

# Statistical mechanics of gravitating systems

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IUCAA, Pune, India

**Les Houches, August 08**

## Some References:

1. T. Padmanabhan, Physics Reports, **188**, 285 (1990).
2. T. Padmanabhan, *Statistical Mechanics of gravitating systems in static and cosmological backgrounds*,  
Les Houches 2001 on “Dynamics and Thermodynamics of Systems with Long Range Interactions”, astro-ph/0206131.
3. T. Padmanabhan, Theoretical Astrophysics Vol 1 (CUP, 2000) Chapter on Gravitational Dynamics.

# SOME COMMENTS ABOUT THE COURSE

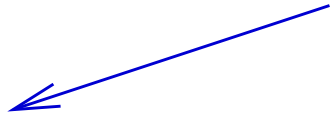
- Blackboard plus Laptop Display: You will find details of the algebra in the references (Or do it as Exercises!)
- Plan:
  - Rapid introduction to concepts;
  - Two talks dealing with static background;
  - Two talks dealing with cosmological background

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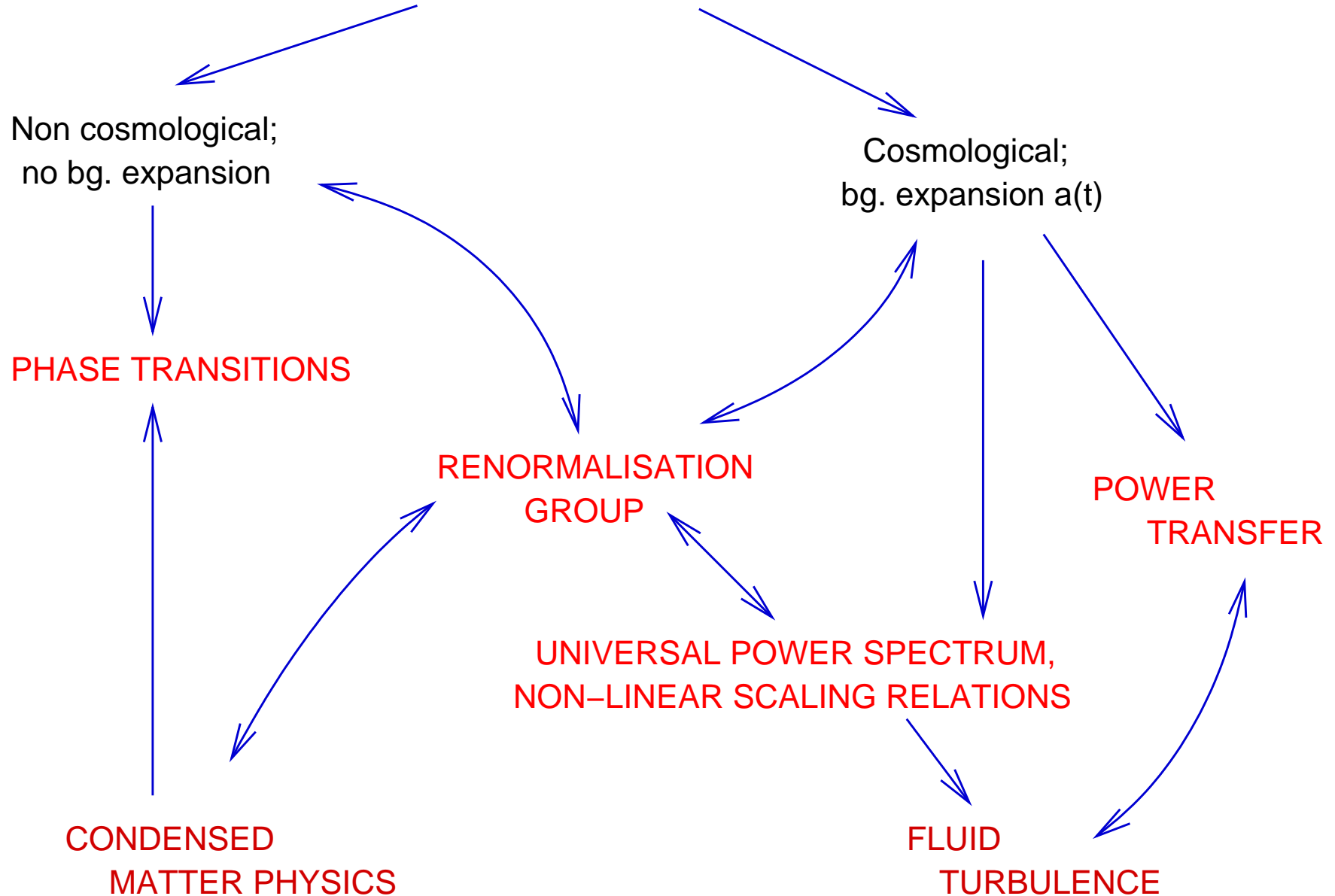
RENORMALISATION  
GROUP

POWER  
TRANSFER

UNIVERSAL POWER SPECTRUM,  
NON-LINEAR SCALING RELATIONS

CONDENSED  
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## FINITE GRAVITATING SYSTEMS

### – GENERAL DESCRIPTION –

- Typical systems have two natural energy scales:  $E_1 = (-Gm^2/a)$  and  $E_2 = (-Gm^2/R)$ ;  $a \ll R$ . Virial theorem gives:  $2K + U = 3PV + U_0$ .
- For  $E \gg E_2$ , gravity is not strong enough and the system behaves like a gas confined by the container. High temperature phase with positive specific heat;  $2K \approx 3PV$ .
- For  $E_1 \ll E \ll E_2$ , the system is unaffected by either the box or the short distance cutoff; dominated entirely by gravity and has **negative** specific heat.  $2K + U \approx 0$ .
- Since canonical ensemble cannot lead to negative specific heat, microcanonical and canonical ensembles differ drastically in this range. Canonical ensemble leads to a phase transition.
- As we go to  $E \simeq E_1$ , the hard core nature of the particles begins to be felt and the gravity is again resisted; low temperature phase with positive specific heat.  $U \approx U_0$ .
- Increasing  $R$  increases the range over which the specific heat is negative.

# TOY MODEL

T. Padmanabhan, Physics Reports, **188**, 285 (1990).

- System with a Hamiltonian

$$H(\mathbf{P}, \mathbf{Q}; \mathbf{p}, \mathbf{r}) = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} - \frac{Gm^2}{r}; \quad (a < r < R)$$

- Volume  $g(E)$  of the constant energy surface  $H = E$ : Entropy  $S(E)$  and the temperature  $T(E)$  of the system is:

$$S(E) = \ln g(E); \quad T^{-1}(E) = \beta(E) = \frac{\partial S(E)}{\partial E}.$$

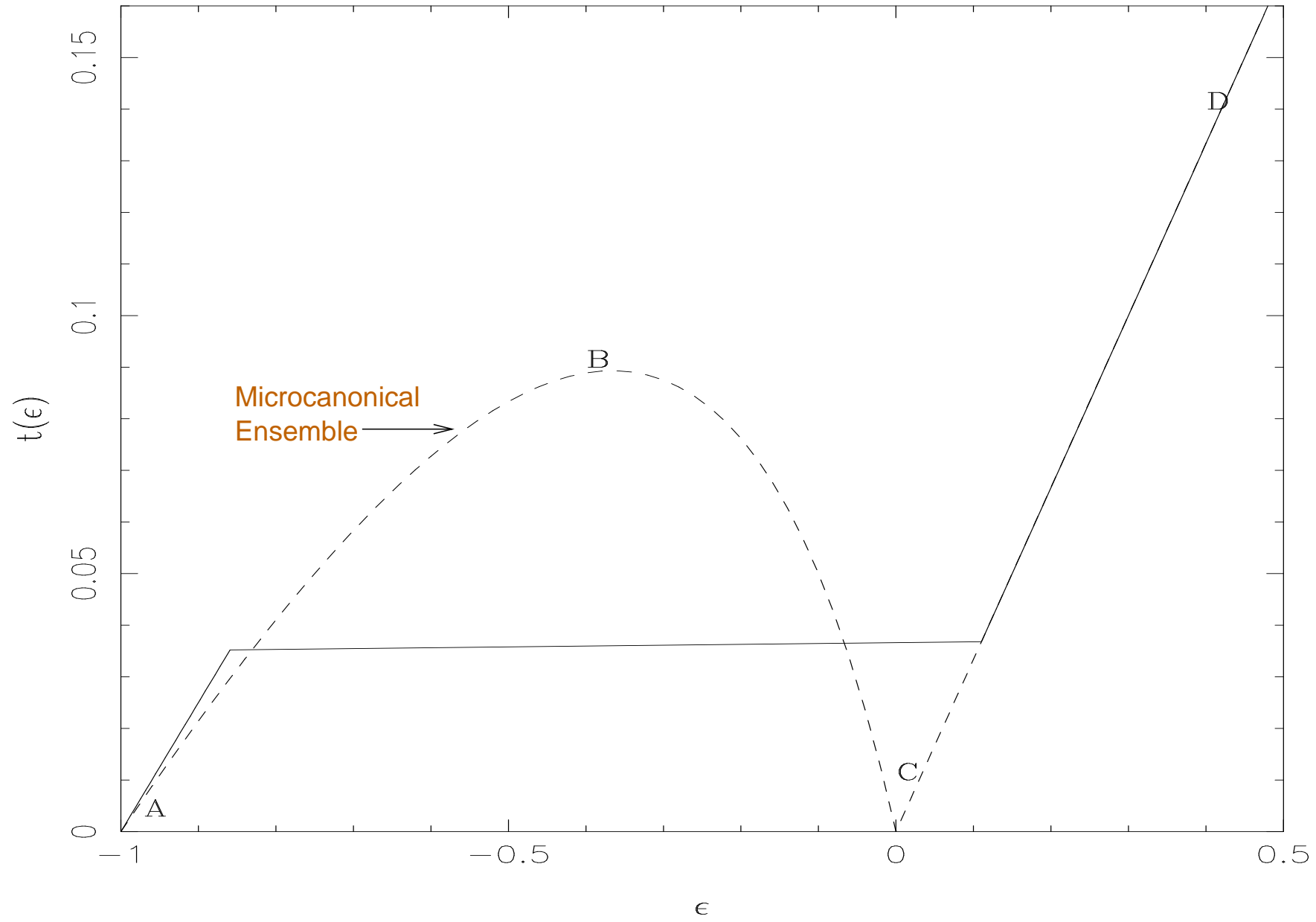
- In this case:

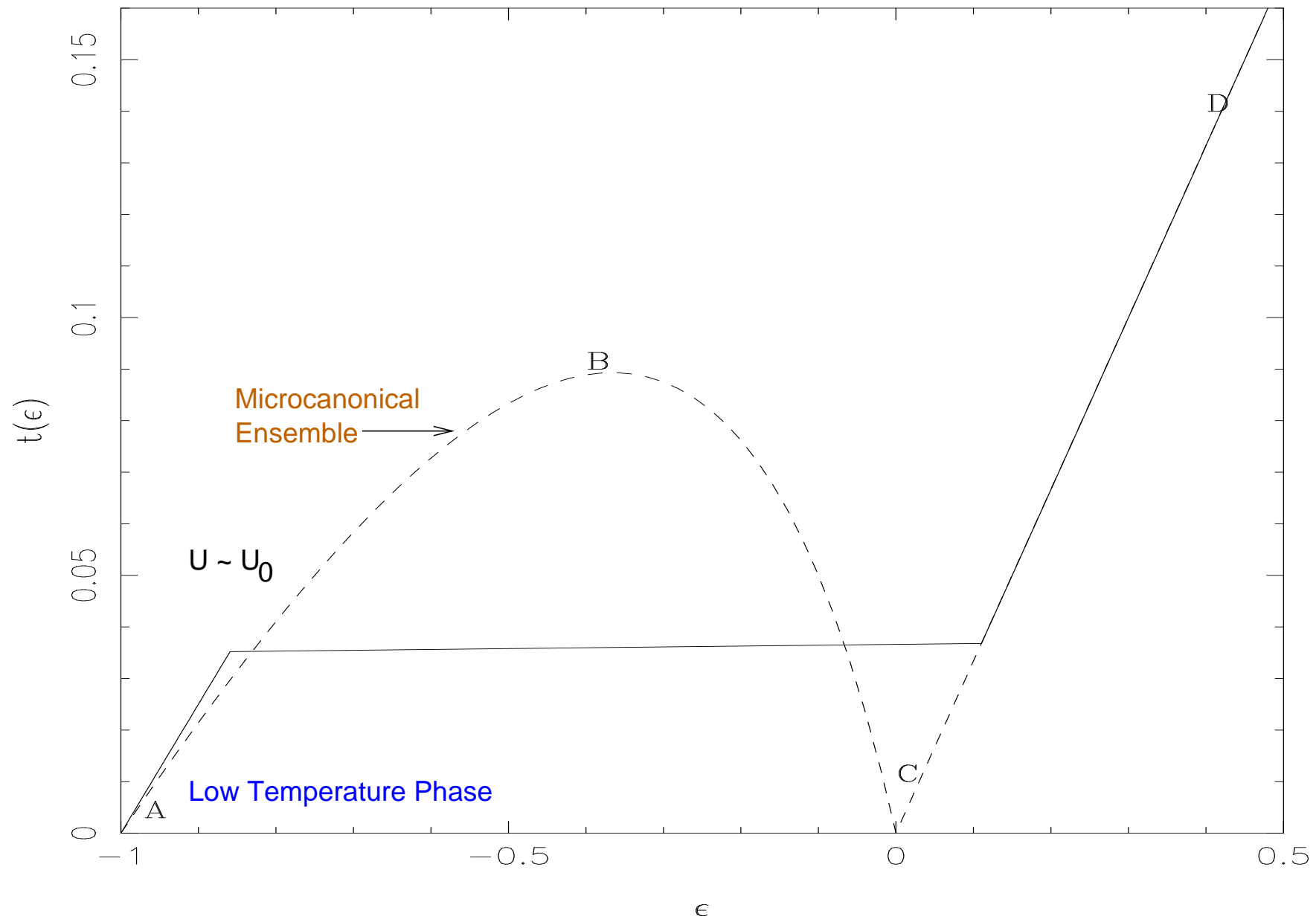
$$g(E) = AR^3 \int_a^{r_{\max}} r^2 dr \left[ E + \frac{Gm^2}{r} \right]^2; \quad A = (64\pi^5 m^3 / 3)$$

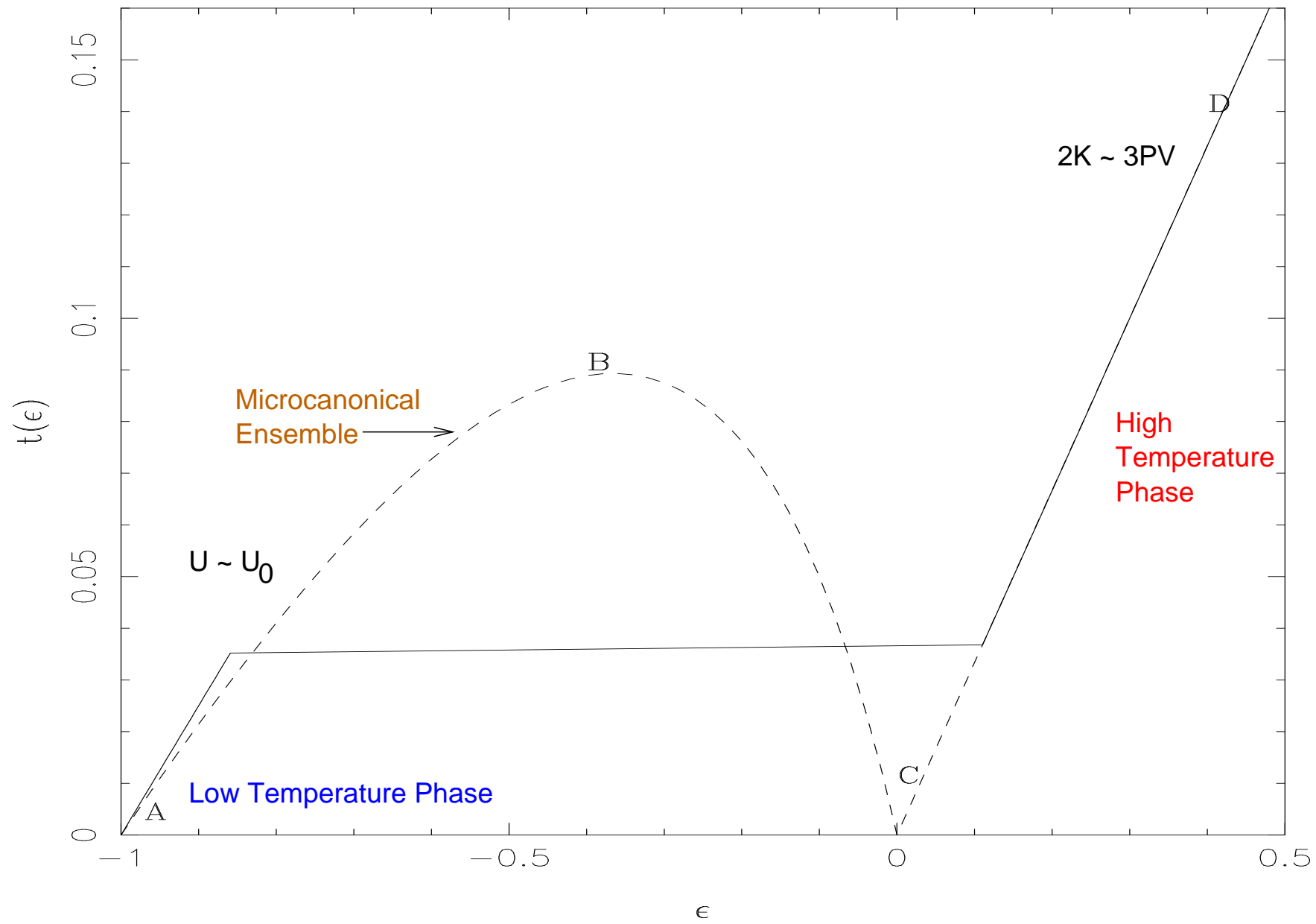
where  $r_{\max} = (Gm^2/|E|)$     if  $(-Gm^2/a) < E < (-Gm^2/R)$ ,  
 $r_{\max} = R$                     if  $(-Gm^2/R) < E < +\infty$

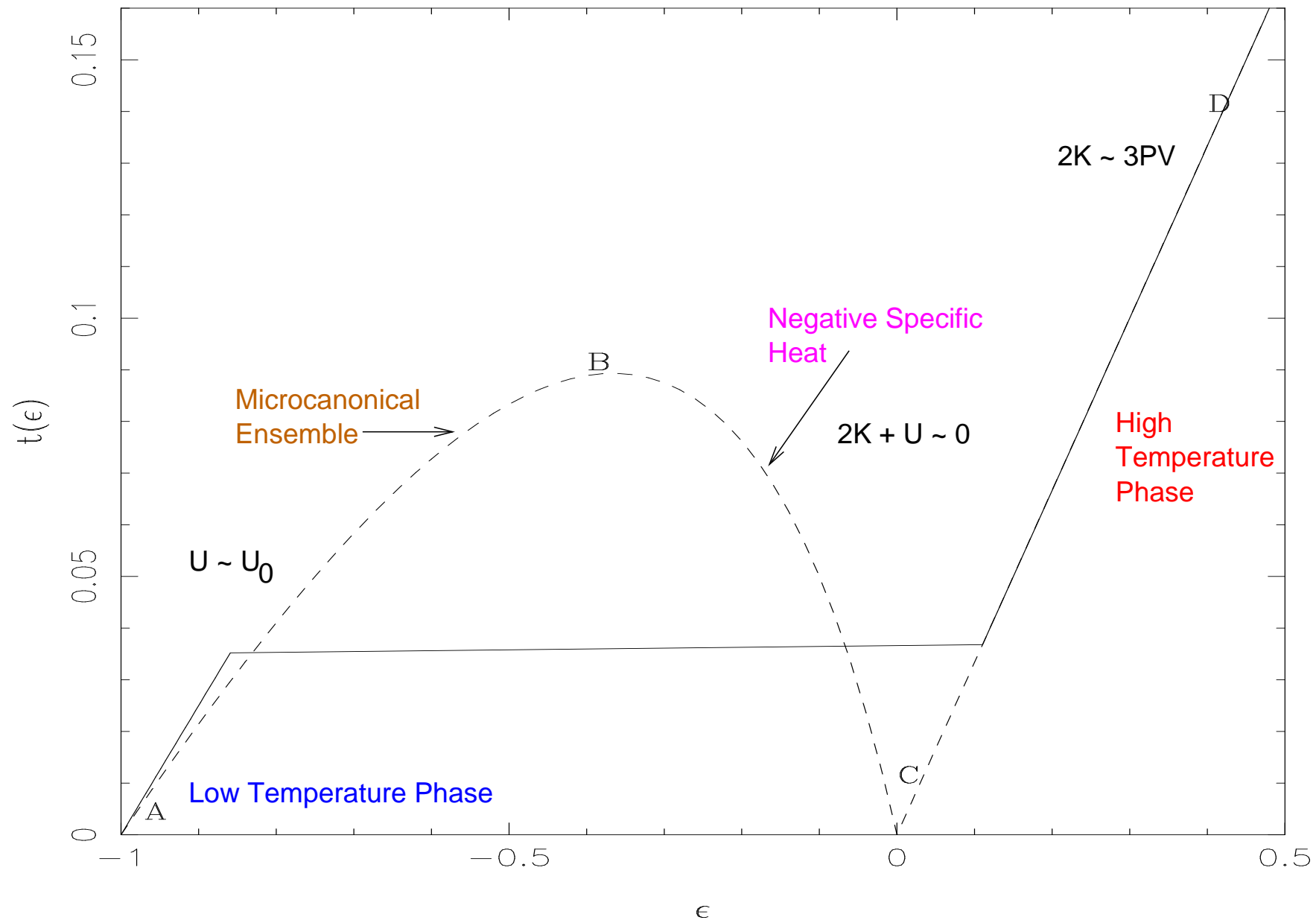
- Phase volume:

$$\frac{g(E)}{(Gm^2)^3} = \begin{cases} \frac{R^3}{3} (-E)^{-1} \left(1 + \frac{aE}{Gm^2}\right)^3, & (-Gm^2/a) < E < (-Gm^2/R) \\ \frac{R^3}{3} (-E)^{-1} \left[ \left(1 + \frac{RE}{Gm^2}\right)^3 - \left(1 + \frac{aE}{Gm^2}\right)^3 \right], & (-Gm^2/R) < E < \infty. \end{cases}$$









## PHASE TRANSITION

- Systems described by canonical distribution cannot exhibit negative specific heat.
- The partition function is:

$$Z(\beta) = \int d^3P d^3p d^3Q d^3r \exp(-\beta H) = t^3 \left(\frac{R}{a}\right)^3 \int_1^{R/a} dx x^2 \exp\left(\frac{1}{xt}\right)$$

- The integrand reaches a minimum for  $x = x_m = (1/2t)$ .
- At high temperatures,  $x_m < 1$  and hence the minimum falls outside the domain of integration and:

$$Z \approx t^3 \left(\frac{R}{a}\right)^3 \int_1^{R/a} dx x^2 \left[1 + \frac{2x_m}{x}\right] = \frac{t^3}{3} \left(\frac{R}{a}\right)^6 \left(1 + \frac{3a}{2Rt}\right)$$

- On the other hand, if  $x_m > 1$  the minimum lies between the limits of the integration and:

$$Z \approx \left(\frac{R}{a}\right)^3 t^4 (1 - 2t)^{-1} \exp\left(\frac{1}{t}\right)$$

- As we lower the temperature, making  $x_m$  cross 1 from below, the contribution switches over. The transition is exponentially sharp and

occurs at

$$t_c = \frac{1}{3} \frac{1}{\ln(R/a)}.$$

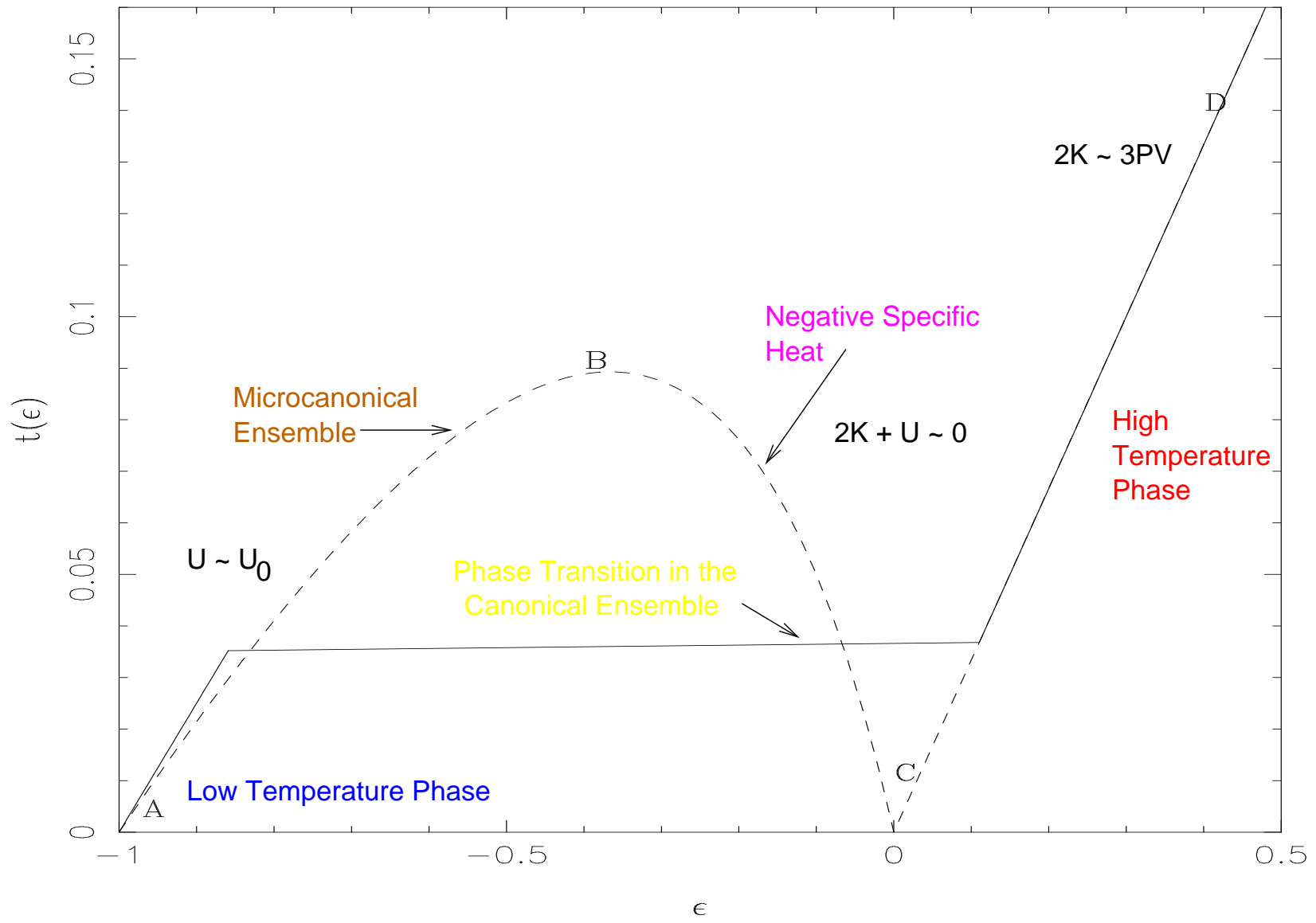
- Mean energy of the system is given by  $E(\beta) = -(\partial \ln Z / \partial \beta)$ . We get,

$$\epsilon(t) = \frac{aE}{Gm^2} = 4t - 1 \quad (\text{for } t < t_c)$$

$$\epsilon(t) = 3t - \frac{3a}{2R}; \quad (\text{for } t > t_c)$$

- Near  $t \approx t_c$ , there is a rapid variation of the energy and system undergoes a phase transition at  $t = t_c$  absorbing a large amount of energy

$$\Delta\epsilon \approx \left( 1 - \frac{1}{3 \ln(R/a)} \right).$$



[T. Padmanabhan, Physics Reports, **188**, 285 (1990).]

## Mean field description of gravitating systems

- A system of  $N$  particles interacting through the two-body potential  $U(\mathbf{x}, \mathbf{y})$ . The entropy  $S$  of this system

$$e^S = g(E) = \frac{1}{N!} \int d^{3N}x d^{3N}p \delta(E - H) = \frac{A}{N!} \int d^{3N}x \left[ E - \frac{1}{2} \sum_{i \neq j} U(\mathbf{x}_i, \mathbf{x}_j) \right]^{\frac{3N}{2}}$$

- Divide spatial volume  $V$  be divided into  $M$  cells of equal size. Integration over the particle coordinates  $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$  = sum over the number of particles  $n_a$  in the cell centered at  $\mathbf{x}_a$  (where  $a = 1, 2, \dots, M$ ).

Integration over  $(N!)^{-1} d^{3N}x$  can be replaced by

$$\sum_{n_1=1}^{\infty} \left( \frac{1}{n_1!} \right) \sum_{n_2=1}^{\infty} \left( \frac{1}{n_2!} \right) \dots \sum_{n_M=1}^{\infty} \left( \frac{1}{n_M!} \right) \delta \left( N - \sum_a n_a \right) \left( \frac{V}{M} \right)^N$$

- This gives:

$$e^S \approx \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \dots \sum_{n_M=1}^{\infty} \delta \left( N - \sum_a n_a \right) \exp S[\{n_a\}]$$

$$S[\{n_a\}] = \frac{3N}{2} \ln \left[ E - \frac{1}{2} \sum_{a \neq b}^M n_a U(\mathbf{x}_a, \mathbf{x}_b) n_b \right] - \sum_{a=1}^M n_a \ln \left( \frac{n_a M}{eV} \right)$$

- The mean field approximation: retain in the sum only the term for which the summand reaches the maximum value

$$\sum_{\{n_a\}} e^{S[n_a]} \approx e^{S[n_{a,\max}]}$$

where  $n_{a,\max}$  is the solution to the variational problem

$$\left( \frac{\delta S}{\delta n_a} \right)_{n_a = n_{a,\max}} = 0 \quad \text{with} \quad \sum_{a=1}^M n_a = N$$

$$\frac{1}{T} \sum_{b=1}^M U(\mathbf{x}_a, \mathbf{x}_b) n_{b,\max} + \ln \left( \frac{n_{a,\max} M}{V} \right) = \text{constant}$$

with

$$\frac{1}{T} = \frac{3N}{2} \left( E - \frac{1}{2} \sum_{a \neq b}^M n_a U(\mathbf{x}_a, \mathbf{x}_b) n_b \right)^{-1} = \beta$$

- Continuum limit with

$$n_{a,\max} \frac{M}{V} = \rho(\mathbf{x}_a); \quad \sum_{a=1}^M \rightarrow \frac{M}{V} \int.$$

gives

$$\rho(\mathbf{x}) = A \exp(-\beta\phi(\mathbf{x})); \quad \text{where} \quad \phi(\mathbf{x}) = \int d^3\mathbf{y} U(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y})$$

- In the case of gravitational interaction:

$$\rho(\mathbf{x}) = A \exp(-\beta\phi(\mathbf{x})); \quad \phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{y}) d^3\mathbf{y}}{|\mathbf{x} - \mathbf{y}|}.$$

gives the configuration of extremal entropy for a gravitating system in the mean field limit.

- For gravitational interactions *without* a short distance cut-off, the quantity  $e^S$  is divergent. A short distance cut-off is needed to justify the entire procedure.

## Isothermal sphere

- Basic equation:

$$\nabla^2 \phi = 4\pi G \rho_c e^{-\beta[\phi(\mathbf{x}) - \phi(0)]}; \quad \rho_c = \rho(0)$$

- Let

$$L_0 \equiv (4\pi G \rho_c \beta)^{1/2}, \quad M_0 = 4\pi \rho_c L_0^3, \quad \phi_0 \equiv \beta^{-1} = \frac{GM_0}{L_0}$$

$$x \equiv \frac{r}{L_0}, \quad n \equiv \frac{\rho}{\rho_c}, \quad m = \frac{M(r)}{M_0}, \quad y \equiv \beta [\phi - \phi(0)].$$

- Then

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = e^{-y}; \quad y(0) = y'(0) = 0$$

- Singular solution:  $n = (2/x^2)$ ,  $m = 2x$ ,  $y = 2 \ln x$

- Invariant under the transformation  $y \rightarrow y + a$  ;  $x \rightarrow kx$  with  $k^2 = e^a$ .
- Reduce the degree by choosing:

$$v \equiv \frac{m}{x}; \quad u \equiv \frac{nx^3}{m} = \frac{nx^2}{v}.$$

$$\frac{u \, dv}{v \, du} = -\frac{(u - 1)}{(u + v - 3)}; \quad v(u = 3) = 0; \quad v'(u = 3) = -\frac{5}{3}$$

- The singular points are:  $u_s = 1$ ,  $v_s = 2$ ; corresponding to the solution  $n = (2/x^2)$ ,  $m = 2x$ .
- All solutions tend to this asymptotically for large  $r$ .

- The potential and kinetic energies are

$$U = - \int_0^R \frac{GM(r)}{r} \frac{dM}{dr} dr = - \frac{GM_0^2}{L_0} \int_0^{x_0} mnxdx$$

$$K = \frac{3}{2} \frac{M}{\beta} = \frac{3}{2} \frac{GM_0^2}{L_0} m(x_0) = \frac{GM_0^2}{L_0} \frac{3}{2} \int_0^{x_0} nx^2 dx; \quad x_0 = R/L_0$$

- So:

$$E = K + U = \frac{GM_0^2}{2L_0} \int_0^{x_0} dx(3nx^2 - 2mnx)$$

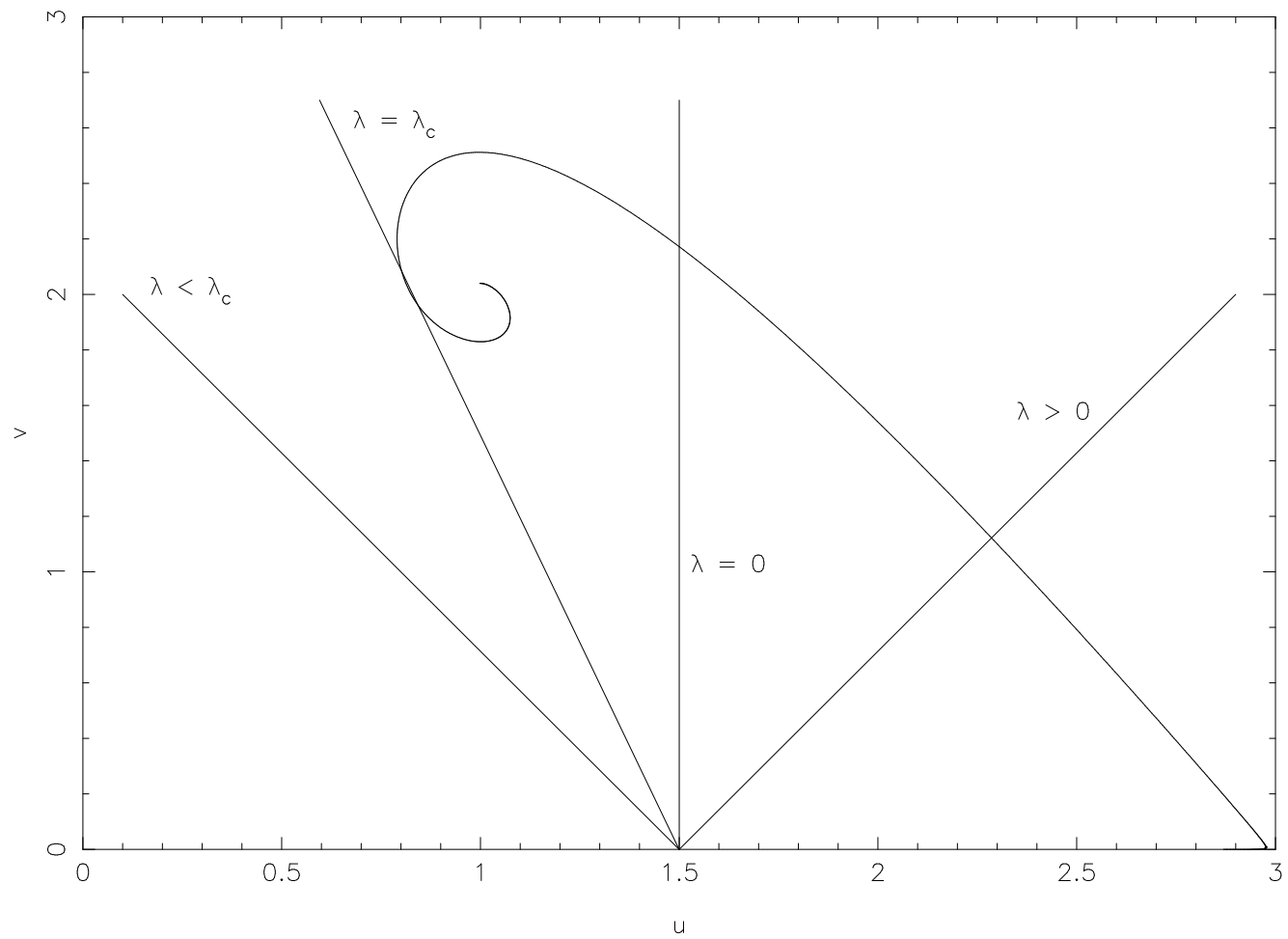
$$= \frac{GM_0^2}{2L_0} \int_0^{x_0} dx \frac{d}{dx} \{2nx^3 - 3m\} = \frac{GM_0^2}{L_0} \{n_0 x_0^3 - \frac{3}{2} m_0\}$$

- *The combination  $(RE/GM^2)$  is a function of  $(u, v)$  alone.*

$$\lambda = \frac{RE}{GM^2} = \frac{1}{v_0} \left\{ u_0 - \frac{3}{2} \right\}.$$

- An isothermal sphere *must* lie on the curve

$$v = \frac{1}{\lambda} \left( u - \frac{3}{2} \right); \quad \lambda \equiv \frac{RE}{GM^2}$$



[T. Padmanabhan, Physics Reports, **188**, 285 (1990).]

## ANTONOV INSTABILITY

### Reference:

- V.A. Antonov, V.A. : Vest. Leningrad Univ. **7**, 135 (1962); Translation: IAU Symposium **113**, 525 (1985).
  - T.Padmanabhan: Astrophys. Jour. Supp. , **71**, 651 (1989).
1. Systems with  $(RE/GM^2) < -0.335$  cannot evolve into isothermal spheres. Entropy has no extremum for such systems.
  2. Systems with  $((RE/GM^2) > -0.335)$  and  $(\rho(0) > 709 \rho(R))$  can exist in a meta-stable (saddle point state) isothermal sphere configuration. The entropy extrema exist but they are not local maxima.
  3. Systems with  $((RE/GM^2) > -0.335)$  and  $(\rho(0) < 709 \rho(R))$  can form isothermal spheres which are local maximum of entropy.

# GRAVITATIONAL CLUSTERING IN EXPANDING UNIVERSE

## – OPEN ISSUES –

- If the initial power spectrum is sharply peaked in a narrow band of wavelengths, how does the evolution transfer the power to other scales?
- What is the asymptotic nature of evolution for the self gravitating system in an expanding background?
- Does the gravitational clustering at late stages wipe out the memory of initial conditions ?
- Do the virialized structures formed in an expanding universe due to gravitational clustering have any invariant properties? Can their structure be understood from first principles?
- How can one connect up the local behaviour of gravitating systems to the evolution of clustering in the universe?

## BASIC DEFINITIONS

- Density:

$$\rho(\mathbf{x}, t) = \frac{m}{a^3(t)} \sum_i \delta_D[\mathbf{x} - \mathbf{x}_i(t)]$$

- Mean density:

$$\rho_b(t) \equiv \int \frac{d^3\mathbf{x}}{V} \rho(\mathbf{x}, t) = \frac{m}{a^3(t)} \left( \frac{N}{V} \right) = \frac{M}{a^3 V} = \frac{\rho_0}{a^3}$$

- Density contrast:

$$1 + \delta(\mathbf{x}, t) \equiv \frac{\rho(\mathbf{x}, t)}{\rho_b} = \frac{V}{N} \sum_i \delta_D[\mathbf{x} - \mathbf{x}_i(t)] = \int d\mathbf{q} \delta_D[\mathbf{x} - \mathbf{x}_T(t, \mathbf{q})].$$

- Density contrast in Fourier space:

$$\delta_{\mathbf{k}}(t) \equiv \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \delta(\mathbf{x}, t) = \int d^3\mathbf{q} \exp[-i\mathbf{k}\cdot\mathbf{x}_T(t, \mathbf{q})] - (2\pi)^3 \delta_D(\mathbf{k})$$

## GRAVITATIONAL CLUSTERING IN EXPANDING BACKGROUND

- Equation of motion:

$$\mathbf{r}(t) = a(t)\mathbf{x}(t); \quad \ddot{\mathbf{r}} = -\nabla_{\mathbf{r}}\Phi; \quad \Phi = \Phi_{\text{FRW}} + \phi$$

$$\ddot{a}\mathbf{x} + 2\dot{a}\dot{\mathbf{x}} + a\ddot{\mathbf{x}} = -\nabla_{\mathbf{r}}\Phi_{\text{FRW}} - \nabla_{\mathbf{r}}\phi = -\nabla_{\mathbf{r}}\Phi_{\text{FRW}} - a^{-1}\nabla_{\mathbf{x}}\phi$$

- This gives:

$$\Phi_{\text{FRW}} = -\frac{1}{2} \frac{\ddot{a}}{a} r^2 = -\frac{2\pi G}{3} \rho_b r^2$$

$$\ddot{\mathbf{x}} + 2\frac{\dot{a}}{a}\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla_x\phi; \quad \nabla_x^2\phi = 4\pi G a^2(\delta\rho) = 4\pi G \rho_b a^2 \delta.$$

## CENTRAL EQUATION

- Density contrast in Fourier space satisfies:

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} = 4\pi G\rho_b\delta_{\mathbf{k}} + A_{\mathbf{k}} - B_{\mathbf{k}}$$

with

$$A_{\mathbf{k}} = 4\pi G\rho_b \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \delta_{\mathbf{k}'} \delta_{\mathbf{k}-\mathbf{k}'} \left[ \frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2} \right]$$

$$B_{\mathbf{k}} = \int d^3\mathbf{q} (\mathbf{k}\cdot\dot{\mathbf{x}}_T)^2 \exp[-i\mathbf{k}\cdot\mathbf{x}_T(t, \mathbf{q})]$$

- Relation to gravitational potential:

$$\delta_{\mathbf{k}} = -\frac{k^2\phi_{\mathbf{k}}}{4\pi G\rho_b a^2} = -\left(\frac{k^2 a}{4\pi G\rho_0}\right)\phi_{\mathbf{k}} = -\left(\frac{2}{3H_0^2}\right)k^2 a\phi_{\mathbf{k}}$$

- Equation satisfied by gravitational potential:

$$\begin{aligned} \ddot{\phi}_{\mathbf{k}} + 4\frac{\dot{a}}{a}\dot{\phi}_{\mathbf{k}} &= -\frac{1}{2a^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}} \phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}} \left[ \left(\frac{k}{2}\right)^2 + p^2 - 2\left(\frac{\mathbf{k}\cdot\mathbf{p}}{k}\right)^2 \right] \\ &+ \left(\frac{3H_0^2}{2}\right) \int \frac{d^3\mathbf{q}}{a} \left(\frac{\mathbf{k}\cdot\dot{\mathbf{x}}}{k}\right)^2 e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned}$$



## Application 1: “Renormalizability” of gravity

- Contribution from a cluster:

$$\mathcal{R} = \int d^3\mathbf{q} e^{-i\mathbf{k}\cdot\mathbf{Q}} i k^a \left\{ \frac{\partial_a \phi}{a^2} + i \dot{Q}^a (\mathbf{k} \cdot \dot{\mathbf{Q}}) \right\}; \quad \mathbf{Q} = \mathbf{x}_T(t, \mathbf{q})$$

- Expand the exponential as  $e^{-i\mathbf{k}\cdot\mathbf{Q}} \approx (1 - i\mathbf{k} \cdot \mathbf{Q}) + \mathcal{O}(k^2 R^2)$ ; the first term, proportional to  $\nabla\phi$ , the sum of the forces acting on all the particles in the cluster (due to self gravity) vanishes.
- The second term gives, on symmetrization,

$$\sum i a^{-2} (\mathbf{k} \cdot \nabla \phi) e^{-i\mathbf{k}\cdot\mathbf{Q}} \approx \frac{k^a k^b}{2a^2} \sum \left( Q^b \frac{\partial \phi}{\partial Q^a} + Q^a \frac{\partial \phi}{\partial Q^b} \right)$$

- From

$$\ddot{Q}^i + 2 \frac{\dot{a}}{a} \dot{Q}^i = - \frac{1}{a^2} \frac{\partial \phi}{\partial Q^i}$$

we have:

$$\frac{d^2}{dt^2} \sum Q^i Q^j - 2 \sum \dot{Q}^i \dot{Q}^j + 2 \frac{\dot{a}}{a} \frac{d}{dt} \sum Q^i Q^j = - \frac{1}{a^2} \sum \left( Q^j \frac{\partial \phi}{\partial Q^i} + Q^i \frac{\partial \phi}{\partial Q^j} \right)$$

- So

$$\begin{aligned} \sum ia^{-2}(\mathbf{k} \cdot \nabla \phi) e^{-i\mathbf{k} \cdot \mathbf{Q}} &= + \sum (\mathbf{k} \cdot \dot{\mathbf{Q}})^2 + \frac{1}{2} \left( \frac{d^2}{dt^2} + 2\frac{\dot{a}}{a} \frac{d}{dt} \right) \sum (\mathbf{k} \cdot \mathbf{Q})^2 \\ &\approx + \sum (\mathbf{k} \cdot \dot{\mathbf{Q}})^2 + \mathcal{O}(k^2 R^2) \end{aligned}$$

- Consider next the second term

$$\begin{aligned} \sum (ik^a) e^{-i\mathbf{k} \cdot \mathbf{Q}} \left[ i\dot{Q}^a k^b \dot{Q}^b \right] &\cong - \sum k^a k^b \dot{Q}^a \dot{Q}^b (1 - i\mathbf{k} \cdot \mathbf{Q}) + \mathcal{O}(k^2 R^2) \\ &= - \sum (\mathbf{k} \cdot \dot{\mathbf{Q}})^2 + \sum (\mathbf{k} \cdot \dot{\mathbf{Q}})^2 k^a Q^a + \mathcal{O}(k^2 R^2) \end{aligned}$$

- The term  $(A_{\mathbf{k}} - B_{\mathbf{k}})$  receives contribution only from particles which are not bound to any of the clusters to the order  $\mathcal{O}(k^2 R^2)$ .

## “Renormalizability” of gravity

- One can then show that: The term  $(A_{\mathbf{k}} - B_{\mathbf{k}})$  receives contribution only from particles which are not bound to any of the clusters to the order  $\mathcal{O}(k^2 R^2)$ . [see e.g., T.P, astro-ph/0206131.]
- Allows one to ignore contributions from virialised systems and treat the rest in Zeldovich (-like) approximation. Then one gets:

$$H_0^2 \left( a \frac{d^2}{da^2} + \frac{7}{2} \frac{d}{da} \right) \phi_{\mathbf{k}} = -\frac{2}{3} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}}^L \phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}}^L \left[ \left( \frac{k}{2} \right)^2 - \left( \frac{\mathbf{k} \cdot \mathbf{p}}{k} \right)^2 \right] \\ - \frac{1}{2} \int' \frac{d^3 \mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}} \phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}} \left[ \left( \frac{k}{2} \right)^2 + p^2 - 2 \left( \frac{\mathbf{k} \cdot \mathbf{p}}{k} \right)^2 \right]$$

## Evolution at large scales

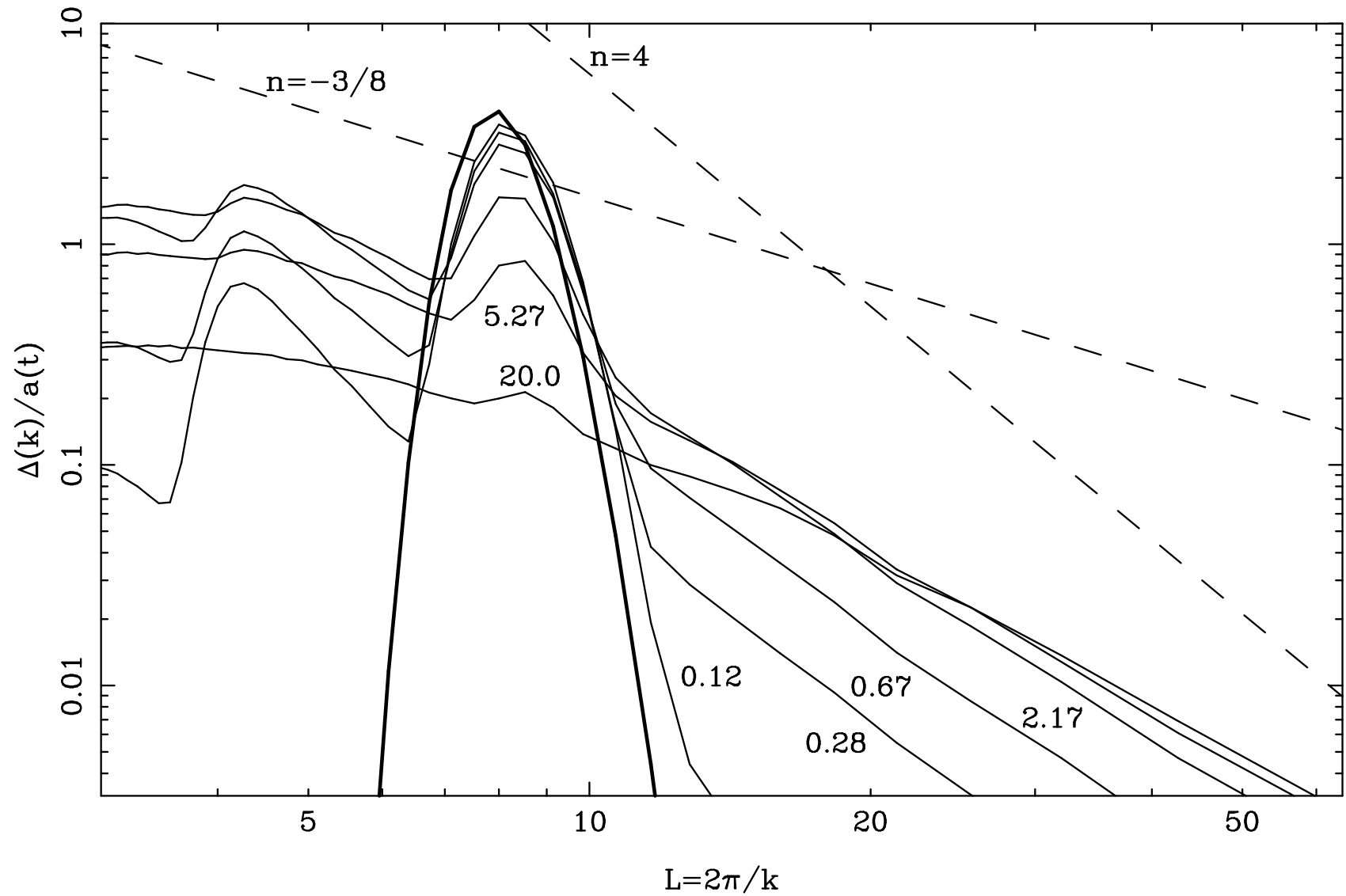
- Ignore the terms  $A_{\mathbf{k}}$  and  $B_{\mathbf{k}}$ . Then:

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} = 4\pi G\rho_b\delta_{\mathbf{k}}$$

- For  $a \propto t^{2/3}$ ,  $\rho_b \propto a^{-3}$ , the growing solution is:

$$\delta_k \propto a; \quad P(k) = |\delta_k|^2 \propto a^2; \quad \xi(a, x) \propto a^2$$

- **BUT:** If  $\delta_{\mathbf{k}} \rightarrow 0$  for certain range of  $\mathbf{k}$  at  $t = t_0$  (but is nonzero elsewhere) then  $(A_{\mathbf{k}} - B_{\mathbf{k}}) \gg 4\pi G\rho_b\delta_{\mathbf{k}}$  and the growth of perturbations around  $\mathbf{k}$  will be entirely determined by nonlinear effects.
- There is inverse cascade of power in gravitational clustering!



Bagla, T.P, (1997) MNRAS, 286, 1023

### Application 3: Spherical approximation

- Take the trajectories to be:  $\mathbf{x}(t, \mathbf{q}) = f(t)\mathbf{q}$ . The density contrast is

$$\delta_{\mathbf{k}}(t) = \int d^3\mathbf{q} e^{-if(t)\mathbf{k}\cdot\mathbf{q}} - (2\pi)^3\delta_D(\mathbf{k}) = (2\pi)^3\delta_D(\mathbf{k})[f^{-3} - 1] \equiv (2\pi)^3\delta_D(\mathbf{k})\delta(t)$$

- In this case:  $A = 4\pi G\rho_b\delta^2(t)[(2\pi)^3\delta_D(\mathbf{k})]$ ; and

$$\begin{aligned} B &= \int d^3\mathbf{q} (k^a q_a)^2 \dot{f}^2 e^{-if(k_a q^a)} = -\dot{f}^2 \frac{\partial^2}{\partial f^2} [(2\pi)^3\delta_D(f\mathbf{k})] \\ &= -\frac{4}{3} \frac{\dot{\delta}^2}{(1+\delta)} [(2\pi)^3\delta_D(\mathbf{k})] \end{aligned}$$

- The closed equation is:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b(1+\delta)\delta + \frac{4}{3} \frac{\dot{\delta}^2}{(1+\delta)}$$

- Let  $(\rho/\rho_b) = (1+\delta) = (R/a)^3$ . This is equivalent to:

$$\ddot{R} = -\frac{GM}{R^2} - \frac{4\pi G}{3}(\rho_b + 3p_b)_{\text{non-dust}}R$$

## Application 4: Formation of first non linear structures

- Trajectories:

- Density contrast:  $\mathbf{x}_T(a, \mathbf{q}) = \mathbf{q} + \mathbf{L}(a, \mathbf{q})$

- For  $\mathbf{k} \cdot \mathbf{L} \ll 1$ , 
$$\delta_{\mathbf{k}}(a) = \int d^3 \mathbf{q} e^{-i\mathbf{k} \cdot \mathbf{q} - i\mathbf{k} \cdot \mathbf{L}(a, \mathbf{q})} - (2\pi)^3 \delta_{\text{Dirac}}[\mathbf{k}]$$

$$\delta_{\mathbf{k}}(a) \cong - \int d^3 \mathbf{q} e^{-i\mathbf{k} \cdot \mathbf{q}} (i\mathbf{k} \cdot \mathbf{L}(a, \mathbf{q})) = - \int d^3 \mathbf{q} e^{-i\mathbf{k} \cdot \mathbf{q}} (\nabla_{\mathbf{q}} \cdot \mathbf{L})$$

so that

$$\nabla \cdot \mathbf{L}(a, \mathbf{q}) = -\delta(a, \mathbf{q}) = -\frac{2}{3} H_0^{-2} a \nabla \cdot (\nabla \phi)$$

giving

$$\mathbf{L}(a, \mathbf{q}) = -(\nabla \psi) a \equiv a \mathbf{u}(\mathbf{q}); \quad \psi \equiv (2/3) H_0^{-2} \phi$$

## ZELDOVICH APPROXIMATION

- Use the prescription:

$$\mathbf{x}_T(a, \mathbf{q}) = \mathbf{q} + a\mathbf{u}(\mathbf{q}); \quad \mathbf{r}(t) \equiv a(t)\mathbf{x}(t) = a(t)[\mathbf{q} + a(t)\mathbf{u}(\mathbf{q})]$$

- Density:

$$\rho(\mathbf{r}, t) = \bar{\rho} \left[ \det \left( \frac{\partial q_i}{\partial r_j} \right) \right]^{-1} = \frac{\bar{\rho}/a^3}{\det(\partial x_j / \partial q_i)} = \frac{\rho_b(t)}{\det(\delta_{ij} + a(t)(\partial u_j / \partial q_i))}$$

$$\rho(\mathbf{r}, t) = \frac{\rho_b(t)}{(1 - a(t)\lambda_1(\mathbf{q}))(1 - a(t)\lambda_2(\mathbf{q}))(1 - a(t)\lambda_3(\mathbf{q}))}$$

- First non linear structures to form are two dimensional pancakes.
- Caustic structure is similar to that in optics, gravitational lensing etc..

## Application 5: A non linear tail at small wavenumber

- Particles obeying ZA has:

$$\mathbf{x}_T(\mathbf{q}, a) = \mathbf{q} + a \nabla \psi; \quad \dot{\mathbf{x}}_T = \left( \frac{2a}{3t} \right) \nabla \psi; \quad \psi = \frac{2}{3H_0^2} \varphi$$

giving

$$\int d^3 \mathbf{q} (\mathbf{k} \cdot \dot{\mathbf{x}}_T)^2 e^{-i\mathbf{k} \cdot (\mathbf{q} + \mathbf{L})} \cong \int d^3 \mathbf{q} (\mathbf{k} \cdot \dot{\mathbf{x}}_T)^2 e^{-i\mathbf{k} \cdot \mathbf{q}}$$

- This leads to a closed equation:

$$\ddot{\phi}_{\mathbf{k}} + 4 \frac{\dot{a}}{a} \dot{\phi}_{\mathbf{k}} = -\frac{1}{3a^2} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k} + \mathbf{p}} \phi_{\frac{1}{2}\mathbf{k} - \mathbf{p}} \mathcal{G}(\mathbf{k}, \mathbf{p});$$
$$\mathcal{G}(\mathbf{k}, \mathbf{p}) = \frac{7}{8} k^2 + \frac{3}{2} p^2 - 5 \left( \frac{\mathbf{k} \cdot \mathbf{p}}{k} \right)^2$$

## PERTURBATIVE SOLUTION

### non linear tail at small wavenumber

- $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^{(1)} + \phi_{\mathbf{k}}^{(2)} + \dots$  where  $\phi_{\mathbf{k}}^{(1)} = \phi_{\mathbf{k}}^{(L)}$
- This gives

$$H_0^2 \left( a \frac{d^2}{da^2} + \frac{7}{2} \frac{d}{da} \right) \phi_{\mathbf{k}}^{(2)} \approx -\frac{2}{3} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}}^L \phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}}^L \mathcal{G}(\mathbf{k}, \mathbf{p})$$

$$\mathcal{G}(\mathbf{k}, \mathbf{p}) = \left[ \left( \frac{k}{2} \right)^2 - \left( \frac{\mathbf{k} \cdot \mathbf{p}}{k} \right)^2 \right]$$

which is solved by:

$$\phi_{\mathbf{k}}^{(2)} \cong - \left( \frac{4a}{21H_0^2} \right) \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}}^L \phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}}^L \mathcal{G}(\mathbf{k}, \mathbf{p})$$

- Note that:

$$\phi_{\mathbf{k} \simeq 0}^{(2)} \approx \frac{2a}{63\pi^2 H_0^2} \int_0^\infty dp p^2 |\phi_{\mathbf{p}}^{(L)}|^2$$

- Corresponds to the  $k^4$  tail in density power spectrum:  
 $P_\delta(k) \propto a^2 k^4 P_\varphi(k) \propto a^4 k^4$ .

## Toy Example: Generation of small scale power

- Inject power at  $k_0$ :

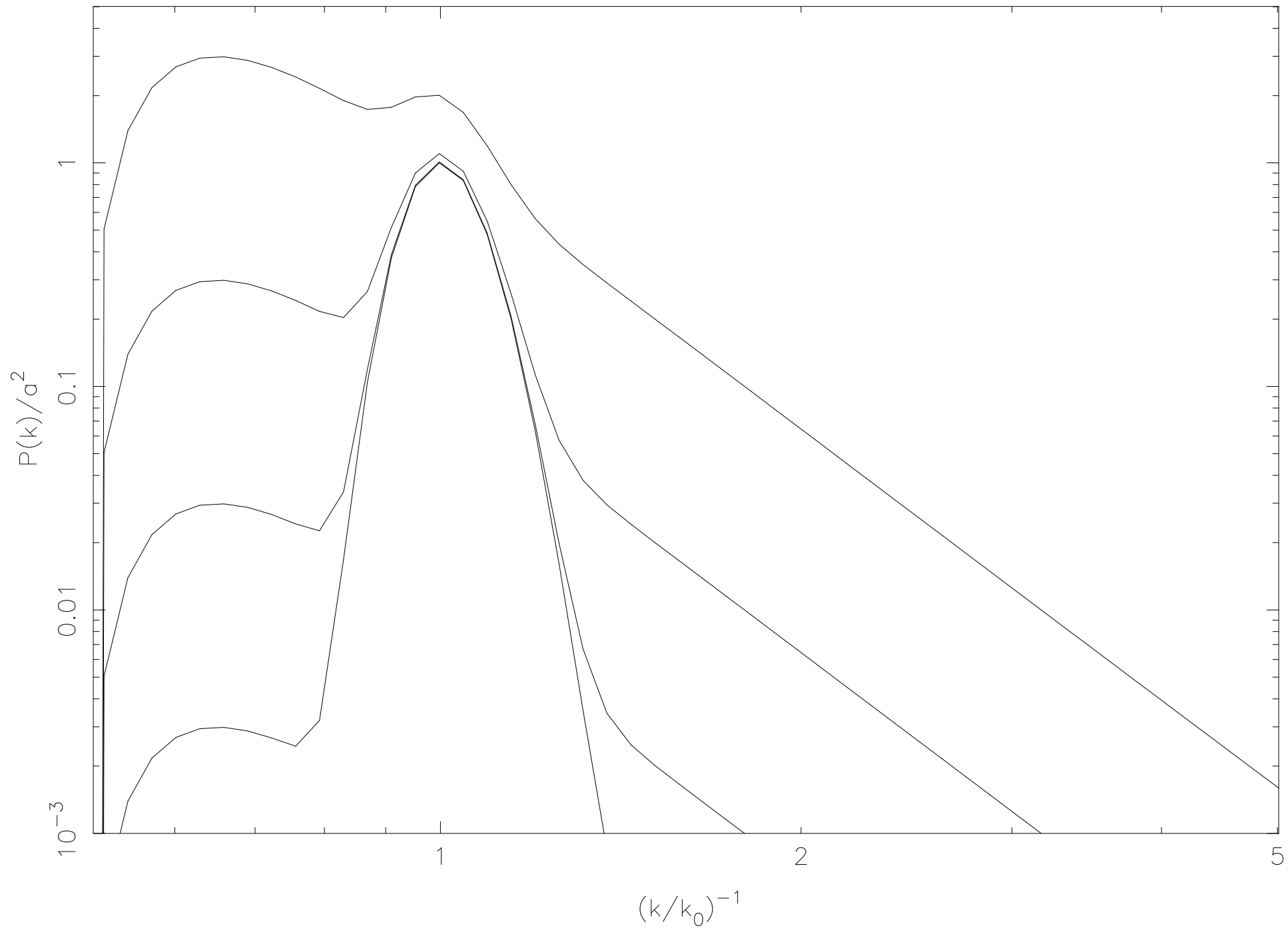
$$\phi_{\mathbf{k}}^L = \mu \frac{H_0^2}{k_0^4} \delta_D[|\mathbf{k}| - k_0]; \quad k_0^3 \delta_{\mathbf{k}}^L = -\frac{2}{3}(\mu a) k_0 \delta_D[|\mathbf{k}| - k_0]$$

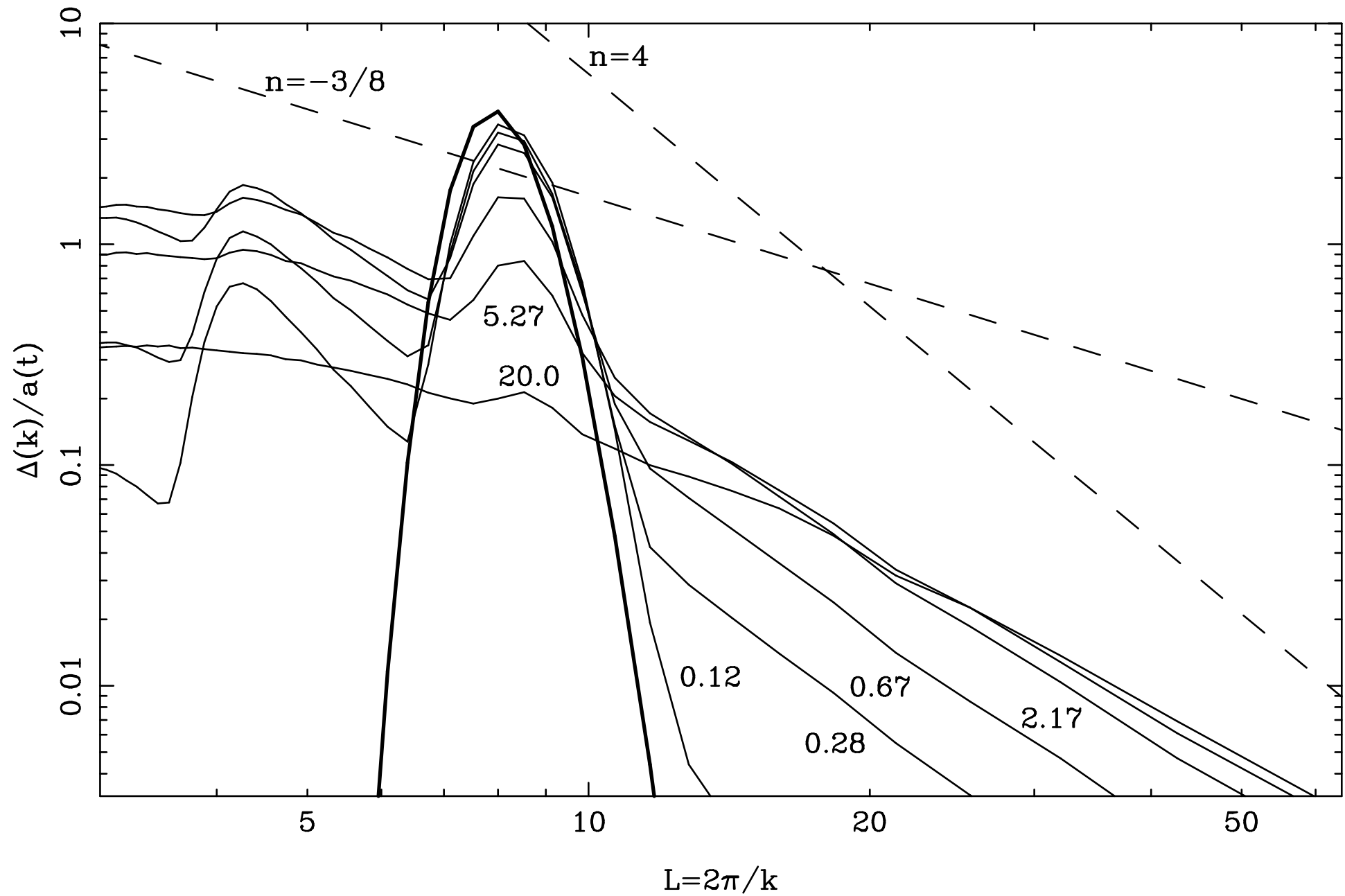
- The coupling term is nonzero only when the *magnitudes* of both the vectors  $[(1/2)\mathbf{k} + \mathbf{p}]$  and  $[(1/2)\mathbf{k} - \mathbf{p}]$  are  $k_0$ . This requires  $\mathbf{k} \cdot \mathbf{p} = 0$ ,  $(k/2)^2 + p^2 = k_0^2$ .
- Solving, we get:

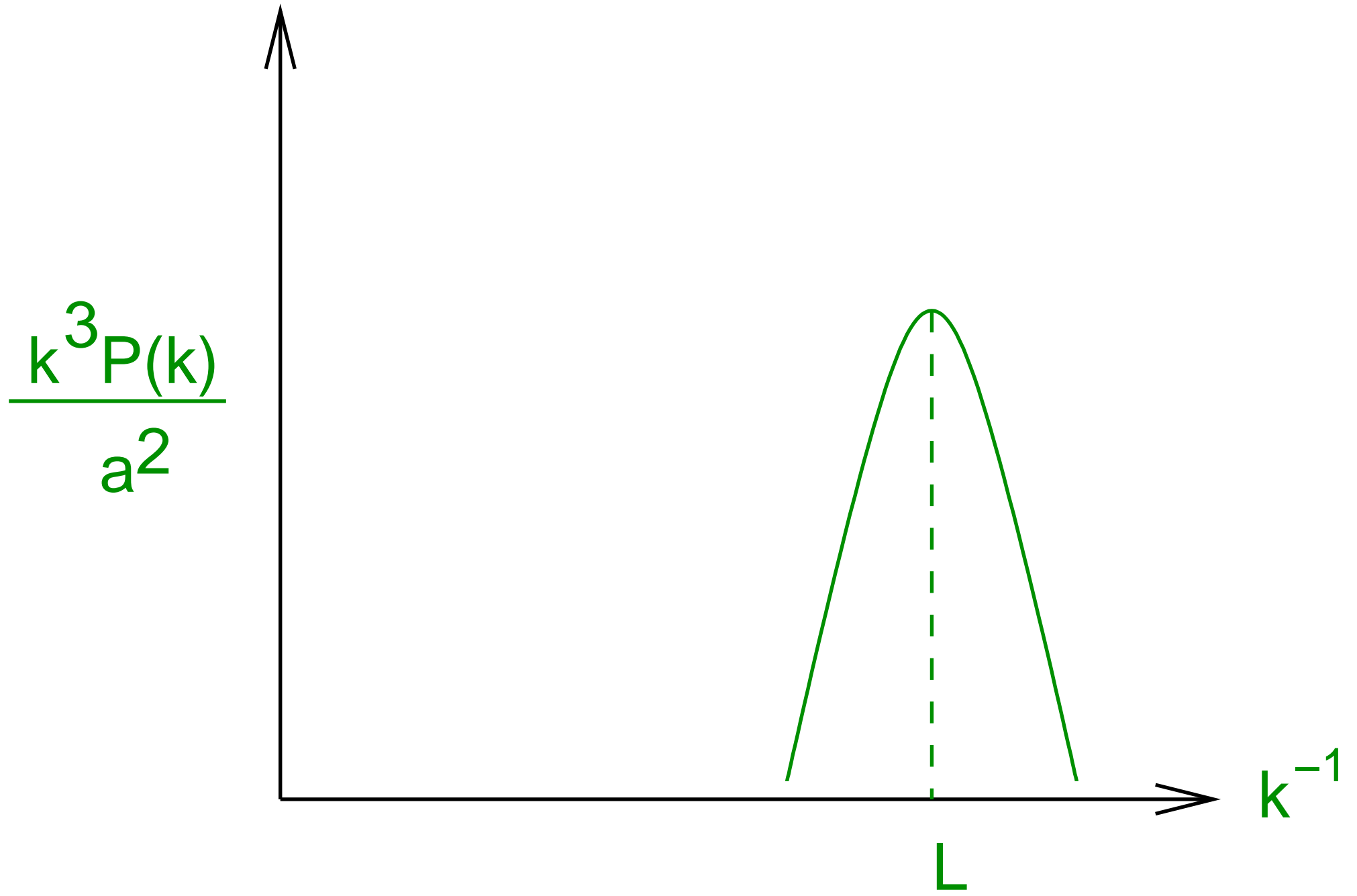
$$\phi_{\mathbf{k}}^{(2)} \propto a \left(1 - \frac{k^2}{4k_0^2}\right) \left(1 + \frac{k^2}{3k_0^2}\right)$$

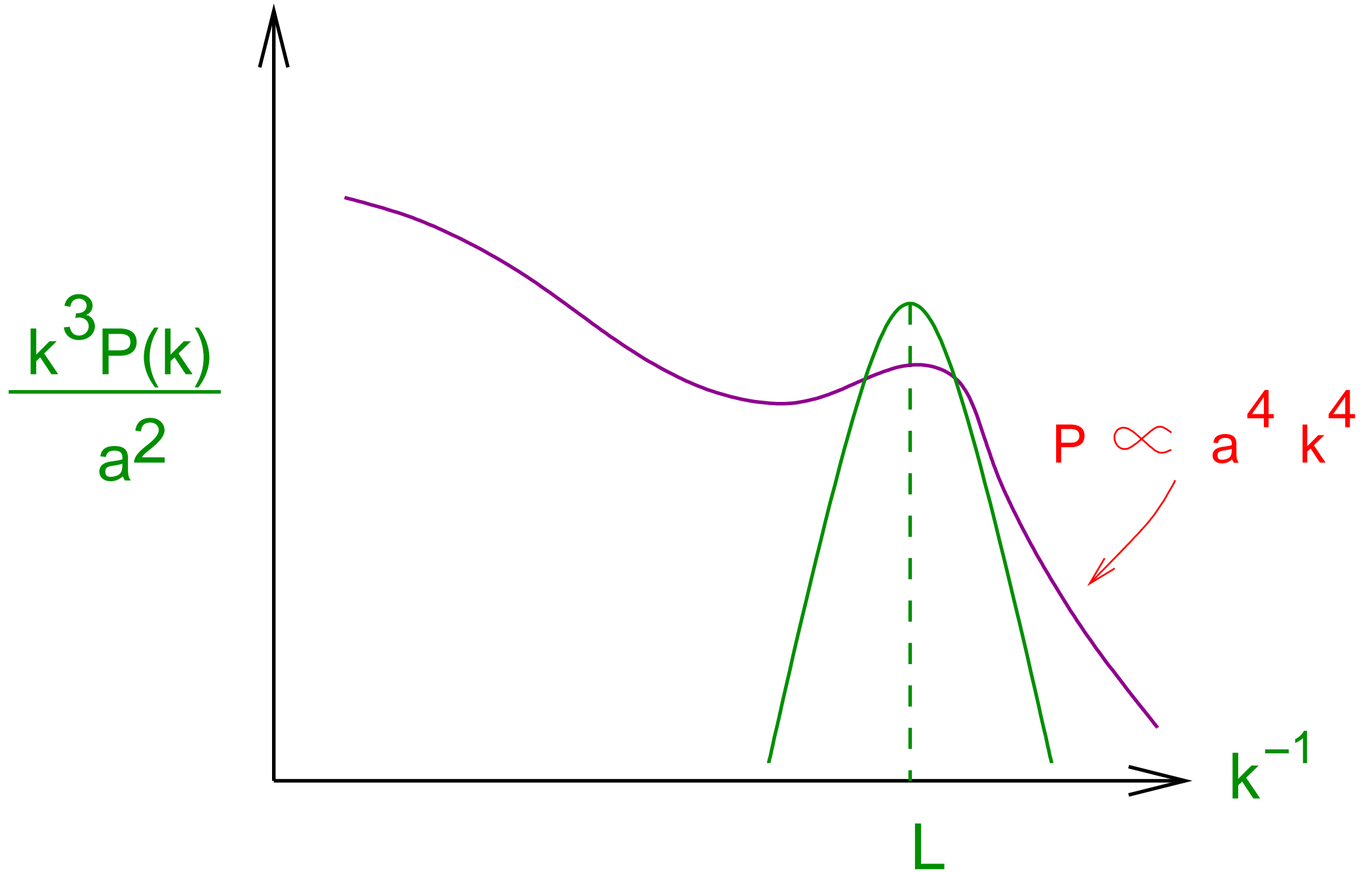
with

$$P^{(2)}(k) \propto (a)^4 q^4 \left(1 - \frac{1}{4}q^2\right)^2 \left(1 + \frac{1}{3}q^2\right)^2; \quad q = \frac{k}{k_0}$$









## ENERGY TRANSFER BETWEEN THE SCALES

- The effective Lagrangian for the system of the particles is

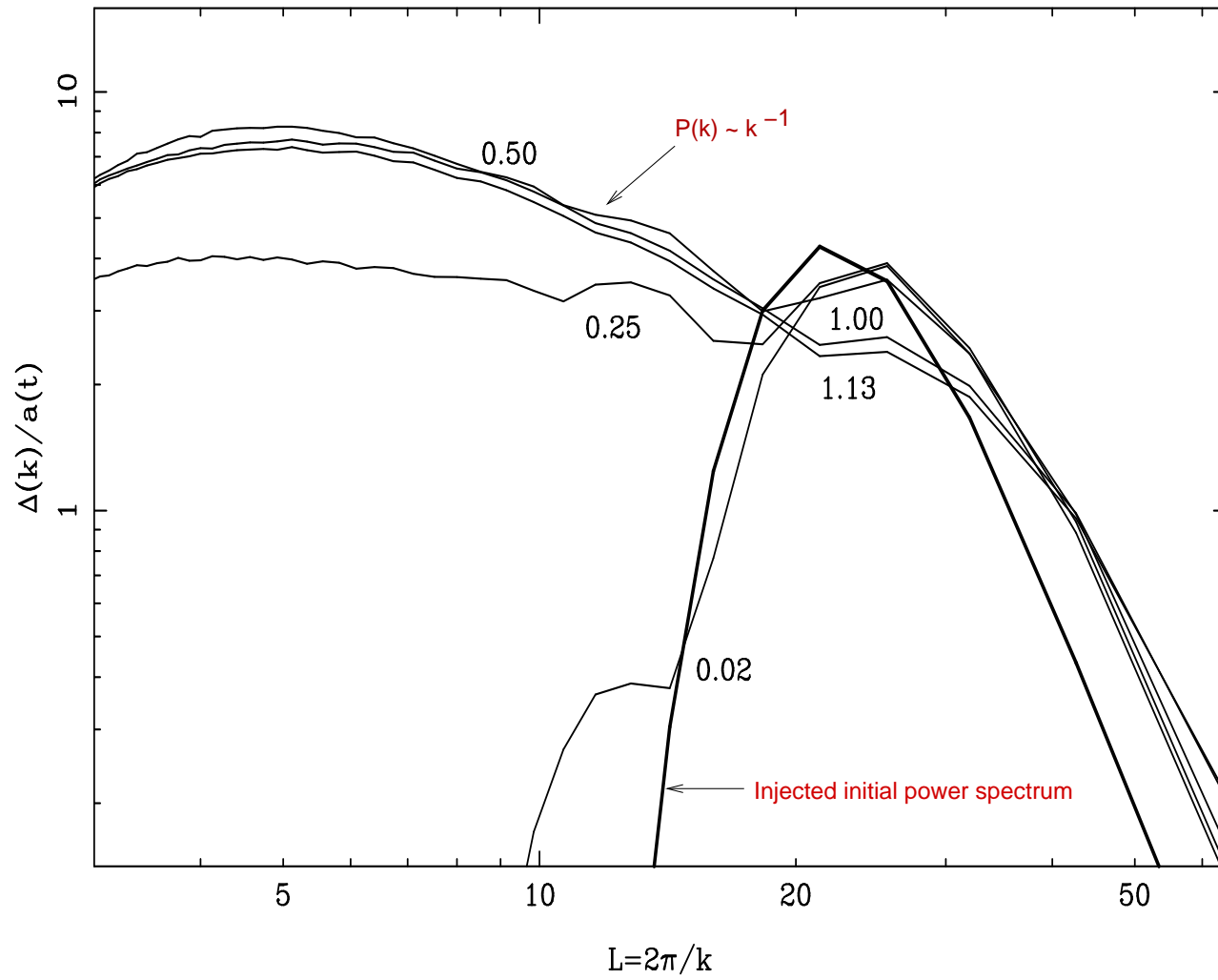
$$L = \sum_i \frac{1}{2} m a^2 \dot{\mathbf{x}}_i^2 - \frac{1}{2} G \rho_0^2 \int d^3 \mathbf{x} d^3 \mathbf{x}' \frac{\delta(a, \mathbf{x}) \delta(a, \mathbf{x}')}{a |\mathbf{x} - \mathbf{x}'|} = T - U$$

- Getting the  $H(\mathbf{p}, \mathbf{x})$  and using  $dH/dt = \partial H/\partial t$ , we get the flow of energy between scales:

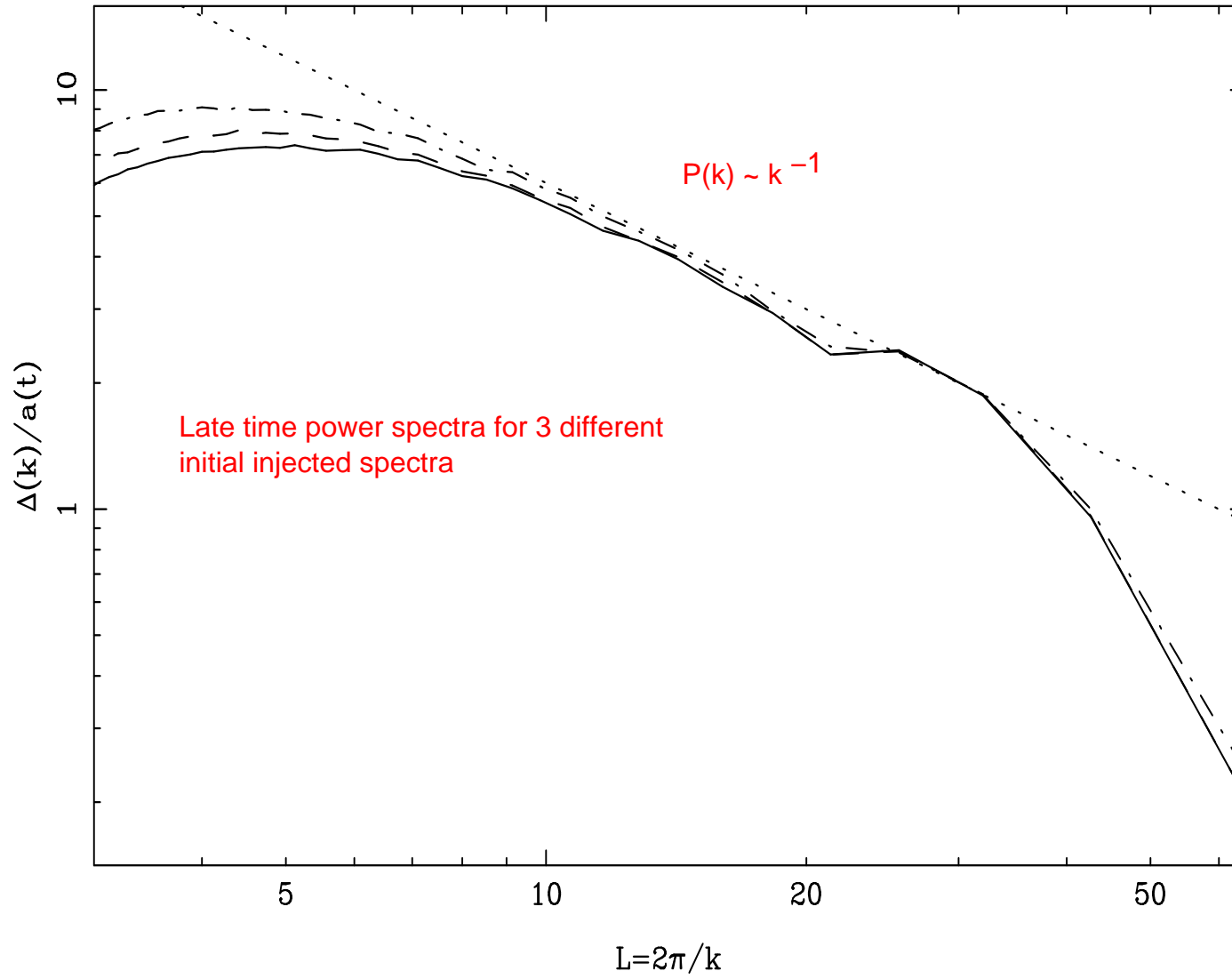
$$\frac{d}{da} a^2 E(a, x) = -2\pi G \rho_0^2 [x \xi(a, x)];$$

$$\frac{d}{da} a^2 E_{\mathbf{k}}(a) = -\frac{G \rho_0^2}{\pi} [P(a, k)]$$

- We can use Nonlinear Scaling Relations to determine the “flow” of  $\xi(a, x)$  and thus determine the flow of  $E(a, x)$ .



[Bagla, T.P, (1997) MNRAS, 286, 1023]



[Bagla, T.P, (1997) MNRAS, 286, 1023]

## NON LINEAR SCALING RELATION

- Define:

$$\bar{\xi}(t, x) = \frac{3}{x^3} \int_0^x \xi(t, y) y^2 dy$$

- Mean number of neighbours:

$$N(x, t) = (na^3) \int_0^x 4\pi y^2 dy [1 + \xi(y, t)]$$

- Conservation of pairs:

$$\frac{\partial \xi}{\partial t} + \frac{1}{ax^2} \frac{\partial}{\partial x} [x^2 (1 + \xi)v] = 0$$

- This gives:

$$\frac{1}{3x^2} \frac{\partial}{\partial x} \left[ x^3 \frac{\partial}{\partial t} (1 + \bar{\xi}) \right] = - \frac{1}{ax^2} \frac{\partial}{\partial x} \left[ \frac{v}{3} \frac{\partial}{\partial x} [x^2 (1 + \bar{\xi})] \right].$$

or:

$$x^3 \frac{\partial}{\partial t} (1 + \bar{\xi}) = - \frac{v}{a} \frac{\partial}{\partial x} [x^3 (1 + \bar{\xi})].$$

- Defining  $h(a, x) = -(v/\dot{a}x)$ , we get:

$$\left( \frac{\partial}{\partial \ln a} - h \frac{\partial}{\partial \ln x} \right) (1 + \bar{\xi}) = 3h (1 + \bar{\xi}).$$

- Assume that  $h(a, x) = h[\bar{\xi}(a, x)]$ . Then the solution is:

$$\bar{\xi}_L(a, l) = \exp \left( \frac{2}{3} \int^{\bar{\xi}(a, x)} \frac{d\mu}{h(\mu)(1 + \mu)} \right); \quad l = x[1 + \bar{\xi}(a, x)]^{1/3}.$$

- It follows that  $\xi_{\text{true}}(a, x)$  is a universal function of  $\xi_L(a, l)$  with  $l = x[1 + \bar{\xi}(a, x)]^{1/3}$ .

## NSR: INTUITIVE DERIVATION OF $\xi(a, x) = U[\xi_L(a, l)]$

- Let  $f(x) \propto x^{-\epsilon}$  be the density profile of individual structures.  
Then  $\xi \sim f \propto x^{-\epsilon}$  in the quasi linear regime while  $\xi \sim f^2 x^3 \propto x^{-2\epsilon+3}$  in the nonlinear regime.
- Evolution must preserve isothermal structures with  $\epsilon = 2$ ;  $\xi_{\text{QL}} \propto x^{-2}$ ;  $\xi_{\text{NL}} \propto x^{-1}$ .
- $\xi_{\text{true}}(x) = U[\xi_L(l)]$  must give  $\xi_{\text{QL}} \propto a^2 x^{-2} = U_{\text{QL}}[a^2 l^{-2}]$  and  $\xi_{\text{NL}} \propto a^2 x^{-1} = U_{\text{NL}}[a^2 l^{-1}]$ ; with  $l = x\xi^{1/3}$ .
- This requires  $U_{\text{QL}}(z) \propto z^3$ ;  $U_{\text{NL}} \propto z^{3/2}$ .
- In 3D:

$$\bar{\xi}(a, x) = \begin{cases} \bar{\xi}_L(a, l) & (\text{for } \bar{\xi}_L < 1, \bar{\xi} < 1) \\ \bar{\xi}_L(a, l)^3 & (\text{for } 1 < \bar{\xi}_L < 5.85, 1 < \bar{\xi} < 200) \\ 14.14 \bar{\xi}_L(a, l)^{3/2} & (\text{for } 5.85 < \bar{\xi}_L, 200 < \bar{\xi}) \end{cases}$$

- Aside: Power injected at a scale cascades to smaller scales leading to

“universal” correlation function:

$$\xi(x) = \frac{1}{x(x + x_c)}$$

## MEMORY OF INITIAL CONDITIONS

- The non linear end with  $\xi_{\text{NL}} \propto \xi_L^{3/2}$  corresponds to stable clustering.
- Stable clustering:  $v = -\dot{a}x$ ;  $r = ax = \text{constant} \Rightarrow$

$$\xi_{\text{NL}}(a, x) \propto \xi_L^{3/2}(a, l); \quad l = a\xi_{\text{NL}}^{1/3}$$

- If  $\xi_L \propto a^2 l^{-(n+3)}$ , then

$$\xi_{\text{NL}} \propto a^{\frac{6}{n+5}} x^{-\frac{3(n+3)}{n+5}}$$

which depends on  $n$ .

- If  $(v/\dot{a}x) = -h$  is a constant asymptotically,

$$\bar{\xi}(a, x) \propto a^{\frac{2\gamma}{n+3}} x^{-\gamma}; \quad \gamma = \frac{3h(n+3)}{2+h(n+3)}$$

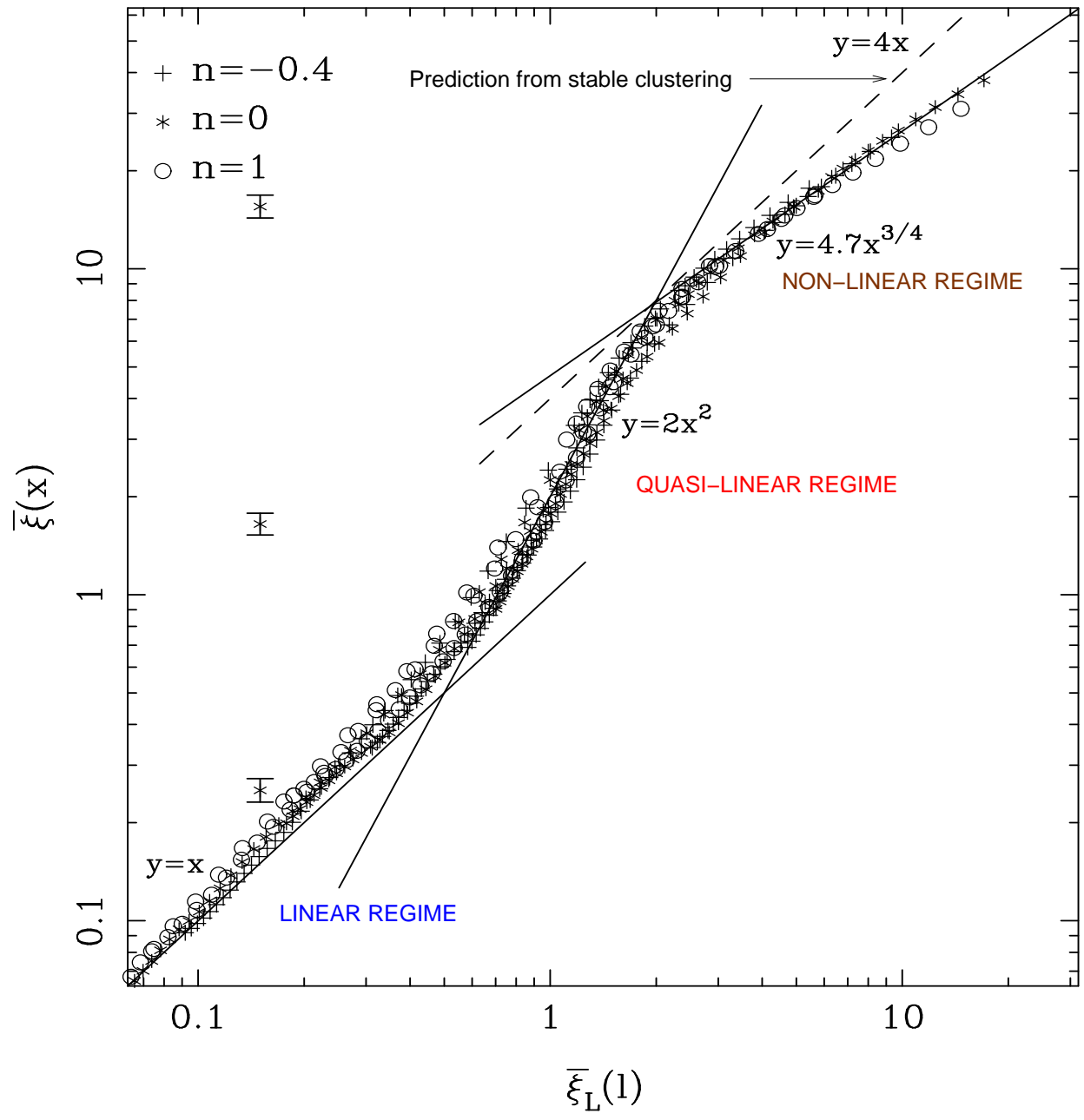
- Asymptotic structure is independent of  $n$  if  $h(n+3) = \text{constant}$ .

- In general,  $\bar{\xi} \propto \xi_L^D$  in the quasi-linear regime and  $\bar{\xi} \propto \xi_L^{Dh/2}$  in the nonlinear regime where  $D =$  the dimension of space;  $h =$  asymptotic value of  $(v/\dot{a}x)$
- In 2D: ( $h \approx 3/4$ )

$$\bar{\xi}(a, x) \propto \begin{cases} \bar{\xi}_L(a, l) & \text{(Linear)} \\ \bar{\xi}_L(a, l)^2 & \text{(Quasi-linear)} \\ \bar{\xi}_L(a, l)^{3/4} & \text{(Nonlinear)} \end{cases}$$

- This gives

$$\bar{\xi}_{NL}(a, x) = a^{2-\gamma} x^{-\gamma}; \quad \gamma = \frac{2h(n+2)}{2+h(n+2)}$$



[Bagla, Engineer, TP (1998), Ap.J., 495, 25]

## HALO STRUCTURE

- Density due to halo profiles:

$$\rho(a, \mathbf{x}) = \sum_i f(\mathbf{x} - \mathbf{x}_i, a)$$

- Power spectrum:

$$P(\mathbf{k}, a) = (a^3 |f(\mathbf{k}, a)|)^2 \left| \sum_i \exp -i\mathbf{k} \cdot \mathbf{x}_i(a) \right|^2 = (a^3 |f(\mathbf{k}, a)|)^2 P_{\text{cent}}(\mathbf{k}, a)$$

- Let:  $f(x) \propto x^{-\epsilon}$ ;  $\xi \propto x^{-\gamma}$ ;  $\xi_c \propto x^{-\gamma_c}$ . Then

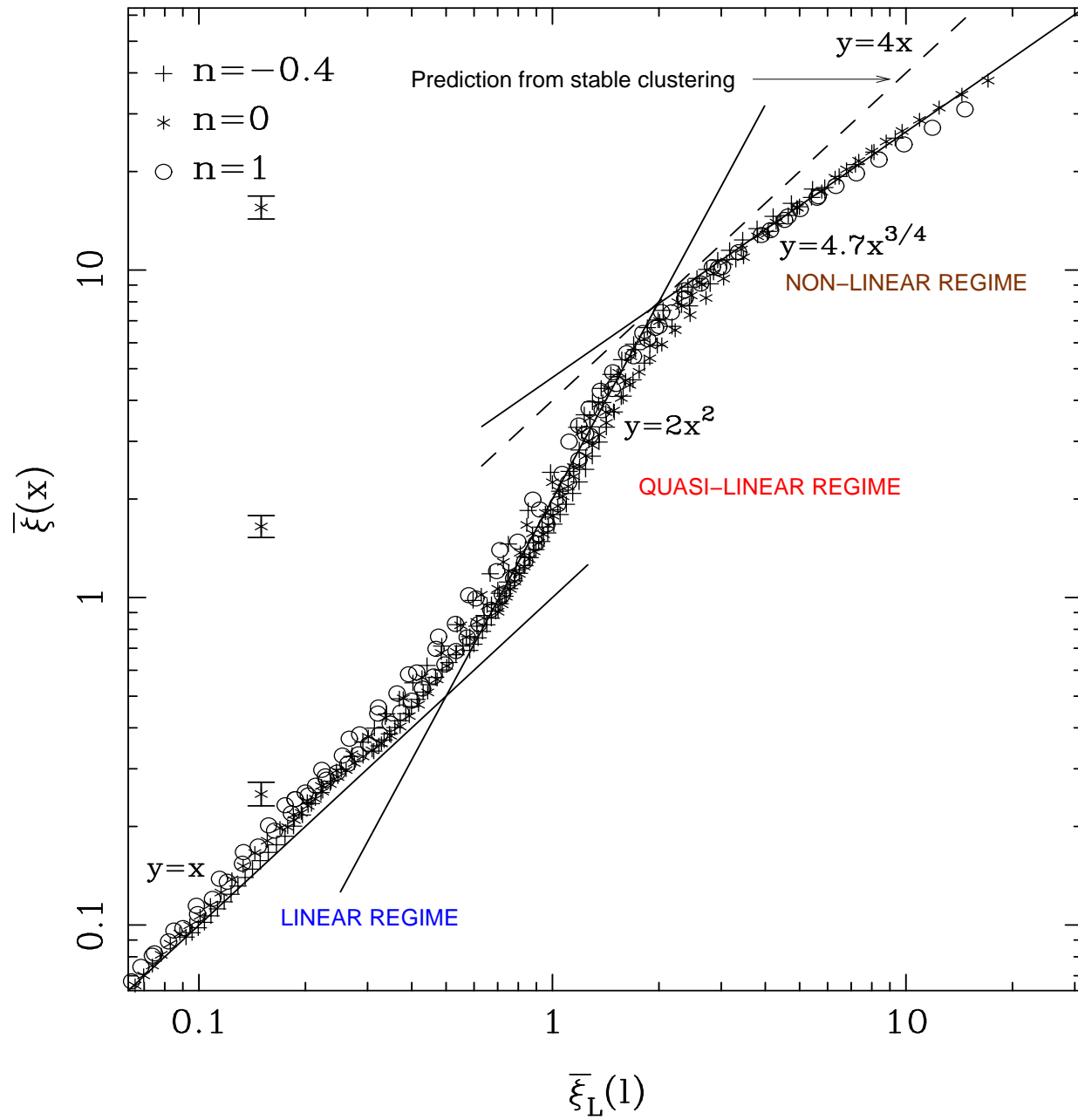
$$\epsilon = 3 + \frac{1}{2}(\gamma - \gamma_c)$$

- If  $P_{\text{cent}} \sim P_\phi$  at non linear scales, then  $\gamma = \gamma_c - 4$ ;  $\epsilon = 3$

- If  $P_{\text{cent}} \sim P$  at quasi linear scales, then  $\gamma = \gamma_c$ ;  $\epsilon = 3$

- This suggests:

$$f(x) \propto \frac{1}{x(x+l)^2}$$



## MEMORY OF INITIAL CONDITIONS

- The non linear end with  $\xi_{\text{NL}} \propto \xi_L^{3/2}$  corresponds to stable clustering.
- Stable clustering:  $v = -\dot{a}x$ ;  $r = ax = \text{constant} \Rightarrow$

$$\xi_{\text{NL}}(a, x) \propto \xi_L^{3/2}(a, l); \quad l = a\xi_{\text{NL}}^{1/3}$$

- If  $\xi_L \propto a^2 l^{-(n+3)}$ , then

$$\xi_{\text{NL}} \propto a^{\frac{6}{n+5}} x^{-\frac{3(n+3)}{n+5}}$$

which depends on  $n$ .

- If  $(v/\dot{a}x) = -h$  is a constant asymptotically,

$$\bar{\xi}(a, x) \propto a^{\frac{2\gamma}{n+3}} x^{-\gamma}; \quad \gamma = \frac{3h(n+3)}{2+h(n+3)}$$

- Asymptotic structure is independent of  $n$  if  $h(n+3) = \text{constant}$ . Does not seem to be true.
- Is there any other universal feature at the asymptotic region ?

## 'EQUIPARTITION' OF ENERGY FLOW

- If the local power law index is  $n$ , then one can show that in the linear regime:

$$E_L(a, x) \approx ax^{-(n+2)}$$

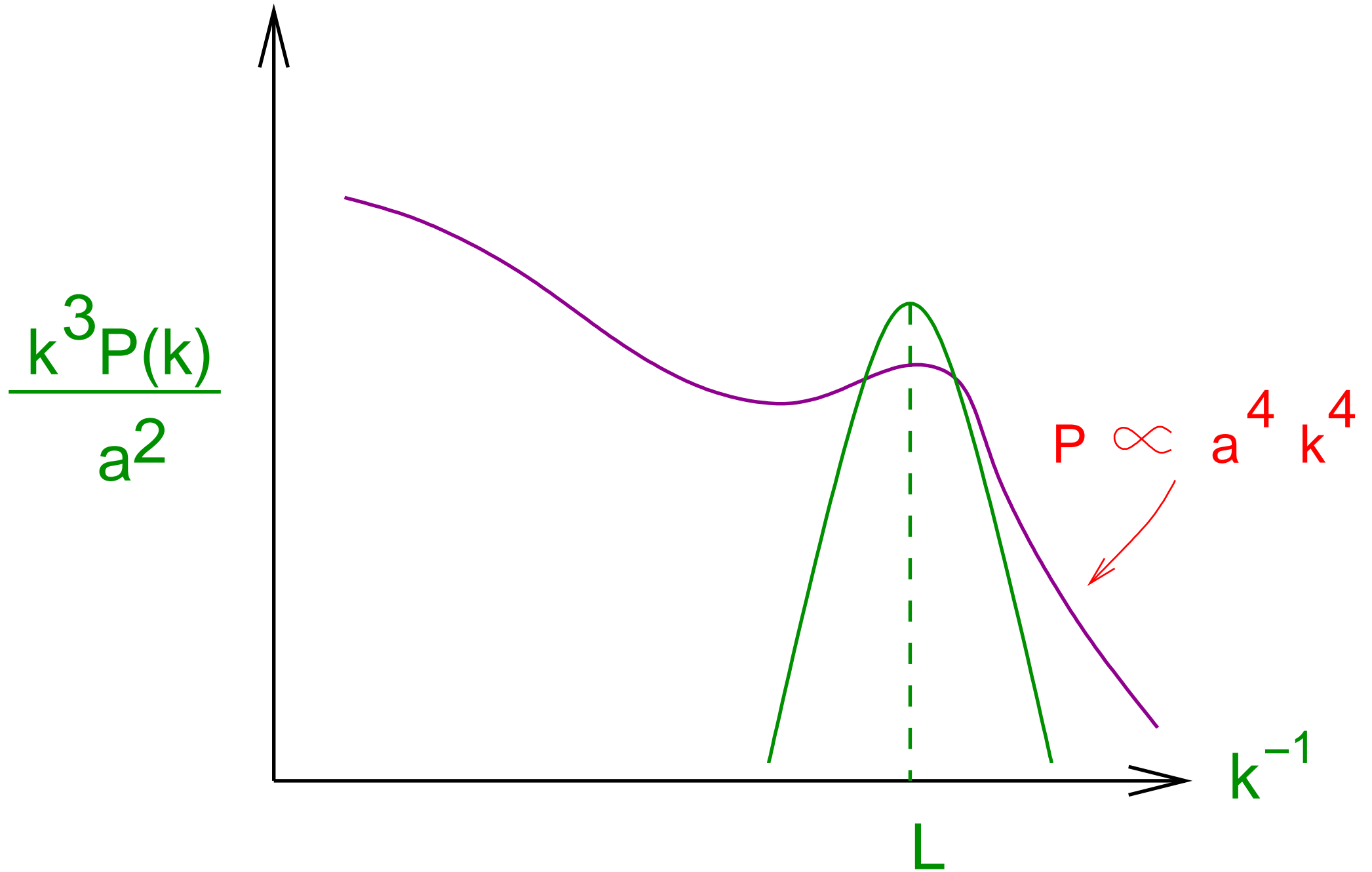
- In the quasilinear regime:

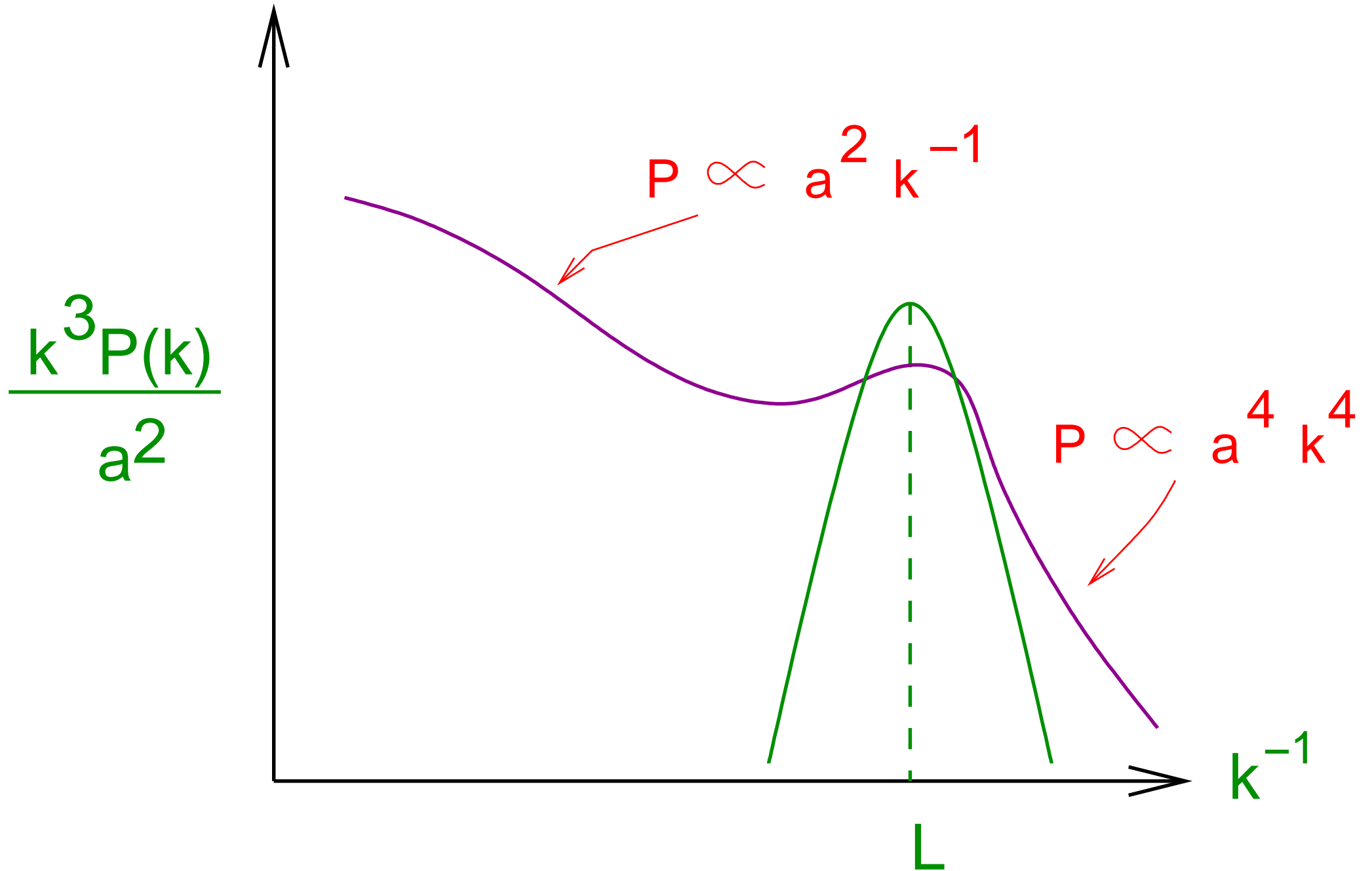
$$E_{QL}(a, x) \approx a^{\frac{2-n}{n+4}} x^{-\frac{2n+5}{n+4}};$$

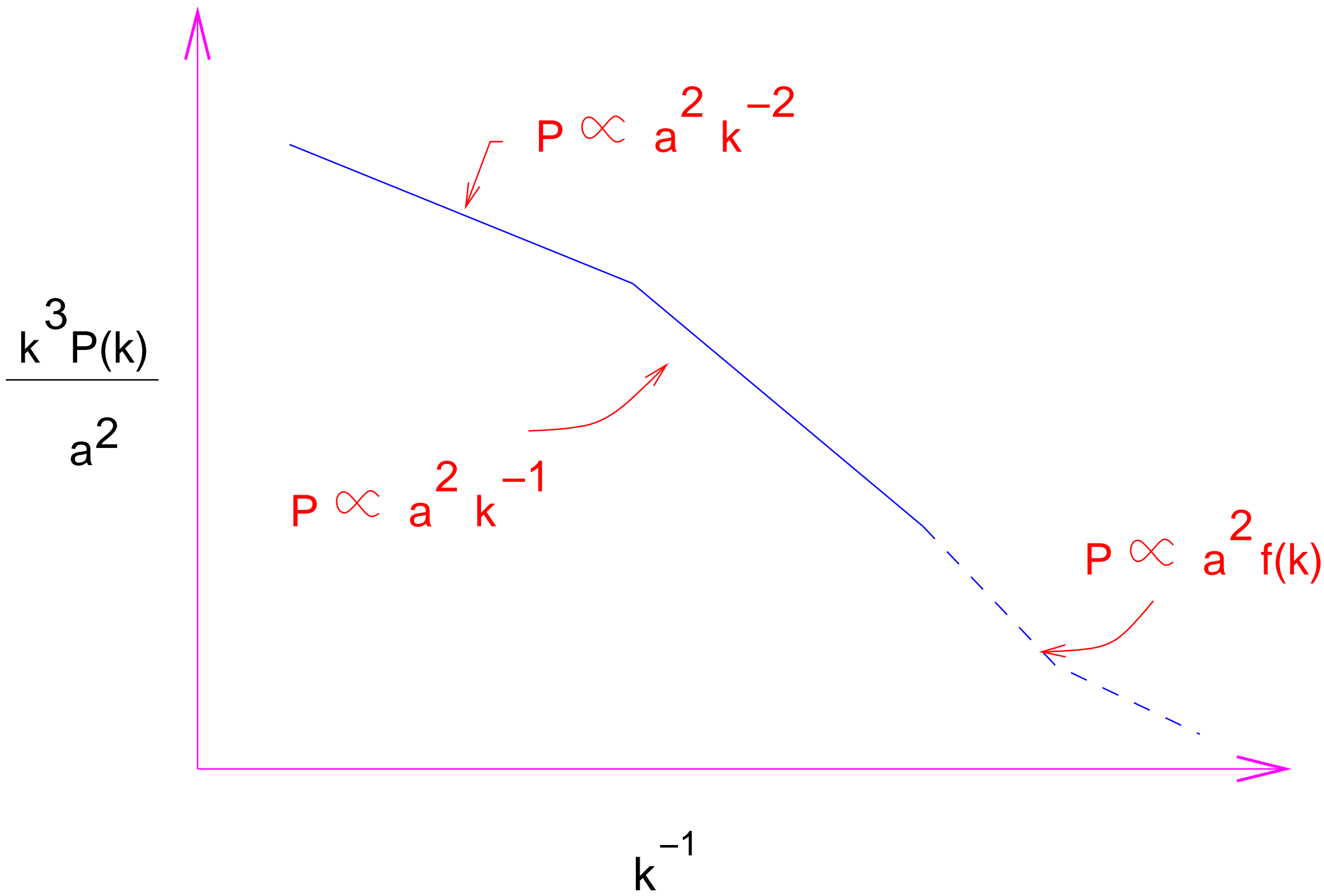
- In the nonlinear regime:

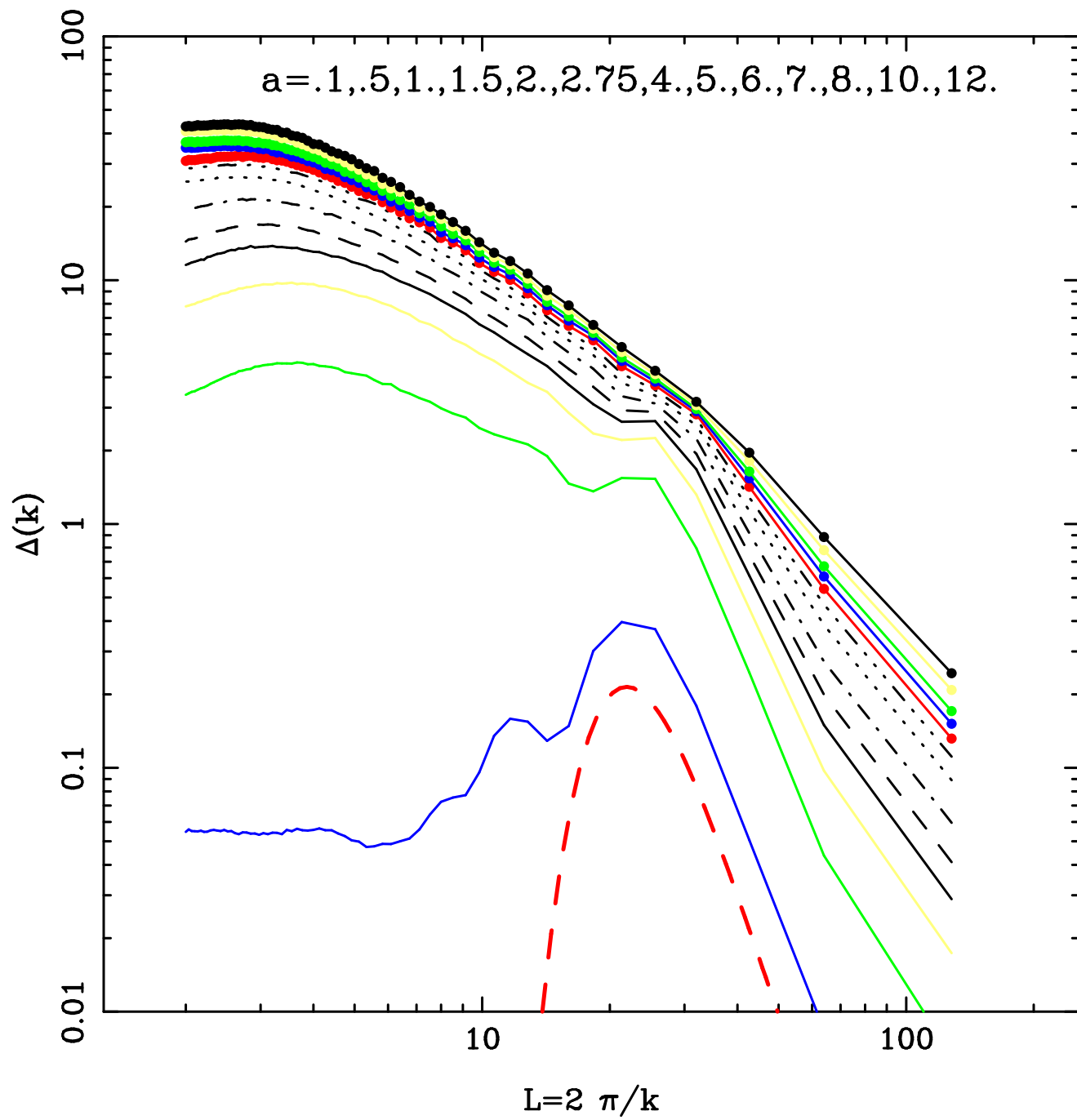
$$E_{NL}(a, x) \approx a^{\frac{1-n}{n+5}} x^{-\frac{2n+4}{n+5}};$$

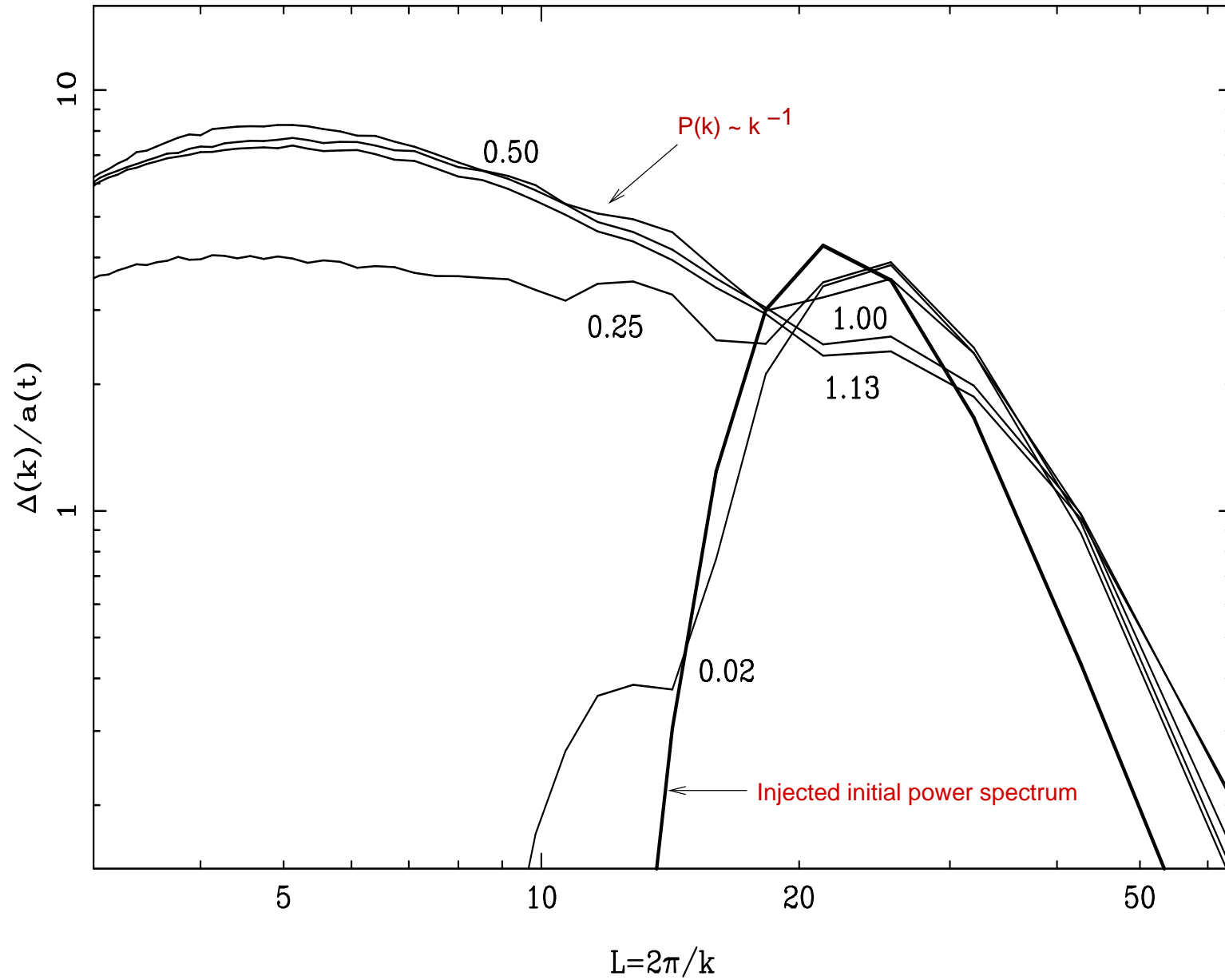
- NEW FEATURE: The energy flow is form invariant (“equipartition”) when  $n = -1$  in QL and  $n = -2$  in the NL regimes! Then  $E \propto a$  in all three regimes.



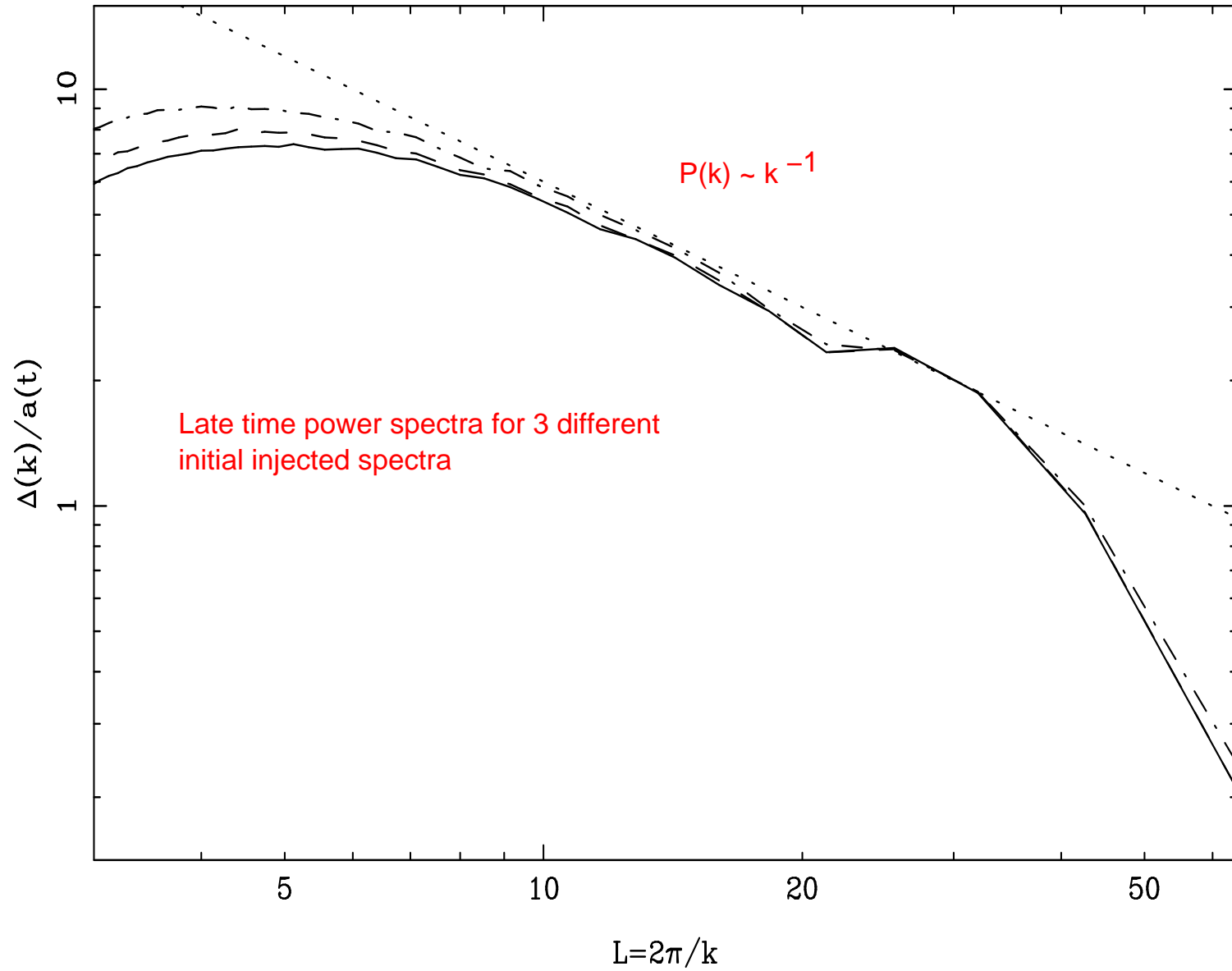




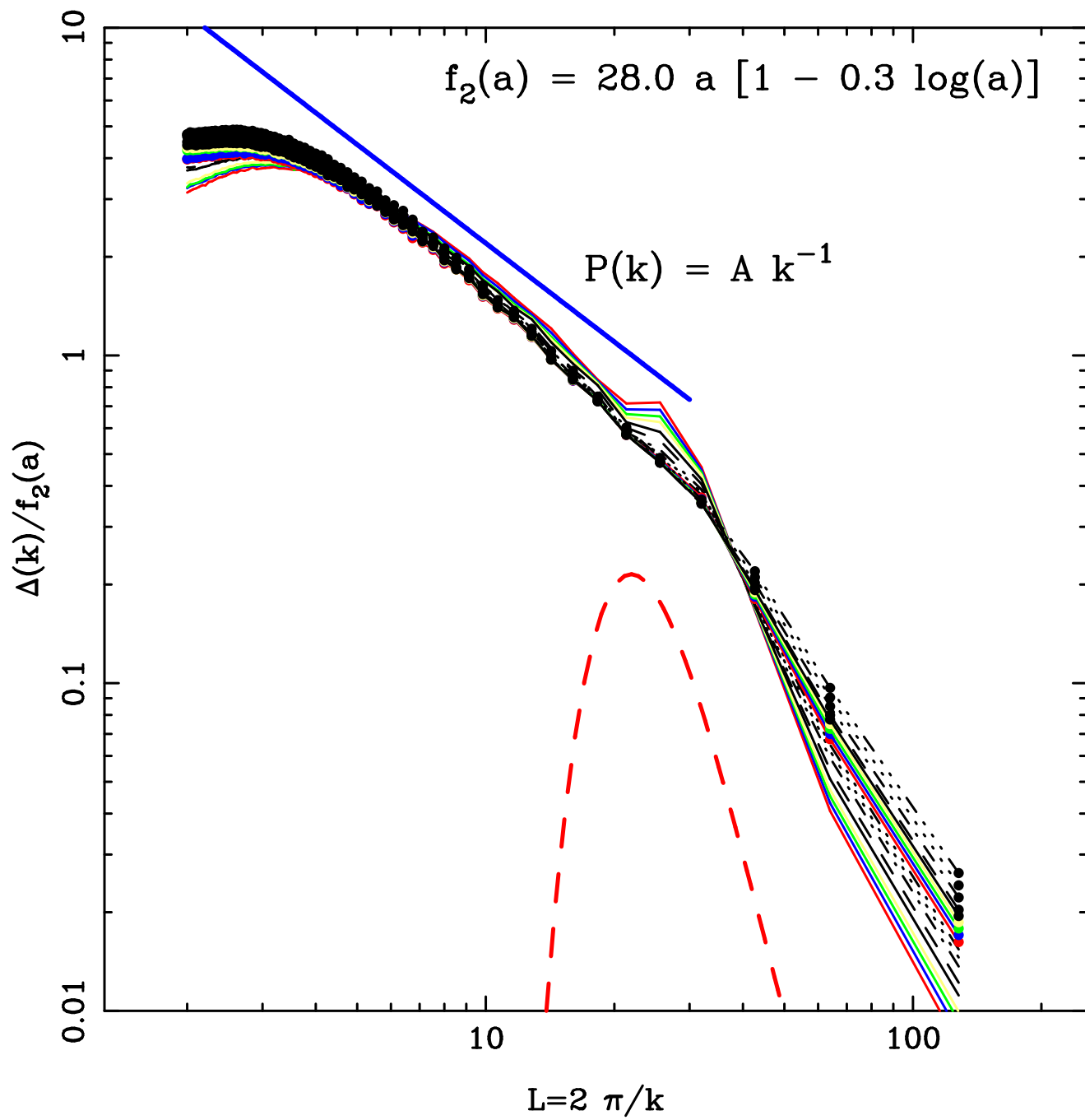




Bagla, T.P, (1997) MNRAS, 286, 1023



Bagla, T.P, (1997) MNRAS, 286, 1023



## POSSIBLE INTERPRETATION ?

- When  $E \propto a$ , we also have  $E_{QL} \propto a/x$  and  $E_{NL}$  = independent of  $x$  in the nonlinear regime.
- Taking  $E \propto GM(x)/x \propto \rho(x)x^2$ , we find that

$$\rho_{QL} \propto \frac{a}{x^3}; \quad \rho_{NL} \propto \frac{a}{x^2}$$

These are the limiting form of

$$\rho(x) = \frac{1}{x^2(l+x)}$$

This is similar – but not quite – the NFW density profile.

## CONCLUSION

- A general formalism to tackle nonlinear gravitational clustering is possible but needs more work.
- If power is injected into a given scale it flows to smaller and larger scales. The large scale  $k^4$  tail is well-understood.
- At small scales there is universality reminiscent of Kolmogorov spectrum in turbulence. Can be understood in terms of equipartition. Initial conditions are not remembered.
- Broad band initial power behaves very differently. Can be understood from NSR (but we still need to obtain NSR from fundamental physics!)
- The initial conditions are remembered in the broad band case. (May be halo structures are still universal but that needs to be demonstrated.)

# GRAVITATIONAL N-BODY PROBLEM

$$( N \text{ particles, mass } m, U(r) = - \frac{Gm^2}{r} )$$

Non cosmological;  
no bg. expansion

Cosmological;  
bg. expansion  $a(t)$

PHASE TRANSITIONS

RENORMALISATION  
GROUP

POWER  
TRANSFER

UNIVERSAL POWER SPECTRUM,  
NON-LINEAR SCALING RELATIONS

CONDENSED  
MATTER PHYSICS

FLUID  
TURBULENCE

