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Quasi characteristic MHD flows

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Abstract. At small magnetic Reynolds number, a two-dimensional model is proposed for MHD flows in a nonuniform magnetic field and in a cavity of nonuniform depth in the direction of the magnetic field. The characteristic surfaces appear when deriving the model and play a crucial role in the resulting solutions. A new type of free shear layers are found for the first time, developing along such surfaces, of thickness of order $Ha^{-1/4}$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

fluid mechanics / magnetohydrodynamics / characteristic surfaces / Hartmann layers / shear layers

Écoulements MHD quasi caractéristiques

Résumé. *Aux petits nombres de Reynolds magnétiques, un modèle MHD bidimensionnel est formulé pour un champ magnétique non uniforme et pour une cavité de profondeur non uniforme dans la direction du champ magnétique. Les surfaces caractéristiques apparaissent naturellement dans le développement du modèle et y jouent un rôle important. Des couches cisaillées se développant le long de certaines surfaces caractéristiques sont décrites pour la première fois : leur épaisseur est de l'ordre de $Ha^{-1/4}$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

mécanique des fluides / magnétohydrodynamique / surfaces caractéristiques / couches de Hartmann / couches cisaillées

Version française abrégée

Hartmann [1] a ouvert la voie aux études asymptotiques portant sur les écoulements de fluides conducteurs de l'électricité en présence d'un champ magnétique statique. Plus tard, Sommeria et Moreau [2] ont écrit les équations du mouvement quasi bidimensionnel lorsque la cavité qui contient le fluide est constituée de deux plans parallèles, perpendiculaires au champ magnétique. L'effet du champ magnétique s'y traduit simplement par un terme de freinage linéaire. Par ailleurs, pour des situations plus complexes, Kulikovskii [3] avait montré que les surfaces caractéristiques, définies comme les surfaces isovaleurs de la fonction $\int \mathrm{d}s/\|B\|$, jouaient un rôle important et qu'à grand nombre de Hartmann, le mouvement du fluide doit leur être tangent (voir également [4]). Dans le présent article, les équations bidimensionnelles sont obtenues, valides même lorsque le champ magnétique n'est pas uniforme et que la cavité, électriquement isolante, n'est pas de profondeur constante. Les surfaces caractéristiques y jouent un grand rôle et des couches cisaillées d'épaisseur de l'ordre de $Ha^{-1/4}$ sont susceptibles de se développer le long de ces surfaces.

Les champs de vecteurs tridimensionnels de vitesse $\mathbf{u}$ et de densité de courant électrique $\mathbf{j}$ sont de divergence nulle et satisfont à la loi d'Ohm (1) et à l'équation de Navier–Stokes (2). Pour simplifier

Note présentée par René MOREAU.

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l'analyse, nous considérons dans cet article seulement des configurations de symétrie régulière [4], c'est-à-dire pour lesquelles la cavité est symétrique par rapport à un plan $\mathcal{P}$, $\mathbf{u}$ et $\mathbf{j}$ possèdent une symétrie miroir par rapport à $\mathcal{P}$ et les lignes de champ magnétiques sont perpendiculaires à $\mathcal{P}$ (voir *figure 1*). L'approximation de lignes magnétiques droites est utilisée, l'intensité du champ magnétique pouvant varier dans les directions transverses : $B_z(x, y)$. L'épaisseur de la cavité dans la direction du champ magnétique est notée $l(x, y)$. On note $K = B_z/l$ la fonction caractéristique. Aux grands nombres de Hartmann, les composantes de $\mathbf{u}$ et $\mathbf{j}$ parallèles à $\mathcal{P}$ deviennent bidimensionnelles dans le cœur : $\mathbf{u}_0(x, y)$ et $\mathbf{j}_0(x, y)$. En outre, les couches de Hartmann conduisent à un très faible déficit de transfert de masse et à un flux bidimensionnel de charge électrique $\mathbf{I}_h$, exprimé par la formule (3). La somme des contributions de cœur et des couches de Hartmann permettent de définir un flux bidimensionnel total de masse $\mathbf{Q}$ et de charge électrique $\mathbf{I}$ par les formules (4). Ces flux satisfont aux lois de conservation de la masse et de la charge, sont donc de divergence nulle et peuvent ainsi être exprimés par des fonctions de courant ψ et h, cf. équation (5). Dans le cœur, la loi d'Ohm et l'équation de Navier–Stokes fournissent des équations bidimensionnelles pour les champs bidimensionnels $\mathbf{u}_0$ et $\mathbf{j}_0$, (6) et (7). Leur rotationnel (8) et (9) conduit à deux équations portant sur ψ et h, après substitution de $\mathbf{u}_0$ et $\mathbf{j}_0$ à l'aide de (3), (4) et (5). Les deux équations ainsi obtenues (10) et (11) – (11) est remplacée par (12) pour l non uniforme – constituent un modèle bidimensionnel des écoulements MHD. Une analyse locale de ces équations montre qu'elles peuvent s'écrire sous la forme (13) et (14), où t est une coordonnée parallèle aux surfaces caractéristiques. En combinant ces deux équations, on obtient une équation qui porte uniquement sur ψ, (15), qui peut s'exprimer finalement sous la forme adimensionnelle (16). L'analyse des ordres de grandeur des termes de cette équations montre que des structures de taille unité dans la direction transverse aux surfaces caractéristiques et de taille d'ordre $Ha^{1/2}$ dans la direction parallèle peuvent exister, conformément à un résultat connu. De plus, des structures d'ordre de grandeur unité dans la direction des surfaces caractéristiques peuvent se développer si leur taille est d'ordre $Ha^{-1/4}$ dans la direction transverse.

Le modèle 2D, (10) et (12), a été résolu numériquement dans un cas test où les fontions B_z et l sont toutes deux sinusoïdales en y et x respectivement. Les isovaleurs de la fonction caractéristique et, pour un nombre de Hartmann de 10^4, les isovaleurs de ψ et h sont montrées sur la *figure 2* pour deux périodes en x et en y. Par symétrie, le reste des calculs est effectué sur le petit carré (une demi période) mis en évidence sur la *figure 2*. La *figure 3* montre l'évolution de ψ et h lorsque Ha varie. La tendance de l'écoulement à suivre les surfaces caractéristiques se manifeste aux très grandes valeurs de Ha. Des zones stagnantes se développent, correspondant aux régions de surfaces caractéristiques fermées. De plus la zone cisaillée entre ces zones stagnantes et la région centrale est visualisée sur la *figure 4* avec le profil des vitesses le long de la ligne verticale médiane du domaine et avec le tracé de l'épaisseur calculée de cette couche cisaillée libre δ. Cette épaisseur varie effectivement comme $Ha^{-1/4}$ pour de grandes valeurs du nombre de Hartmann. De telles couches cisaillées d'épaisseur $Ha^{-1/4}$ seront invoquées (publication ultérieure) pour séparer la zone stagnante du fluide en mouvement dans un écoulement en conduite avec une marche de champ magnétique transverse (voir [5]). Le modèle présenté ici est comparable au modèle quasi géostrophique pour les fluides dans un référentiel tournant. Les surfaces caractéristiques et les contours géostrophiques jouent un rôle équivalent, d'où l'usage du terme « quasi caractéristique » dans le titre de cet article.

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1. Introduction

In the analysis of the melt flow during crystal growth under a DC magnetic field or in the study of liquid metal cooling blankets for fusion reactor concept designs, it is difficult to predict the flow intensity or pressure drop induced. The difficulty lies both in the presence of a strong magnetic field and in the presence of the nontrivial container geometry. Under a strong DC magnetic field, or large Hartmann number, asymptotic methods of analysis have been developed for a long time since the work of Hartmann [1] and, in a simple geometry like between two parallel walls perpendicular to a uniform magnetic field, two-dimensional equations governing the flow have been derived by Sommeria and Moreau [2]. In this case, the magnetic field just brings a linear 'Hartmann' friction term to the 2D Navier–Stokes equations. However when the magnetic field is not uniform or when the geometry is less simple, the so-called characteristic surfaces introduced by Kulikovskii [3] come into play. This characteristic surfaces are defined as the set of all magnetic lines having the same value for the function $\int \mathrm{d}s/\|B\|$, where the integration is performed along the part of the magnetic line contained in the cavity. In the case of pressure driven flows in electrically insulating cavities, Kulikovskii showed that the flow should not cross the characteristic surfaces in the limit of large Hartmann numbers. This result was generalized by Alboussière et al. [4] for other driving forces like buoyancy. We do not know how quickly this asymptotic state is reached when the Hartmann number is increased and there is no two-dimensional model that shows clearly the effect of characteristic surfaces on MHD flows. An answer is provided to these questions in this paper. In the next Section 2, a two-dimensional inertialess flow model is proposed in the case of nonuniform magnetic field and cavity of nonuniform depth in the direction of the magnetic field. From this model, the possible flow structures are analyzed. In Section 3, the 2D model is solved numerically for a particular cavity shape and magnetic field distribution. Finally, Section 4 is devoted to a discussion of the model presented and its future extensions.

2. Two-dimensional MHD equations

Our attention will be restricted to inertialess pressure driven flows in electrically insulating cavities. Magnetic field $\mathbf{B}$, divergenceless velocity $\mathbf{u}$, divergenceless electrical current density $\mathbf{j}$, pressure p and electric potential φ satisfy Ohm's law and Navier–Stokes equation in their three-dimensional version:

$$\frac{\mathbf{j}}{\sigma} = -\nabla\varphi + \mathbf{u} \times \mathbf{B} \tag{1}$$

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho\nu\nabla^2\mathbf{u} \tag{2}$$

The symbols σ, ρ, ν denote electrical conductivity, density and kinematic viscosity. Here, only regularly symmetrical configurations will be considered (see [4]), namely with a plane of symmetry $\mathcal{P}$ for the cavity and magnetic lines, velocity and electric current density fields with mirror symmetrical properties and $\mathbf{B}$ perpendicular to $\mathcal{P}$ (see *figure 1*). The approximation of straight magnetic lines is made, with the magnetic intensity depending on the plane directions x and y. Hence, magnetic field and geometry are defined by two 2D functions, the magnetic intensity $B_z(x, y)$ and the cavity depth along the direction of magnetic lines $l(x, y)$. At large Hartmann number, in the core of the flow, the components of the velocity and electric current density parallel to $\mathcal{P}$ are uniform along z: $\mathbf{u}_0(x, y)$ and $\mathbf{j}_0(x, y)$. In the Hartmann layers at the top and bottom, there is a negligible mass transfer deficit and a well-known electric current surface flux depending on the tangent core velocity $\mathbf{u}_t$: $\mathbf{I}_h = -\operatorname{sgn}(\mathbf{B}\cdot\mathbf{n})\sqrt{\sigma\rho\nu}\,\mathbf{u}_t \times \mathbf{n}$. This surface flux, within the

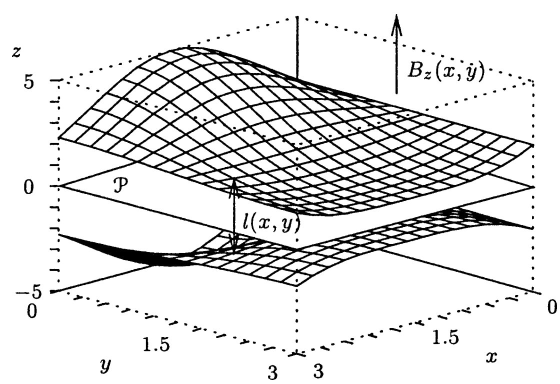

Figure 1. Cavity geometry and magnetic field.

(x, y) coordinate system, can be expressed in the following way:

$$\mathbf{I}_h = \sqrt{\sigma\rho\nu}\begin{bmatrix} -u_{0y} \\ u_{0x} \end{bmatrix} \quad \text{or} \quad \mathbf{I}_h = \sqrt{\sigma\rho\nu}\begin{bmatrix} -u_{0y} - \frac{1}{4}\left(\frac{\partial l}{\partial y}\right)^2 u_{0y} - \frac{1}{4}\frac{\partial l}{\partial y}\frac{\partial l}{\partial x}u_{0x} \\ u_{0x} + \frac{1}{4}\left(\frac{\partial l}{\partial x}\right)^2 u_{0x} + \frac{1}{4}\frac{\partial l}{\partial y}\frac{\partial l}{\partial x}u_{0y} \end{bmatrix} \tag{3}$$

the first expression being correct only for constant thickness l. Global 2D mass and electric current fluxes are defined, taking into account the core and Hartmann layers contributions:

$$\mathbf{Q} = l\mathbf{u}_0, \qquad \mathbf{I} = l\mathbf{j}_0 + 2\mathbf{I}_h \tag{4}$$

When no current or mass flow is provided through the cavity wall, global mass and electric current conservation provide two equations: $\operatorname{div}\mathbf{Q} = 0$ and $\operatorname{div}\mathbf{I} = 0$. It is then possible to define a velocity streamfunction ψ and a global electric current streamfunction h, such that

$$\mathbf{Q} = \nabla\psi \times \mathbf{e}_z, \qquad \mathbf{I} = \nabla h \times \mathbf{e}_z \tag{5}$$

Now, we consider the governing equations (1) and (2) in the core of the flow where the viscous term is neglected. Their horizontal components on the plane $\mathcal{P}$ take the form

$$\frac{\mathbf{j}_0}{\sigma} = -\nabla\varphi_0 + B_z\mathbf{u}_0 \times \mathbf{e}_z \tag{6}$$

$$0 = -\nabla p_0 + B_z\mathbf{j}_0 \times \mathbf{e}_z \tag{7}$$

Taking the curl of these 2D equations governing the 2D fields $\mathbf{u}_0$ and $\mathbf{j}_0$ gives:

$$\operatorname{curl}\left(\frac{\mathbf{j}_0}{\sigma}\right) = -\operatorname{div}(B_z\mathbf{u}_0)\mathbf{e}_z \tag{8}$$

$$0 = -\operatorname{div}(B_z\mathbf{j}_0)\mathbf{e}_z \tag{9}$$

The last step consists in substituting ψ and h for $\mathbf{u}_0$ and $\mathbf{j}_0$ in (8) and (9) using (3), (4) and (5). When evaluating the left-hand side term of (8), the contribution coming from $\mathbf{I}_h$ is neglected because it is of order Ha times smaller and of the same derivative order as the right-hand side of (8). Two coupled linear 2D equations governing ψ and h are thus obtained:

$$\operatorname{div}\left(\frac{\nabla h}{\sigma l}\right)\mathbf{e}_z = \nabla\left(\frac{B_z}{l}\right) \times \nabla\psi \tag{10}$$

$$2\sqrt{\rho\nu\sigma}\operatorname{div}\left(\frac{B_z}{l^2}\nabla\psi\right)\mathbf{e}_z = \nabla\left(\frac{B_z}{l}\right) \times \nabla h \tag{11}$$

when l is uniform. When taking the more precise second expression for $\mathbf{I}_h$ in (3) valid for any function l, equation (11) becomes:

$$2\sqrt{\rho\nu\sigma}\operatorname{div}\left(\frac{B_z}{l^2}\begin{bmatrix} \frac{\partial\psi}{\partial x} + \frac{1}{4}\left(\frac{\partial l}{\partial y}\right)^2\frac{\partial\psi}{\partial x} - \frac{1}{4}\frac{\partial l}{\partial y}\frac{\partial l}{\partial x}\frac{\partial\psi}{\partial y} \\ \frac{\partial\psi}{\partial y} + \frac{1}{4}\left(\frac{\partial l}{\partial x}\right)^2\frac{\partial\psi}{\partial y} - \frac{1}{4}\frac{\partial l}{\partial y}\frac{\partial l}{\partial x}\frac{\partial\psi}{\partial x} \end{bmatrix}\right)\mathbf{e}_z = \nabla\left(\frac{B_z}{l}\right) \times \nabla h \tag{12}$$

The two-dimensional model is given by equations (10) and (12). Its features are similar to the simpler model defined by equations (10) and (11). It can be first noticed that the characteristic function $K = B_z/l$, using

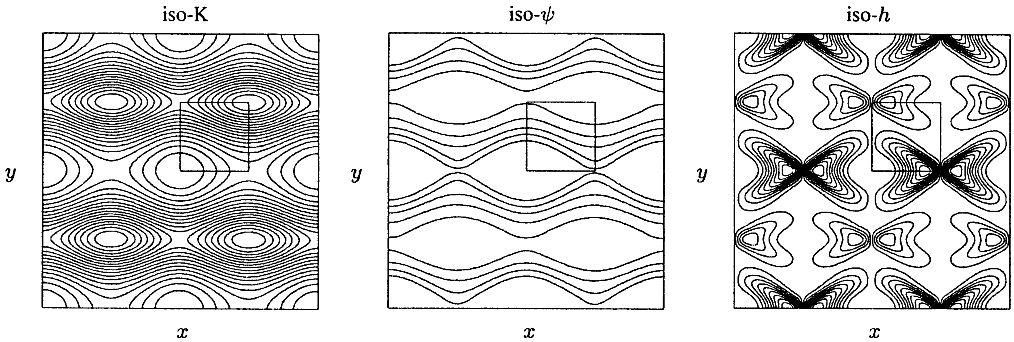

Figure 2. Isovalues of K, ψ and h for $Ha = 10^4$, showing two periods along x and y.

the inverse of the classical definition, plays an obvious role on both right-hand sides. Equation (10) tells us that when the flow crosses the characteristic surfaces, this constitutes a source for the curl of the electric current. Conversely, equation (11) or (12) tells us that when this current crosses the characteristic surfaces, this results in a source term for the vorticity of the flow.

Our purpose now will be to examine the structure of these coupled equations to find out which flow and current structures can be expected to develop. One needs to consider these equations locally and the simplified set of equations (10) and (11) will be used. Two new rectangular coordinates (s,t) are defined, with $\mathbf{e}_s$ the unit vector associated to s being parallel to the gradient of the characteristic function $K = B_z/l$. The unit vector $\mathbf{e}_t$ associated to t is directed along the characteristic surfaces. Let us denote G_l the local intensity of the gradient of K and B_l and l_l the local values of B_z and l. The two governing equations (10) and (11) take the local form:

$$\frac{1}{\sigma l_l}\nabla^2 h = G_l \frac{\partial \psi}{\partial t} \tag{13}$$

$$2\sqrt{\rho\nu\sigma}\frac{B_l}{l_l^2}\nabla^2\psi = G_l\frac{\partial h}{\partial t} \tag{14}$$

Taking the partial derivative of (13) with respect to t and substituting $\partial h/\partial t$ using (14), an equation governing ψ alone is obtained (a similar equation can be obtained for h)

$$\frac{\partial^2\psi}{\partial t^2} = \frac{2}{G_l^2}\sqrt{\frac{\rho\nu}{\sigma}}\frac{B_l}{l_l^3}\left(\nabla^2\right)^2\psi \tag{15}$$

Using l_l to generate dimensionless coordinates (also denoted s and t) and defining

$$Ha_l = \sqrt{\frac{\sigma}{\rho\nu}}\frac{G_l^2 l_l^5}{2B_l}$$

equation (15) takes the following dimensionless form

$$Ha_l\frac{\partial^2\psi}{\partial t^2} - \left(\nabla^2\right)^2\psi = 0 \tag{16}$$

By scaling analysis, it can be seen that for a typical length scale unity across the characteristic surfaces, a length scale of order $Ha_l^{1/2}$ will develop along these surfaces. Conversely, for a typical length scale unity along the characteristic surfaces, it is found that a small scale of order $Ha_l^{-1/4}$ can develop across these surfaces, forming free shear layers along particular characteristic surfaces.

3. Numerical analysis of the 2D model on an example

We specify a particular cavity geometry in dimensionless coordinates using a reference length l_0: $l = 1.5 + 0.5\cos(2\pi y)$, and a magnetic field distribution using a reference intensity B_0: $B_z = 1.5 + 0.2\cos(2\pi x)$. The boundary conditions are periodic and such that the flow is along the x direction on average. A Hartmann number is defined as $Ha = \sqrt{\sigma/\rho\nu}B_0 l_0$. A finite element software, FreeFem+, has been used to solve the two-dimensional model (10) and (12). *Figure 3* shows two periods along x and y, but symmetries are invoked to restrict the computational domain to the small square shown of size half a period. The tendency for the flow to follow the characteristic surfaces is visible. Closed characteristic surfaces prevent any flow to occur. On *figure 3*, the evolution of the velocity and electric current streamlines can be seen when Ha is increased. The flows follow more and more closely the characteristic surfaces and thin structures develop along those particular characteristic surfaces that possess a singularity (bottom right corner and upper left corner) where the gradient of K vanishes and two characteristic surfaces cross each other. As can be seen on *figure 4*, the stagnant and central regions are separated by a free shear layer. Its thickness δ is defined from space averages of the vorticity distribution ω, by the equation $\delta = \langle\omega^2\rangle^2/\langle\omega^4\rangle$. At large Ha, it scales as $Ha^{-1/4}$ (right-hand side of *figure 4*), in accordance with our expectations at the end of Section 2.

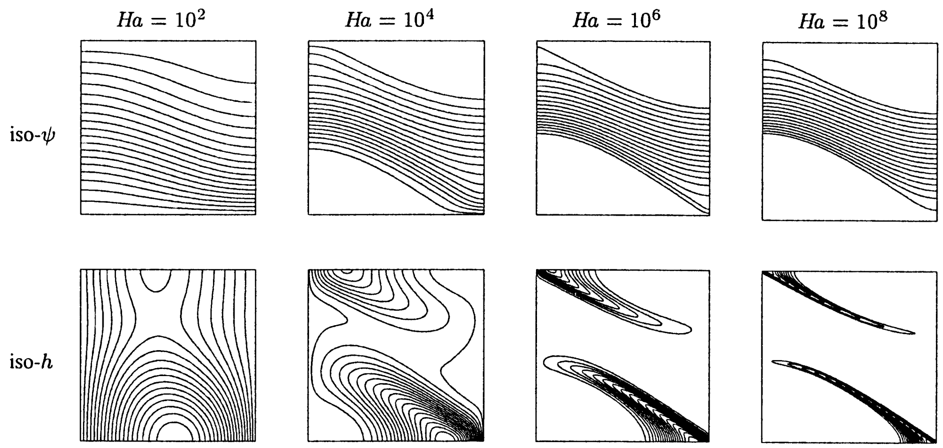

Figure 3. Velocity and current streamlines on the small computational domain (half a period).

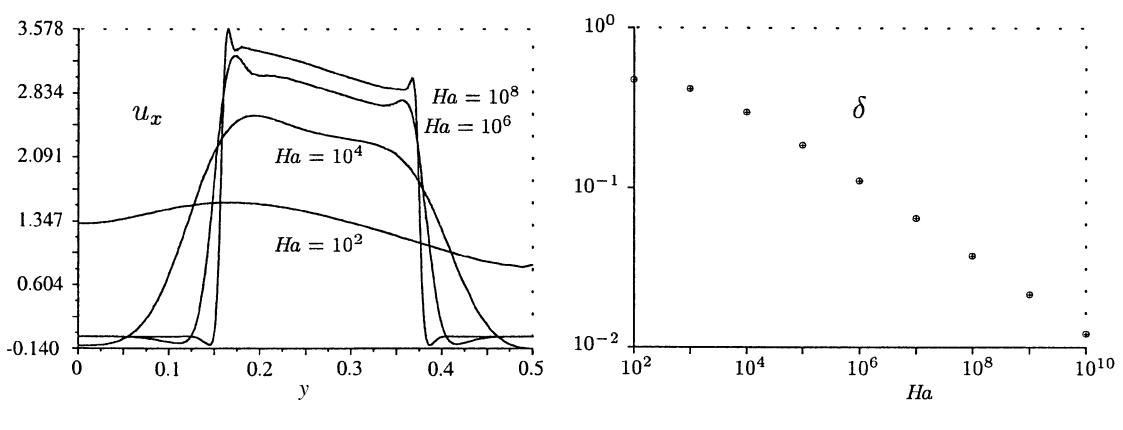

Figure 4. Velocity profile along the mid vertical line (left). Calculated shear layer thickness (right).

4. Concluding remarks

In Section 2, the analysis of the differential operator associated to our 2D MHD model, (10) and (12), has highlighted the possibility of a $Ha^{1/2}$ development length scale along characteristic surfaces. This effect had been predicted before in the problem of a step transverse magnetic field along a duct flow [5]. Moreover, free shear layers of thickness $Ha^{-1/4}$ developing along characteristic surfaces along a length scale of order unity are predicted for the first time in Section 2 and put in evidence by solving the two-dimensional model on a particular test case in Section 3. The author believes that such a free shear layer of thickness $Ha^{-1/4}$ also develops in the case of the step magnetic field in a pipe flow along the characteristic surfaces that merge in the nonuniform magnetic field region. Such shear layers provide an alternative, somehow intermediate between the proposals of Holroyd and Walker [5], one author suggesting that a parallel layer of thickness $Ha^{-1/2}$ should develop, the other one suggesting that a region of constant (small) thickness should exist at large Hartmann number.

Our model assumes negligible viscosity effects in the core. This is true until structures of thickness $Ha^{-1/2}$ develop. Much thicker free shear layers are obtained of thickness $Ha^{-1/4}$, which are compatible with negligible viscous effects. The present work has been extended in two directions. First inertial effects can be added, and secondly, the analysis can be made more rigorous by considering true magnetic field distributions (with curved magnetic lines). These extensions will be published later. The case of the magnetic field step along a pipe flow has been analyzed in the light of this model and will also be published in the near future.

In engineering applications, the maximal value of Hartmann number is around 10^4. This paper shows that the asymptotic high Hartmann number regime can hardly be reached for the 2D flow, since $Ha^{-1/4}$ is not very small. This gives extra importance to the 2D model (10) and (12): one needs to solve it to get correct predictions of the flow and electric current. Finally, there is a strong parallel between this study and the studies on rotating flows, in particular the quasi geostrophic model. Characteristic surfaces are the counterpart of geostrophic contours. This is a justification for the term ‘quasi characteristic’ used in the title to qualify this model.

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References

[1] Hartmann J., Hg-dynamics I. Theory of the laminar flow of an electrically conductive liquid in an homogeneous magnetic field, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. 15 (7) (1937) 1–27.

[2] Sommeria J., Moreau R., Why, how, and when, MHD turbulence becomes two-dimensional, J. Fluid Mech. 118 (1982) 507–518.

[3] Kulikovskii A.G., Slow steady flows of a conducting fluid at high Hartmann numbers, Izv. Akad. Nauk. SSSR Mekh. Zhidk. i Gaza 3 (1968) 3–10.

[4] Alboussière T., Garandet J.P., Moreau R., Asymptotic analysis and symmetry in MHD convection, Phys. Fluids 8 (8) (1996) 2215–2226.

[5] Holroyd R.J., Walker J.S., A theoretical study of the effect of wall conductivity, non-uniform magnetic fields and variable-area ducts on liquid-metal flows at high Hartmann numbers, J. Fluid Mech. 84 (3) (1978) 471–495.