Transition to Turbulence in the Hartmann Boundary Layer

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Key words Magnetohydrodynamics, liquid metals, magnetic field, Hartmann layer, turbulence, transition.

The Hartmann boundary layer is a paradigm of magnetohydrodynamic (MHD) flows. Hartmann boundary layers develop when a liquid metal flows under the influence of a steady magnetic field. The present paper is an overview of recent successful attempts to understand the mechanisms by which the Hartmann layer undergoes a transition from laminar to turbulent flow.

1 What is a Hartmann boundary layer?

When a viscous incompressible fluid flows laminarly in the gap between two unbounded plates separated from each other by a distance d, its velocity profile, shown in Figure 1a, is well known to be parabolic. When the fluid is electrically conducting and when a uniform steady magnetic field acts perpendicular to the channel walls, the structure of the flow changes drastically, as shown in Figure 1b. The profile becomes flat in the so-called core as a result of the electromagnetic braking effect. This braking is due to the interaction of the induced electric current with the applied magnetic field. Moreover, two boundary layers develop in the vicinity of the walls. These layers have been theoretically predicted and experimentally characterised by Julius Hartmann in 1937 [1], [2] and represent one of the most important characteristic features of MHD flows.

The thickness δ of a Hartmann boundary layer (or "Hartmann layer" for simplicity) is of the order

$$\delta = \frac{1}{B}\sqrt{\frac{\rho\nu}{\sigma}} \tag{1}$$

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where B is the strength of the magnetic field, and ρ, ν, σ are the density, kinematic viscosity, and electrical conductivity of the liquid metal [3]. The thickness is independent of the channel width and decreases with increasing magnetic field. To give a numerical example, liquid metals like mercury, gallium or steel in laboratory experiments or industrial applications are characterised by $\rho\nu \sim 10^{-3} kg/ms$, $\sigma \sim 10^{6}\Omega^{-1}m^{-1}$, $B \sim 1T$ which gives $\delta \sim 30\mu m$. Thus, Hartmann layers are usually very thin. The ratio between the channel width d and the thickness of the Hartmann layer

$$Ha = Bd\sqrt{\frac{\sigma}{\rho\nu}} \tag{2}$$

is called Hartmann number. It is a nondimensional measure for the strength of the electro-magnetic forces in relation to the viscous forces.

The Hartmann layer is one of the few MHD flows that are amenable to rigorous analytic treatment. Starting from the incompressible Navier-Stokes equations supplemented by Maxwell's equations and Ohm's law one can show [3] that the unidirectional flow $\mathbf{v} = v(z)\mathbf{e}_x$ is an exact solution to the full MHD problem provided that the profile $v(z)$ satisfies the equation

$$\delta^2 \frac{d^3v}{dz^3} - \frac{dv}{dz} = 0 \tag{3}$$

subject to the no-slip condition $v(-d/2) = v(+d/2) = 0$ and to the condition $d^2 \int v(z)dz = Q$ where Q (with unit m^3/s) is a prescribed volume flux through a section of the channel with spanwise length d. The reader can readily verify that the solution to this equation is given by the Hartmann profile

$$v(z) = \frac{QHa}{2d^2} \left\{ \frac{\cosh\left(\frac{Ha \cdot z}{d}\right) - \cosh\left(\frac{Ha}{2}\right)}{\sinh\left(\frac{Ha}{2}\right) - \frac{Ha}{2}\cosh\left(\frac{Ha}{2}\right)} \right\} \tag{4}$$

An example of this profile is shown in Figure 1b. The Hartmann flow (4) is an exact solution of the basic equations of MHD but is not necessarily stable.
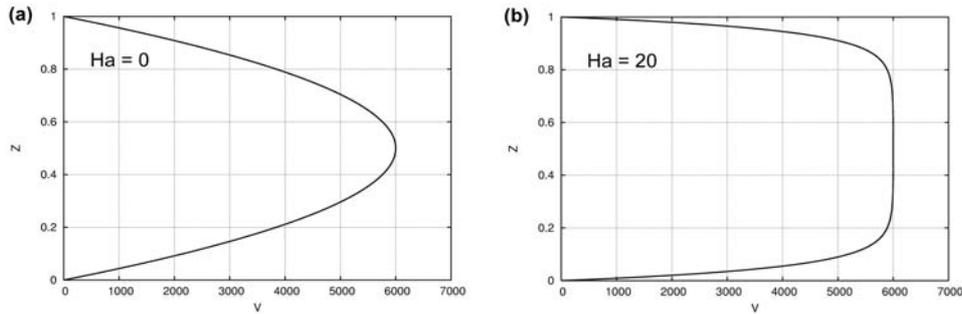
Fig. 1 Parabolic velocity profile (a) and Hartmann profile (b) for laminar flow in a channel.

The goal of this paper is to familiarize the reader with the mechanisms responsible for the loss of stability of this flow and for the ways in which the Hartmann flow becomes turbulent. Our paper is not intended as a review but will rather focus on the results of a recent joint experimental-numerical work whose details are given in [4] and [5].

2   A brief history of the Hartmann layer

The Hartmann layer was theoretically predicted and experimentally investigated in the seminal work of Hartmann and Lazarus [1], [2]. These authors demonstrated that in the case of a laminar flow the hydraulic resistance of the considered channel was in good agreement with the theoretical prediction as obtained from the Hartmann profile (4). Moreover, they showed that the transition between turbulent and laminar flow states is governed by the parameter

$$R = \frac{U\delta}{\nu} \tag{5}$$

and occurs in the range

$$150 < R < 250. \tag{6}$$

This observation was later refined and confirmed by a number of other experimental studies [6], [12], [13]. Notice that U is the velocity in the middle of the channel (centerline velocity) and that R can be expressed through the "conventional" Reynolds number $Re = Ud/\nu$ as $R = Re/Ha$. Therefore, R can be interpreted as a Reynolds number based on the thickness of the Hartmann layer. The understanding of the threshold (6) for this transition remained a puzzle until recently.

Early results for the linear instability of the flow (4) were obtained by Lock [7], who found $R_c \approx 50000$, neglecting Lorentz forces acting on the disturbances, and by Roberts [8], who corrected the limit to $R_c \approx 46200$. More recently, the stability analysis using numerical techniques for modified plane Poiseuille flow and modified plane Couette flow in the presence of a transverse magnetic field ([9], [10]) produced a critical Reynolds number of $R_c = 48311.016$ for sufficiently high Hartmann number. An isolated Hartmann layer was investigated numerically by Lingwood & Alboussière [11] who studied the cases of electrically insulating and conducting walls with normal and arbitrarily oriented magnetic field. For the flow considered here, i.e. for the case of insulating walls and vertical magnetic field, they found $R_c = 48250$, which differs only slightly from the results of Takashima [9]. In summary, the critical Reynolds number for the linear instability of the Hartmann flow is much higher than the observed one. This implies that linear stability theory is unable to explain the transition to turbulence in the Hartmann flow.

The failure of linear stability theory led Lingwood & Alboussière [11] to perform an energy stability analysis of the Hartmann flow. They demonstrated that for $R < 25.6$ a single Hartmann layer is stable with respect to arbitrary perturbations. Although is was assuring that the observed critical Reynolds number was above the critical Reynolds number for energy stability, the latter is an order of magnitude below R_c. Thus, energy stability does not provide a key for understanding transition to turbulence in the Hartmann flow.

This unsatisfactory state of affairs prompted researchers to perform new experiments [4] and direct numerical simulations [5] which finally led to a better understanding of the instability of the Hartmann flow. The main results of these investigations will be summarized next.

3   Experiments of the Cambridge-Grenoble group [4]

The experiments were carried out in an annular channel with an outer diameter of 10cm and a channel width $d = 1cm$ filled with mercury and placed in a magnetic field with up to 13 Tesla. The flow was driven by passing a radial electric current through the mercury layer which interacted with the applied magnetic field to produce an azimuthal driving force. By varying the voltage V and measuring the total electric current I the authors could obtain the friction factor F as a function of the Reynolds number R. Figure 2 shows a result of the experiment.
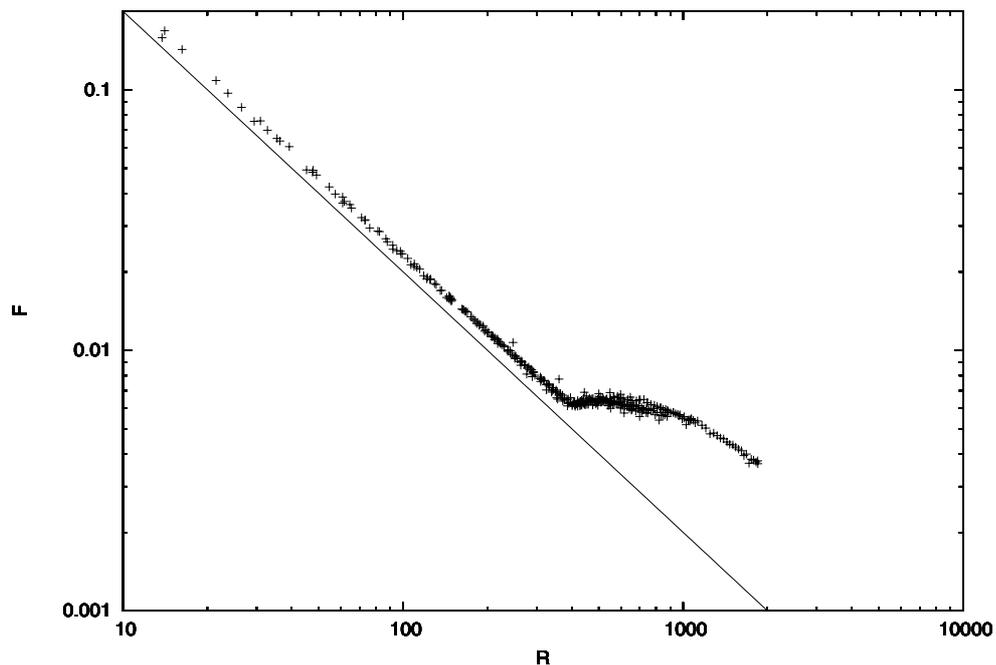
Fig. 2 Friction factor versus R as obtained in the experiments of Moresco & Aboussière [4]. The straight line corresponds to the friction factor for laminar Hartmann layers.

The straight line $2/R$ in the plot corresponds to the value of F in the laminar case [14]. It can be seen that the experimental results follow this curve well up to $R \approx 380$, where a marked transition occurs and the friction factor takes higher values than in the laminar case. The critical Reynolds number was found to be the same when the current was decreased until the flow became laminar, i.e. no hysteresis was observed.

4   Numerical simulations of the Ilmenau-Dearborn group [5]

The Hartmann flow is similar to plane Poiseuille and pipe Poiseuille flow in that its instability appears far below the threshold of linear instability. In the last decade, considerable advances have been made towards a better theoretical understanding of this kind of transition (see [15]

or [16] for a review). It has been shown that a strong transient growth of certain perturbations is possible because of the non-normality of the linear stability (Orr-Sommerfeld) operator of the flow (see [16] and references therein). The perturbations are two-dimensional in the sense that they do not vary in the streamwise (flow) direction. The largest growth is provided by the perturbations that initially have the form of streamwise vortices (rolls) and grow by the mechanism of redistribution of the mean flow energy to form two-dimensional "streaks". In the linearized formulation, the perturbations would eventually decay after the transient growth because all eigenvalues of the system correspond to linearly stable modes. But, if the transient growth is large enough, the non-linearity has to be taken into consideration. It can be shown that the modulated flow is unstable to three-dimensional (3D) perturbations with their amplitude being dependent, in general, on flow parameters. Thus, a two-step mechanism was proposed to explain the features of transition to turbulence in shear flows consisting of (i) a large transient growth of small (but not infinitely small) two-dimensional (2D) disturbances leading to a modulation of the basic flow, and (ii) the linear instability of the modulated flow with respect to 3D perturbations.

We have performed a series of numerical simulations in order to test whether the foregoing two-step scenario can explain the experimental threshold $R_c = 380$ for the instability. We use a pseudospectral code with periodic boundary conditions in both horizontal direction and split the numerical experiments into two steps to be carried out separately: 2D and full 3D simulations. A similar approach was employed in earlier stability investigations of other shear flows such as, for example channel flow [17]. We specify the initial energy $E(0)$ of the 2D streamwise vortices and calculate the 2D evolution until the energy of vortices grows to the maximum level of amplification. At this moment, 3D random noise with given amplitude is imposed, while the artificial zeroing of streamwise Fourier coefficients is switched off and the simulation is continued as fully three-dimensional.
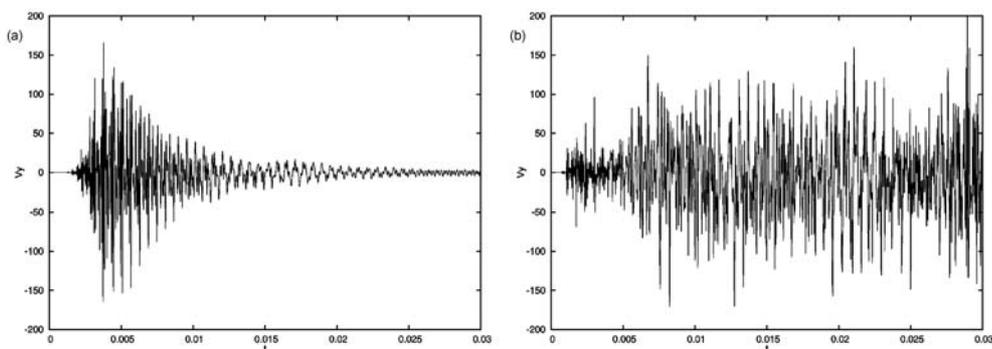
Fig. 3 Temporal evolution of the spanwise velocity component for $R = 350$ and $Ha = 20$ for a subcritical (a) and supercritical (b) perturbation as obtained in the numerical simulations of [5].

The results of our simulations can be summarized as follows. No transition to turbulence was found at $R < 350$. At higher values of R, approximately $R > 400$, the transition occurred every time the amplitude of 2D perturbations was sufficient for the inflection points in the mean flow profile to develop. It was necessary to exceed some minimum amplitude of 3D noise to trigger the instability. This amplitude, which was determined in the calculations,

varied with R and $E(0)$. For the intermediate values $350 \leq R < 400$, a peculiar behaviour of modulated 2D flow was observed. If the initial energy of 2D flow was below a certain level, no transition occurred regardless of the amplitude of 3D perturbations imposed afterwards. A slight increase of E(0) above this level made it possible to find the amplitude of 3D noise that triggered the transition. It is important to remark that visual inspection of both "stable" and "unstable" 2D modulated flow revealed similar streaks with well developed inflection points so that the instability was not correlated directly with the presence of these features.

The difference between the stable and unstable evolutions in the intermediate range of R is seen in Figure 3 which shows the signals of a local spanwise velocity "measurement". Both correspond to the case with $R = 350$. The initial energy $E(0)/E_{\text{basic flow}}$ of the 2D modulation is 8×10^{-3} and 10^{-2} respectively, and the initial energy of the random noise is 10^{-4} for both cases. Despite the fact that the magnitude of the 2D perturbations is strong enough to form the inflection points in both cases, the evolution yields two separate possibilities, i.e. re-laminarization (Figure 3a) and transition to turbulence (Figure 3b). During the initial phase both flows experience a transient growth accompanied by the oscillations, whose frequency is apparently a characteristic for a given set of parameters. The subsequent evolution shows the decay of perturbations in Figure 3a and stochastic behaviour in Figure 3b with all time-scales involved. The critical flow modulation, i.e. the initial amplitude of 2D energy necessary to trigger the energy transfer from 2D to 3D perturbations, exists for any $R > 25.6$.

In conclusion of our numerical study we can say that, for $R \approx 350$ the Hartmann layer becomes unstable at the initial amplitude 10^{-2}. This result is in remarkable agreement with the experimental finding that instability occurs for $R \approx 380$. To demonstrate the robustness of the threshold of transition we have performed additional simulations using random initial conditions instead of two-dimensional streamwise vortices. The results which illustrate the formation of streaks are shown in Figure 4. The set of frames (a) - (f) shows the temporal evolution of an arbitrary three-dimensional random noise imposed at the initial state. The isosurfaces of streamwise velocity fluctuations demonstrate that the flow transformation is accompanied by the appearance of structures elongated in the streamwise direction.

5   Summary and conclusions

The transition to turbulence in the Hartmann layer which is observed to take place for $R = 380$ [4] can be understood in terms of a two-step scenario. The transition consists of a growth of streamwise two-dimensional perturbations which subsequently become unstable with respect to fully three-dimensional perturbations. The numerical simulations [5] yield a range of $350 < R < 400$ for the critical Reynolds number which is in good agreement with the experimental findings. Thereby a long-standing open problem of MHD has been solved. Future experimental and numerical work is necessary in order to refine our understanding of the transition process and to better characterize the fully developed turbulent regime.
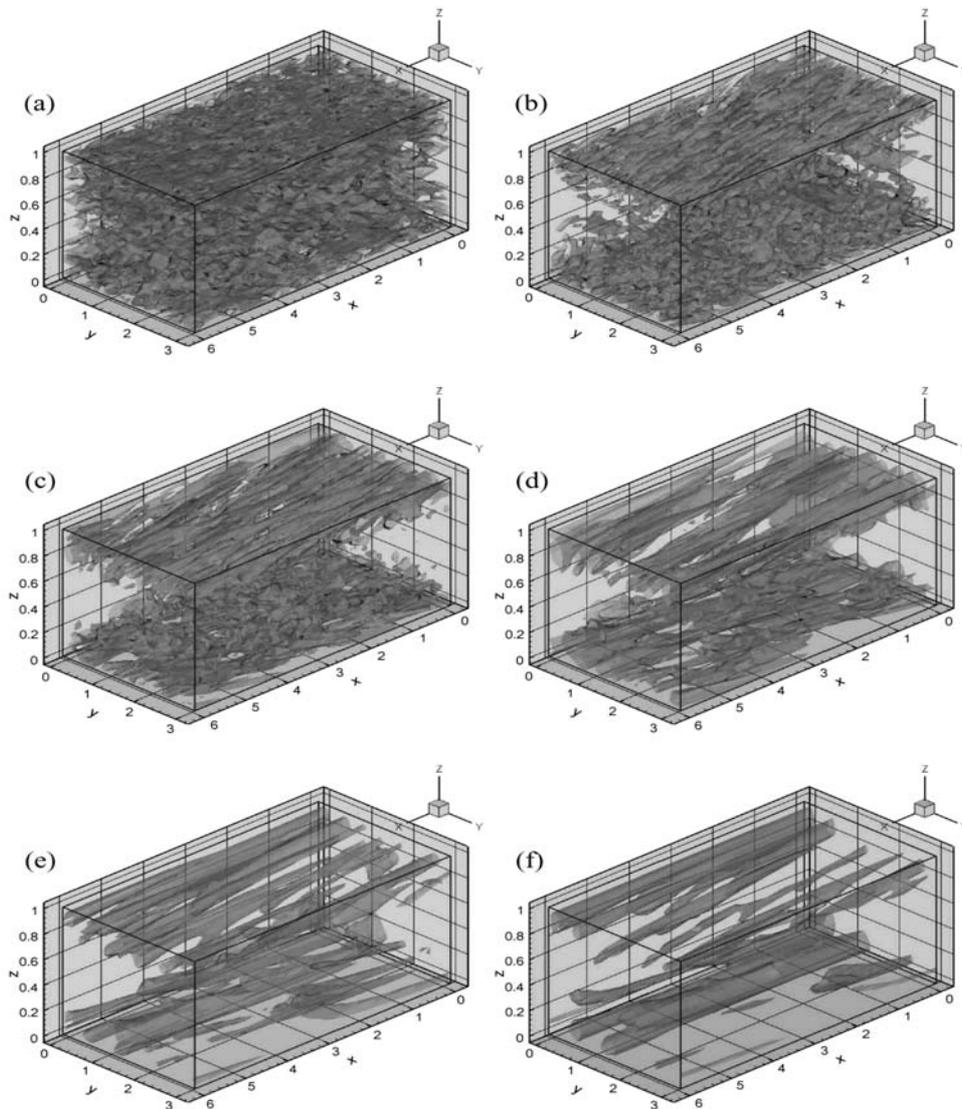
Fig. 4 Temporal evolution of the Hartmann flow with initially imposed random three-dimensional noise. Shown are the isosurfaces of the streamwise velocity fluctuations for $R = 400$ and $Ha = 15$ as obtained in the numerical simulations of [5].

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