2003), we have

$$U_s(z) = \frac{Gr}{Ha^2}\left(\frac{\sinh(Ha\,z)}{2\sinh(Ha/2)} - z\right), \tag{20}$$

$$T_s(x,z) = x + \frac{Gr\,Pr}{Ha^2}\left[\frac{1}{2Ha^2}\frac{\sinh(Ha\,z)}{\sinh(Ha/2)} - \frac{z^3}{6} + \left(\frac{1}{24} - \frac{1}{Ha^2}\right)z\right], \tag{21}$$

expressions which, in the limit of small Ha, can be written as

$$U_s(z) = Gr\left[\left(\frac{z^3}{6} - \frac{z}{24}\right) + Ha^2\left(\frac{z^5}{120} - \frac{z^3}{144} + \frac{7z}{5760}\right) + O(Ha^4)\right], \tag{22}$$

$$T_s(x,z) = x + Gr\,Pr\left[\left(\frac{z^5}{120} - \frac{z^3}{144} + \frac{7z}{5760}\right) + Ha^2\left(\frac{z^7}{5040} - \frac{z^5}{2880} + \frac{7z^3}{34560} - \frac{31z}{967680}\right) + O(Ha^4)\right]. \tag{23}$$

We verify that, in any of these cases, the basic flow can be expressed as even powers of Ha, with expressions similar to (6). For the horizontal magnetic field along x or y, the expressions (given by (18) and (19)) are in fact limited to zero order terms. For the vertical magnetic field, a comparison between the contributions at order 2 and at order 0 can be obtained by calculating the ratio between the maxima reached by the expressions in z at the different orders. This ratio (order 2/order 0 at $Ha = 1$) is 0.02542 for the horizontal velocity U_s and 0.02537 for the temperature T_s, indicating in both cases a relatively small influence of the magnetic field (through the leading order 2 terms) at small Ha.

The stability of the basic flow is investigated through a linear stability analysis, the perturbations being taken as normal modes of the form:

$$(\boldsymbol{v}, \boldsymbol{j}, p, \phi, \theta)(z)\mathrm{e}^{\mathrm{i}(hx+ky)+qt}, \tag{24}$$

where h and k are real wavenumbers in the longitudinal, x, and transverse, y, directions, respectively. The eigenvalue problem which is obtained is solved with the spectral Tau Chebyshev method (Kaddeche et al., 2003). From the thresholds $Gr_0(Pr, Ha, h, k)$ (values of Gr for which an eigenvalue has a real part equal to zero whereas all the other eigenvalues have negative real parts), the critical Grashof number Gr_c has been obtained after minimization along h and k ($Gr_\mathrm{c} = \inf_{(h,k)\in\mathscr{R}^2} Gr_0(Pr, Ha, h, k)$).

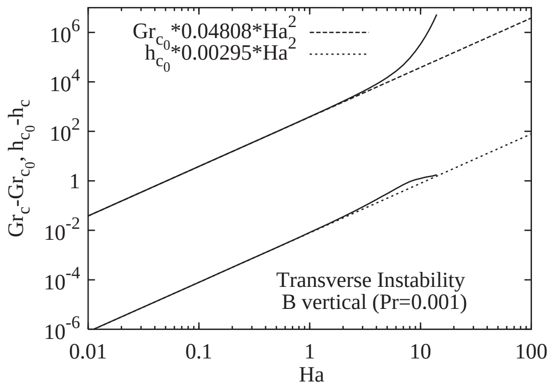

Fig. 1. Variation of the critical thresholds (Gr_c) and wavenumbers (h_c) as a function of Ha for the transverse instability under vertical magnetic field. Comparison with characteristic curves fitting the results at small Ha.

For conducting materials (small Pr numbers), the main instabilities correspond either to steady transverse instabilities (wavenumber along x) or to oscillatory longitudinal instabilities (wavenumber along y). It has been found in a previous study (Kaddeche et al., 2003) that the vertical magnetic field strongly stabilizes both instabilities, even at moderate Ha: the increase of the thresholds has been characterized by fitted laws which scale as $Gr_c - Gr_{c_0} \sim Ha^2$ for the longitudinal instabilities and still more efficiently as $Gr_c \sim Gr_{c_0} \exp(Ha^2/K)$ (where K is a constant) for the transverse instabilities. The horizontal field (along x or y) only stabilizes the instability with a wavenumber parallel to its direction. Moreover, it is efficient only at large Ha with asymptotic stabilization laws which scale as $Gr_c \sim Ha$ (Kaddeche et al., 2003).

In this paper, we focus on the variation of the instability characteristics at small Ha. For all the different cases mentioned just above, we have carefully looked at the initial variation with Ha of the critical instability thresholds (Gr_c), wavenumbers (h_c or k_c), and pulsation (ω_c). The results are shown in Figs. 1–4, each figure giving the instability characteristics as a function of Ha (Log–Log scale) for a given case, Fig. 1 for the transverse instability ($Pr = 0.001$) under vertical magnetic field, Fig. 2 for the oscillatory longitudinal instability ($Pr = 0.02$) under vertical magnetic field, Fig. 3 for the transverse instability under horizontal magnetic field along x, and Fig. 4 for the oscillatory longitudinal instability under horizontal magnetic field along y. In each figure, the numerical stability results are fitted at small Ha by characteristic curves whose expressions are given explicitly. We see that in any cases the initial variation with Ha of the different instability characteristics scales as Ha^2, for the thresholds ($Gr_c - Gr_{c_0}$) as well as for the wavenumbers ($h_c - h_{c_0}$ or $k_c - k_{c_0}$) and pulsations ($\omega_c - \omega_{c_0}$). This Ha^2 variation is in general valid up to some units of Ha (limit between 1 and 3), except for the thresholds in the case of Fig. 2 where the limit is a little higher (this case was found to scale as Ha^2 even for moderate Ha). Concerning the thresholds, we see that the normalized coefficients of the Ha^2 variation (normalization by the value at $Ha = 0$, Gr_{c_0}) are stronger for the vertical magnetic field cases (Figs. 1 and 2) where the further variation with Ha is still more pronounced, than for the horizontal magnetic field cases (Figs. 3 and 4) where the further variation is less steep and evolves towards an asymptotic Ha variation. In any case, these coefficients are rather small, less than 0.05, indicating that, for $Ha = 1, Gr_c$ will increase from Gr_{c_0}

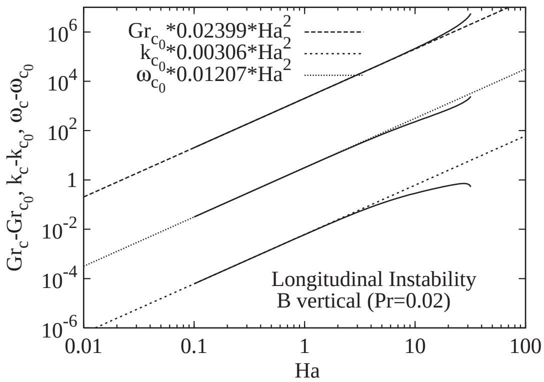

Fig. 2. Variation of the critical thresholds (Gr_c), wavenumbers (k_c) and pulsations (ω_c) as a function of Ha for the oscillatory longitudinal instability under vertical magnetic field. Comparison with characteristic curves fitting the results at small Ha.

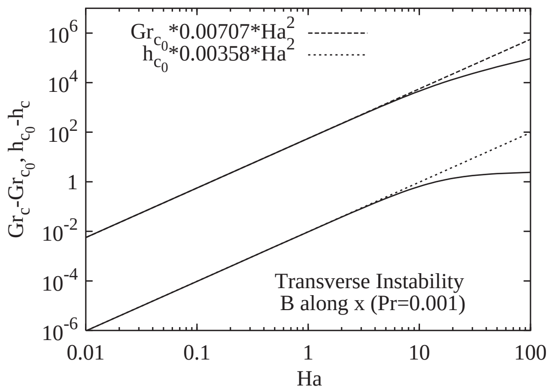

Fig. 3. Variation of the critical thresholds (Gr_c) and wavenumbers (h_c) as a function of Ha for the transverse instability under horizontal magnetic field along x. Comparison with characteristic curves fitting the results at small Ha.

by 5% at most. Concerning the evolution of the wavenumbers, the normalized coefficients of the Ha^2 variations are similar in all cases. We can remark that the wavenumbers reach an asymptotic value at large Ha in the case of the horizontal fields: this is due to the fact that h_c (respectively k_c) tends towards zero so that $h_{c_0} - h_c$ (respectively $k_{c_0} - k_c$) tends towards h_{c_0} (respectively k_{c_0}). At last, concerning the pulsation, the normalized coefficients of the Ha^2 variations are stronger for the oscillatory longitudinal instability under vertical magnetic field (Fig. 2) than under horizontal magnetic field (Fig. 4), the asymptotic value reached by $\omega_c - \omega_{c_0}$ at large Ha in this last case being connected to the asymptotic value reached by ω_c.

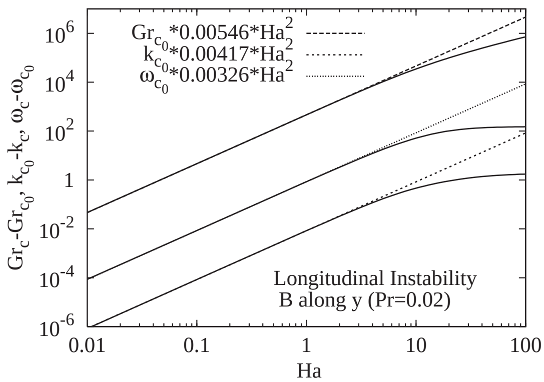

Fig. 4. Variation of the critical thresholds (Gr_c), wavenumbers (k_c) and pulsations (ω_c) as a function of Ha for the oscillatory longitudinal instability under horizontal magnetic field along y. Comparison with characteristic curves fitting the results at small Ha.

5. Conclusion

It has been shown in this note that the braking of buoyant flows under a magnetic field scales at small Ha as even powers of Ha, the initial braking then scaling as Ha^2. Moreover the instabilities which are triggered present characteristics (critical thresholds, eigenvectors and pulsations) which also scale at small Ha as even powers of Ha giving an initial stabilization as Ha^2. These findings have been illustrated by the case of the flow in an infinite layer.

These results show that the specific form of the equations under a magnetic field with a Ha^2 coefficient in the Lorentz force term can induce specific behaviours for the variation at small Ha of the flow intensity and of the instability characteristics. Nevertheless these universal evolutions at small Ha do not anticipate what is obtained at larger Ha which is often connected to specific evolutions of the flow structure. Moreover, the evolutions obtained in this small Ha domain are rather small. In the case of the flow in an infinite layer, at most an increase of the thresholds by 20% with respect to Gr_{c_0} is obtained at the limit of the Ha^2 variation ($Ha \sim 2$) for the transverse instabilities under a vertical magnetic field, a case for which the further increase with Ha is very strong, reaching a factor 42 for $Ha = 10$ (4200%) and even a factor 667 for $Ha = 14$.

The results obtained for linear stability can be extended to the case of other eigenvalue problems. For instance, energetic stability (Lingwood and Alboussière, 1999) leads to an eigenvalue problem distinct but of the same form as the linear stability problem. It can be shown that the energetic threshold will also follow a Ha^2 law at weak magnetic fields.

Acknowledgements

This work was initiated when T. Alboussière was on sabbatical leave at the Laboratoire de Mécanique des Fluides et d'Acoustique at the Ecole Centrale de Lyon.

References

Alboussière, T., Garandet, J.P., Moreau, R., 1993. Buoyancy-driven convection with a uniform magnetic field. Part 1: Asymptotic analysis. J. Fluid Mech. 253, 545–563.

BenHadid, H., Henry, D., 1996. Numerical simulation of convective three-dimensional flows in a horizontal cylinder under the action of a constant magnetic field. J. Cryst. Growth 166, 436–445.

BenHadid, H., Henry, D., Kaddeche, S., 1997. Numerical study of convection in the horizontal Bridgman configuration under the action of a constant magnetic field. Part 1: Two-dimensional flows. J. Fluid Mech. 333, 23–56.

BenHadid, H., Henry, D., 1997. Numerical study of convection in the horizontal Bridgman configuration under the action of a constant magnetic field. Part 2: Three-dimensional flows. J. Fluid Mech. 333, 57–83.

Davoust, L., Cowley, M.D., Moreau, R., Bolcato, R., 1999. Buoyancy-driven convection with a uniform magnetic field. Part 2. Experimental investigation. J. Fluid Mech. 400, 59–90.

Gelfgat, A.Yu., Bar-Yoseph, P.Z., 2001. The effect of an external magnetic field on oscillatory instability of convective flows in a rectangular cavity. Phys. Fluids 13, 2269–2278.

Hof, B., 2001. A study of magnetohydrodynamic convection in liquid gallium. Thesis, University of Manchester, Manchester, UK.

Hof, B., Juel, A., Mullin, T., 2003. Magnetohydrodynamic damping of convective flows in molten gallium. J. Fluid Mech. 482, 163–179.

Hurle, D.T.J., 1966. Temperature oscillations in molten metals and their relationship to growth striae in melt-grown crystals. Philos. Mag. 13, 305–310.

Hurle, D.T.J., Jakeman, E., Johnson, C.P., 1974. Convective temperature oscillations in molten gallium. J. Fluid Mech. 64, 565–576.

Juel, A., Mullin, T., BenHadid, H., Henry, D., 1999. Magnetohydrodynamic convection in molten gallium. J. Fluid Mech. 378, 97–118.

Kaddeche, S., Henry, D., BenHadid, H., 2003. Magnetic stabilization of the buoyant convection between infinite horizontal walls with a horizontal temperature gradient. J. Fluid Mech. 480, 185–216.

Kim, D.H., Adornato, P., Brown, R., 1988. Effect of vertical magnetic field on convection and segregation in vertical Bridgman crystal growth. J. Cryst. Growth 89, 339–356.

Lingwood, R.J., Alboussière, T., 1999. On the stability of the Hartmann layer. Phys. Fluids 11, 2058–2068.

Okada, K., Ozoe, H., 1992. Experimental heat transfer rates of natural convection of molten gallium suppressed under an external magnetic field in either the X, Y, or Z direction. J. Heat Transfer 114, 107–114.

Oreper, G.M., Szekely, J., 1983. The effect of an externally imposed magnetic field on buoyancy driven flow in a rectangular cavity. J. Cryst. Growth 64, 505–515.

Oreper, G.M., Szekely, J., 1984. The effect of a magnetic field on transport phenomena in a Bridgman-Stockbarger crystal growth. J. Cryst. Growth 67, 405–419.

Ozoe, H., Okada, K., 1989. The effect of the direction of the external magnetic field on the three-dimensional natural convection in a cubical enclosure. Int. J. Heat Mass Transfer 32, 1939–1954.

Pimputkar, S.M., Ostrach, S., 1981. Convective effects in crystals grown from melts. J. Cryst. Growth 55, 614–646.

Touihri, R., BenHadid, H., Henry, D., 1999. On the onset of convective instabilities in cylindrical cavities heated from below. II. Effect of a magnetic field. Phys. Fluids 11, 2089–2100.