Collective Instabilities In Light-Matter Interactions

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Outline

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2. Collective interaction between Light and **free electrons**
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3. Collective interaction between Light and **cold atoms**
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4. Collective interaction between Light and **ultracold atoms**
   – Quantum CARL
   4.1 The effect of atom-atom interactions on CARL
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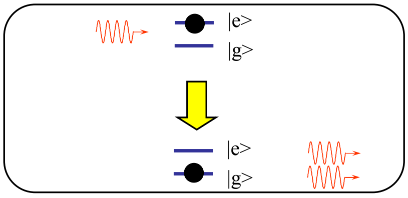
1. Introduction

The collective interactions we will look at can be considered as "free particle lasing" processes.

Generally speaking, a laser is a device which amplifies light

In "conventional" lasers, light is amplified at the expense of the **internal energy** of the atoms which make up the laser medium.



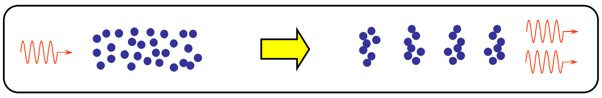
In conventional lasers, atomic motion plays no active role, even in gas lasers.

The subject of this seminar is a completely different type of laser where the laser medium is a gas of free particles.

Here light amplification is **not** due to changes in the internal energy of each atom/particle of the laser medium.

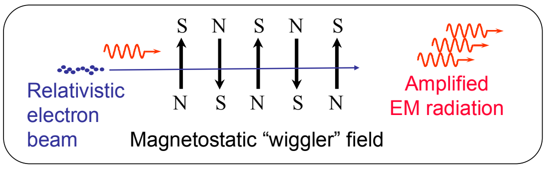
It is instead due to changes in each particle's **kinetic energy** or **momentum**.

We shall see that light amplification occurs due to the development of a periodic density modulation or **bunching** in the gas of particles.



We will investigate two cases :

- The **Free Electron Laser (FEL)**
  - light amplification in a beam of relativistic free electrons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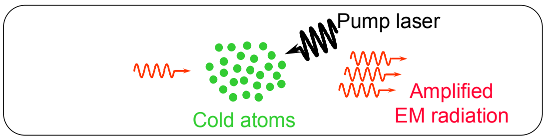
N S N S N

Relativistic electron beam

Magnetostatic “wiggler” field

Amplified EM radiation

- **Collective Atomic Recoil Lasing (CARL)**
  - light amplification in a gas of cold/ultracold atoms


Pump laser

Cold atoms

Amplified EM radiation

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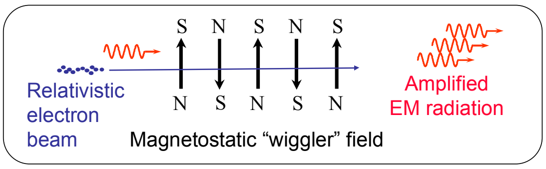
2. Collective Interactions Between Light and Free Electrons - the Free Electron Laser (FEL)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Relativistic electron beam

Magnetostatic “wiggler” field

Amplified EM radiation

Laser medium is a relativistic beam of electrons ($v \approx c$) moving through a spatially periodic magnetic field (wiggler).

Useful features of FELs :

- tunability – coherent light from microwaves through to hard X-rays (λ<1Å)
- very high power – medium cannot be damaged like laser crystals

Wide range of applications (spectroscopic, medical etc.)

2.1 FEL Radiation – Spontaneous Emission

An electron moving in a magnetic field experiences a force

$$\vec{F} = -e\vec{v} \times \vec{B}$$

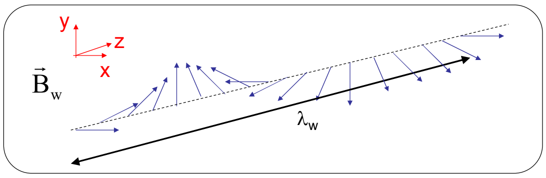
and consequently an acceleration.

An accelerated charge **radiates**.

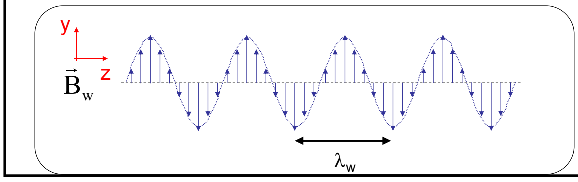
Here we will deduce some properties of the spontaneous emission by electrons moving through a wiggler magnet.

There are two different types of wigglers commonly used :

- **Helical wigglers** : wiggler field is circularly polarised



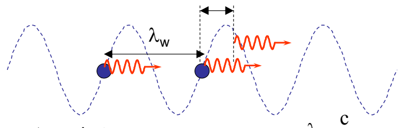
- **Planar wigglers** : wiggler field is linearly polarised



FEL Spontaneous Emission – Simple picture

Wiggler causes electron to oscillate with spatial period $\lambda_{w.}$

Consider a section of the electron trajectory in the y-z plane :



When electron travels $\lambda_{w,}$ light wave travels $\lambda_w \frac{c}{v_z}$

Constructive interference if $d = \lambda_w \left(\frac{c}{v_z} - 1 \right) = \lambda$

As $\frac{c}{v_z} - 1 \approx \frac{1}{2\gamma^2}$

then would expect strongest emission at $\boxed{\lambda \approx \frac{\lambda_w}{2\gamma^2}}$

As the beam energy is increased, the FEL emission moves to shorter wavelengths.

For λ_w=2cm :

For mildly relativistic beams ($\gamma \approx 3$) : λ ~ 1mm (microwaves)

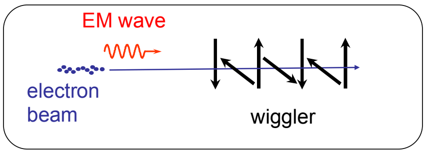
more relativistic beams ($\gamma \approx 30$) : λ ~ 10μm (infra-red)

ultra-relativistic beams ($\gamma \approx 30000$) : λ ~ 0.1nm (X-ray)

2.2 FEL Stimulated Emission

Spontaneous emission is incoherent as electrons are randomly distributed.

Now we consider an electron beam moving in both a strong magnetostatic (helical) wiggler field and an electromagnetic wave.



Wiggler field : $\vec{B}_w = B_w \left(\cos(k_w z)\hat{x} + \sin(k_w z)\hat{y} \right)$

Radiation field : $\vec{E}_r = -E_r \left(\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y} \right)$

(circularly polarised plane wave) $\vec{B}_r = \frac{E_r}{c} \left(\cos\alpha\,\hat{x} - \sin\alpha\,\hat{y} \right)$

where $\alpha = kz - \omega t + \phi$

We want to know the beam-wave energy exchange :

Energy of the electrons is $\mathrm{E} = \gamma m c^2$

Rate of electron energy change is $\dfrac{\mathrm{dE}}{\mathrm{dt}} = mc^2 \dfrac{\mathrm{d}\gamma}{\mathrm{dt}}$

This must be equal to work done by EM wave on electrons i.e.

$$mc^2 \frac{\mathrm{d}\gamma}{\mathrm{dt}} = -|e|\vec{E}\cdot\vec{v}$$

Problem : What is $\vec{v}$?

We can use the fact that canonical momentum is a conserved quantity.

Canonical momentum : $\vec{\Pi} = \vec{p} - |e|\vec{A} = 0$ $\quad \vec{A}$ = vector potential

Consequently : $\vec{v} = \dfrac{|e|\vec{A}}{\gamma m}$

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{dt}} &= -\frac{|e|}{mc^2}\vec{E}\cdot\vec{v} \\ &= -\frac{|e|}{mc^2}\left(-\frac{\partial \vec{A}_\perp}{\partial t}\right)\cdot\left(\frac{|e|\vec{A}_\perp}{\gamma m}\right) \\ &= \frac{|e|}{mc^2}\frac{1}{2}\frac{\partial\left(\vec{A}_\perp\cdot\vec{A}_\perp\right)}{\partial t}\end{aligned}$$

where $\vec{A}_\perp = \vec{A}_w + \vec{A}_r$

(wiggler + EM field)

Now $\vec{A}_\perp\cdot\vec{A}_\perp = \cancel{\vec{A}_w\cdot\vec{A}_w} + 2\vec{A}_w\cdot\vec{A}_r + \cancel{\vec{A}_r\cdot\vec{A}_r}$

no time dependence

EM field << wiggler

so the only term of interest is

$$2\vec{A}_w\cdot\vec{A}_r \propto \cos\left(\left(k_w + k\right)z - \omega t\right)$$

so $\quad \dfrac{d\gamma}{dt} \propto \sin\left(\left(k_w + k\right)z - \omega t\right)$

$$\frac{d\gamma}{dt} \propto \sin\left(\left(k_w + k\right)z - \omega t\right)$$

Whether electron gains or loses energy depends on the value of the phase variable

$$\theta = \left(k_w + k\right)z - \omega t$$

The EM wave (ω,k) and the wiggler “wave” (0,k$_w$) interfere to produce a “ponderomotive wave” with a phase velocity

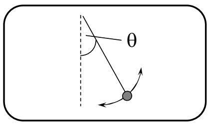
$$v_{ph} = \frac{\omega}{k_w + k}$$

From the definition of θ, it can be shown that : $\dfrac{\mathrm{d}^2\theta}{\mathrm{dt}^2} = \left(k_w + k\right)\dfrac{\mathrm{dv}_z}{\mathrm{dt}}$

And from the definition of γ : $\dfrac{\mathrm{dv}_z}{\mathrm{dt}} = \dfrac{2c}{\gamma^3}\dfrac{\mathrm{d}\gamma}{\mathrm{dt}}$

So we eventually obtain : $\boxed{\dfrac{\mathrm{d}^2\theta}{\mathrm{dt}^2} = -\Omega^2 \sin\theta}$ $\quad (\Omega^2 \propto E_r)$

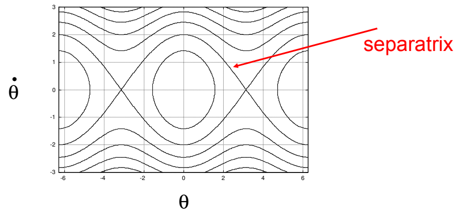
$$\frac{\mathrm{d}^2\theta}{\mathrm{dt}^2} = -\Omega^2 \sin\theta$$

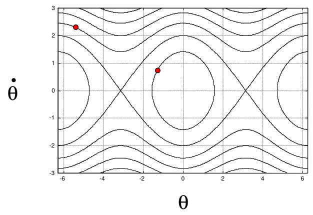


This is the equation of a pendulum, with period of oscillation Ω

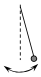
Wave electric field plays an analogous role to gravity.

Pendulum phase space : $\left(H = \dfrac{\dot{\theta}^2}{2} - \Omega^2\cos\theta\right)$

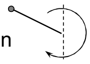




Electrons inside separatrix are trapped : libration



Electrons outside separatrix are untrapped : rotation



Trapped electrons undergo maximum energy change
Untrapped electrons energy oscillates rapidly – little net change

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Therefore, for electrons to be trapped, we require $\frac{d\theta}{dt} \approx 0$

i.e. $$v_z \approx v_{ph} = \frac{\omega}{k_w + k}$$

i.e. Electron velocity ≈ phase velocity of ponderomotive wave

so that it remains **resonant** with the ponderomotive wave

Exercise : Show that the resonance condition above
is the same as the condition for
the spontaneous emission peak i.e.

$$v_z = \frac{\omega}{k_w + k} \quad \text{implies} \quad \lambda = \frac{\lambda_w}{2\gamma^2}$$

High gain regime – collective behaviour

So far, we've assumed that the EM field amplitude and phase remain approximately constant
i.e. height and position of separatrix is constant.

We now relax this restriction, allowing us to study high-gain amplification i.e.
$\vec{E}_r = -E_r(z)(\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y})$ where $\alpha = kz - \omega t + \phi(z)$

The EM field is determined by Maxwell's wave equation

$$\nabla^2 \vec{E}_r - \frac{1}{c^2}\frac{\partial^2 \vec{E}_r}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t}$$

The (transverse) current density is due to the electron motion
In the wiggler magnet.

$$\vec{J} = -|e| \sum_j \vec{v}_j\, \delta\left(\vec{r} - \vec{r}_j(t)\right)$$

Outline of derivation procedure :
For full derivation see articles by e.g. Bonifacio et al. or Murphy & Pelligrini.

- Express $\vec{E}_r(z,t)$ in terms of a slowly-varying envelope, a(z)

  i.e. $\vec{E}_r(z,t) = \left(a(z)e^{i(kz-\omega t)} + c.c.\right)\vec{e}$

  where $\omega = kc$ and $\left|\frac{da(z)}{dz}\right| << \left|\lambda a(z)\right|$

  Slowly-varying
  Envelope
  Approximation
  (SVEA)

- Substituting for $\vec{E}_r(z,t)$ in Maxwell's wave equation
  -reduces 2nd order wave equation in $\vec{E}_r(z,t)$
  to 1st order equation in a(z)

  -Spatially average over wavelength to remove δ functions and ensure field is a smoothly varying function.

The end result is the high gain FEL equations :

$$\frac{d\theta_j}{d\bar{z}} = \bar{p}_j$$

$$\frac{d\bar{p}_j}{d\bar{z}} = -(Ae^{i\theta_j} + c.c.)$$

$$\frac{dA}{d\bar{z}} = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j} \equiv \left\langle e^{-i\theta} \right\rangle$$

Pendulum Equations + Wave equation

$$\theta_j = (k_w + k)z - \omega t_j$$ Ponderomotive phase

$$\bar{p}_j = \frac{\gamma_j - \gamma_R}{\rho\gamma_R}$$ Scaled energy change

$$|\mathrm{A}|^2 = \frac{\varepsilon_0 |E|^2}{\rho n \gamma_R mc^2}$$ Scaled EM field intensity

$$\bar{z} = \frac{z}{L_g} = \frac{4\pi\rho z}{\lambda_w}$$ Scaled position in wiggler

Interaction characterised by FEL parameter : $\rho = \frac{1}{\gamma_R}\left(\frac{a_w \omega_p}{4ck_w}\right)^{2/3}$

We will now use these equations to investigate the high-gain regime.

We solve the equations with initial conditions

$\theta_j = (0, 2\pi]$ (uniform distribution of phases)

$p_j = 0$ (cold, resonant beam)

$A << 1$ (weak initial EM field)

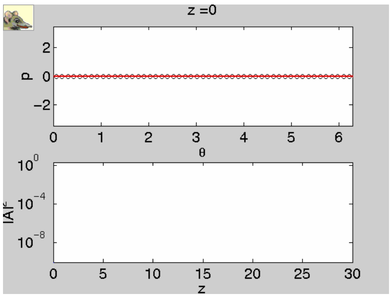
and observe how the EM field and electrons evolve.

For linear stability analysis see :

J.B. Murphy & C. Pelligrini
"Introduction to the Physics of the Free Electron Laser"
Laser Handbook, vol. 6 p. 9-69 (1990).

R. Bonifacio et al
"Physics of the High-Gain Free Electron Laser & Superradiance"
Rivista del Nuovo Cimento Vol. 13, no. 9 p. 1-69 (1990).

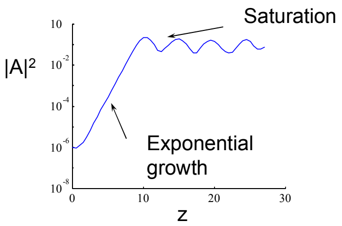
High-gain FEL



Both height and position of separatrix now evolve in time

Height depends on EM field amplitude

Position depends on EM field phase/frequency



Field evolution can be divided into 3 regions :

1. A short initial period where $|A|^2 \approx |A^{(z=0)}|^2$ (low gain)
2. A period of exponential growth i.e. $|A|^2 \propto \exp(gz)$
3. saturation

At saturation, $|A|^2 >> |A^{(z=0)}|^2$ - high gain

Strong amplification of field is closely linked to phase bunching of electrons.

Bunched electrons mean that the emitted radiation is **coherent.**

For randomly spaced electrons : intensity ∝ N as $\left|\sum_{j=1}^{N} e^{-i\theta_j}\right|^2 = N$
For bunched electrons : intensity up to ~ N^2

As N ~10^8, this is a very large enhancement over incoherent (spontaneous) emission.

At saturation in our model, scaled intensity $|A|^2 \approx 1.4$

As there are no free parameters in equations, this implies that the real EM field intensity, $|E|^2 \propto N^{4/3}$

As intensity scales > N, this indicates collective behaviour

Exponential amplification in high-gain FEL is an example of a collective instability.

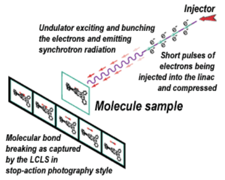
Current FEL research

Major efforts underway to generate coherent XUV/X-ray radiation.

Must be single-pass, high-gain devices as no good quality mirrors exist at short wavelengths.

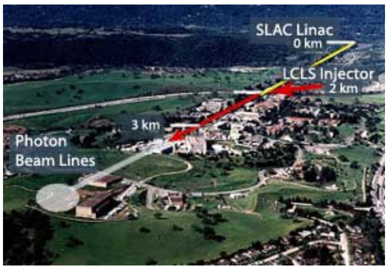
Powerful coherent X-ray source will have many applications e.g. X-ray holography

X-ray FELs will have sufficiently high spatial resolution (λ<1A) and temporal resolution (<fs) to follow chemical & biological processes in “real time” e.g. stroboscopic “movies” of molecular bond breaking



DESY (Hamburg) project (FLASH) is currently lasing at λ=7nm.

Projects in USA (LCLS,Stanford, below), next DESY project (XFEL) underway to produce coherent hard X-rays with λ=0.1nm.



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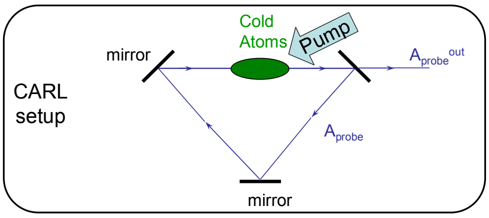
3. Collective Interactions Between Light and Cold Atoms - Collective Atomic Recoil Lasing (CARL)

Over the last 20 years, major advances in the cooling and trapping of atomic gases have been made.

Now “trivial” to cool >10^6 atoms to temperatures <1mK

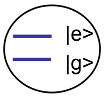


Probe frequency ≈ pump frequency

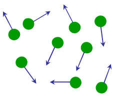
3.1 CARL Model

There are three parts to the description of the atom+field system :

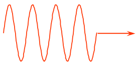
1. Internal atomic degrees of freedom (dipole moment, population difference)



2. External atomic degrees of freedom (position, momentum)

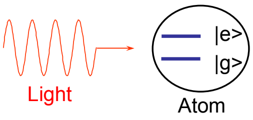


3. Probe EM field evolution



R. Bonifacio & L. DeSalvo, Nucl. Inst. Meth. **A 341**, 360 (1994)
R.Bonifacio, L. DeSalvo, L.M. Narducci & E.J. D'Angelo PRA **50**, 1716 (1994).
R. Bonifacio, G.R.M. Robb & B.W.J. McNeil, PRA **56**, 912 (1997)

3.1.1 Internal atomic evolution



Proper treatment needs quantum description (Bloch equations)
See e.g. "Optical Resonance and Two-Level Atoms" by Allen & Eberly or similar

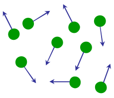
As long as light frequency, ω, is far from any atomic resonance frequency, ω_a, then atom behaves as a linear dipole.

i.e. induced dipole moment is proportional to electric field

$$\vec{d} = \alpha\vec{E}$$ α=polarisability

Full quantum treatment gives same result as classical one, with different definition of α .

3.1.2. External atomic evolution



We consider atoms as classical point particles (for now)
i.e. point dipoles

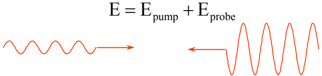
Force on each atom is the dipole force :

$$\mathrm{F_z} = \vec{\mathrm{d}}\cdot\frac{\partial\vec{E}}{\partial \mathrm{z}}$$

d = dipole moment of atom
E = electric field

If E is the sum of two counterpropagating fields (as in CARL)

$$\vec{\mathrm{E}} = \vec{\mathrm{E}}_{\mathrm{pump}} + \vec{\mathrm{E}}_{\mathrm{probe}}$$

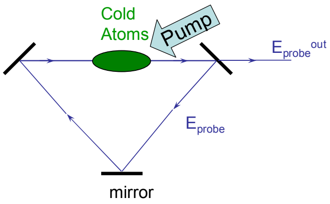


$$\vec{\mathrm{E}}_{\mathrm{probe}} = \left(E_{probe}e^{i(kz-\omega t)} + c.c.\right)\hat{x} \qquad \vec{\mathrm{E}}_{\mathrm{pump}} = \left(E_{pump}e^{-i(kz+\omega t)} + c.c.\right)\hat{x}$$

then $$\mathrm{F_z} = \vec{\mathrm{d}}\cdot\frac{\partial\vec{E}}{\partial \mathrm{z}} \propto \left(E_{pump}E_{probe}e^{2ikz} + c.c.\right)$$

Spatially periodic by λ/2

3.1.3. EM Field evolution



Maxwell's wave equation :

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial^2\vec{P}}{\partial t^2}$$

If E_{probe} consists of a single cavity mode then

$$\vec{E}(z,t) = \left(A(t)e^{i(kz-\omega t)} + c.c.\right)\hat{x}$$

where A(t) = mode amplitude

Polarisation is a collection of point dipoles

$$\underline{P}(t) = \sum_j \underline{d_j}(t)\,\delta\left(\underline{x} - \underline{x_j}(t)\right) \quad \text{where} \quad \vec{d}_j = \alpha\vec{E}\left(z = z_j, t\right)$$

Performing slowly varying approximation for A(t) reduces 2nd order equation for E(z,t) to first order equation for A(t).

The equations for position, momentum and probe amplitude can be written in the same form as the high gain FEL equations with time as independent variable :

$$\frac{d\theta_j}{d\tau} = \bar{p}_j$$

$$\frac{d\bar{p}_j}{d\tau} = -(Ae^{i\theta_j} + c.c.)$$

$$\frac{dA}{d\tau} = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j} - \cancel{\kappa A}$$

for κ<<1 (good cavity)

$\theta_j = \frac{4\pi z_j}{\lambda}$ Atom position in optical potential

$\bar{p}_j = \frac{mv_j}{\hbar k \rho}$ Scaled momentum

$A = \sqrt{\frac{2\varepsilon_0 |E_{\mathrm{probe}}|^2}{\hbar\omega\rho}}$ Scaled EM field intensity

$\tau = \omega_r \rho t$ Scaled time variable

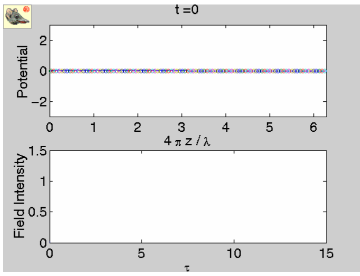
$\kappa = \frac{K}{\omega_r \rho}$ Scaled field decay rate

CARL parameter : $\rho \propto \frac{(\mathrm{I}_{\mathrm{pump}}\mathrm{n})^{1/3}}{\Delta_a^{2/3}}$

CARL instability animation

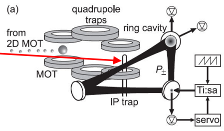
Animation shows evolution of atomic positions in the dynamic optical potential together with the scaled probe field intensity.

Good cavity limit (κ<<1) – CARL is atomic analogue of FEL



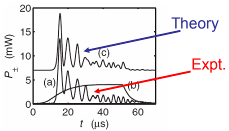
Results from CARL model agree well with experimental results :

*S. Slama, S. Bux, G. Krenz, C. Zimmermann, and Ph.W. Courteille, PRL **98** 053603 (2007).*



10^7 Rb atoms
T~1μK

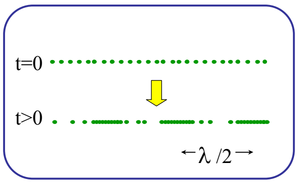
Probe intensity evolution



Probe intensity ∝ $N^{4/3}$ in good cavity limit (κ<<1)

∝ N^2 in superradiant limit (κ>1)

3.2 CARL and Collective Synchronisation



CARL (and FEL) can be interpreted as spontaneous ordering due to global coupling by light.

CARL is of practical interest in optical physics

as e.g. a new nonlinear optical phenomenon

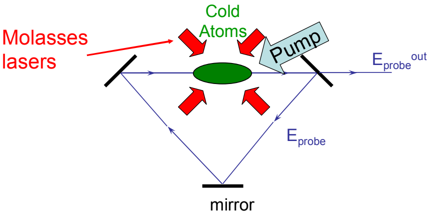
Also has wider relevance :

Versatile experimental setup makes CARL a testing ground for various global coupling/mean-field models used in

e.g. plasma physics, condensed matter, mathematical biology & neuroscience

e.g.could we use CARL-like setup to measure plasma diffusion coefficients?

Example : “Viscous CARL” & the Kuramoto Model



Same setup as
for CARL,
but with additional
“molasses”
lasers to damp
atomic momentum

For details see :

- G.R.M. Robb, N. Piovella, A. Ferraro , R. Bonifacio, Ph. W. Courteille and C. Zimmermann, Phys. Rev. A 69, 041403(R) (2004)
- J. Javaloyes, M. Perrin , G. L. Lippi, and A. Politi, Phys. Rev. A **70**, 023405 (2004)
- C. Von Cube, S. Slama, Ph. W. Courteille, C. Zimmermann, G.R.M. Robb, N. Piovella & R. Bonifacio **PRL 93**, 083601 (2004).
- Y. Kuramoto, Prog. Theor. Phys. Suppl. 79, 223 (1984).

In order to model experiment we need to add effects of :

- momentum damping (cooling)
- momentum diffusion (temperature)

Momentum damping/cooling

$$\frac{d\vartheta_j}{d\tau} = \bar{p}_j$$

$$\frac{d\bar{p}_j}{d\tau} = -(Ae^{i\vartheta_j} + c.c.) \boxed{-\gamma \bar{p}_j} + \boxed{F(\tau)}$$

scattering
/heating

$$\frac{dA}{d\tau} = \frac{1}{N}\sum_{j=1}^{N} e^{-i\vartheta_j} - \kappa A$$

Alternatively, these can be expressed in terms of a Fokker-Planck equation for the atomic distribution function P(θ,τ).

$$\frac{\partial P(\theta,\tau)}{\partial \tau} = \frac{\partial}{\partial \theta}\left[\left(Ae^{i\theta} + c.c.\right)P\right] + D\frac{\partial^2 P}{\partial \theta^2}$$

Diffusion ∝ T

$$\frac{dA(\tau)}{d\tau} = \int_0^{2\pi} P(\theta,\tau)\, e^{-i\theta}\, d\theta - \kappa A$$

- G.R.M. Robb, N. Piovella, A. Ferraro , R. Bonifacio, Ph. W. Courteille and C. Zimmermann, Phys. Rev. A 69, 041403(R) (2004)
- J. Javaloyes, M. Perrin , G. L. Lippi, and A. Politi, Phys. Rev. A **70**, 023405 (2004)

The viscous CARL experiments demonstrate behaviour similar to the **Kuramoto model of collective synchronization in large systems of globally coupled oscillators.**

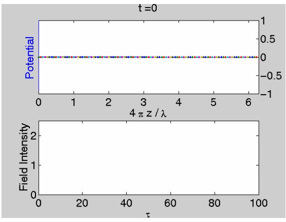
$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_i - \theta_j\right) \qquad j = 1..N$$

θ_j is the phase of oscillator j **= atomic position**
ω_j is its (random) natural frequency **= thermal velocity**
Coupling constant K ∝ **pump power**

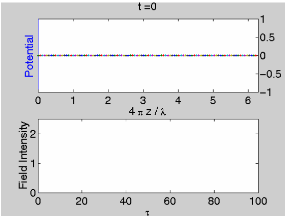
Coupling constant K ∝ **pump power**

Synchronisation transition occurs when K exceeds a threshold, K_c

$K<K_c$ (weak pump)



$K>K_c$ (strong pump)



The Kuramoto model has been used to model a wide range of synchronisation phenomena in physics and mathematical biology.

Same equations describe synchronization of cold atoms in coupled by light, flashing fireflies and rhythmic applause!

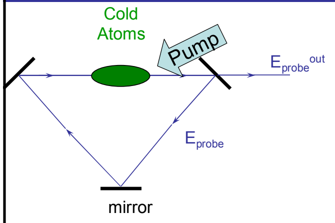
Synchronously flashing fireflies

“The Trials of Life”
BBC (1990)

Rhythmic applause

Neda et al., Nature **405**, 849 (2000)
SH Strogatz, Nature **410**, 268 (2001)

3.3 CARL and Chaos



Now we consider a pump field which is **phase modulated**.

Pump field is of the form

$$e^{-i\left(k_{pump}z+\omega_{pump}t+\alpha_m \sin \omega_m t\right)}$$

where α_m = modulation amplitude
ω_m=modulation frequency

Incorporating a phase-modulated pump field into the CARL model, we obtain :

$$\frac{d\theta_j}{d\tau} = p_j$$

$$\frac{dp_j}{d\tau} = -(Ae^{i(\theta_j-\alpha_m \sin \Omega_m \tau)} + c.c.) \qquad (j=1..N)$$

$$\frac{dA}{d\tau} = \left\langle e^{-i(\theta-\alpha_m \sin \Omega_m \tau)} \right\rangle$$

where $\Omega_m = \frac{\omega_m}{\omega_r \rho}$

is the scaled

modulation frequency

Using the identity $\exp(i\alpha_m \sin \Omega_m \tau) = \sum_{-\infty}^{\infty} J_n(\alpha_m) e^{in\Omega_m \tau}$

interaction involves many resonances with phase velocities

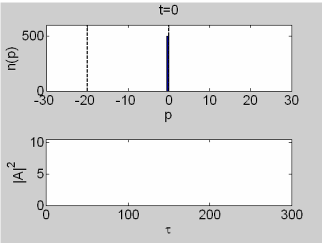
separated by Ω_m and width $\propto \sqrt{|A| J_n(\alpha_m)}$

As probe is amplified, resonance overlap can occur,

causing chaotic diffusion of atomic momentum.

α_m =10

Ω_m =1



G.R.M.Robb & W.J. Firth, to appear in Phys. Rev. A

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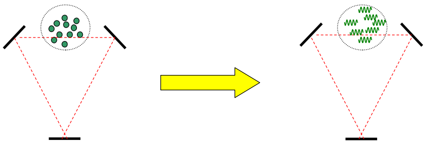
4. Quantum CARL

So far we have treated the atomic gas as
a collection of classical point particles

If we cool the gas sufficiently (T<1μK)
so that it forms a BEC then this description fails
- we must then describe the atomic motion quantum mechanically



How does the transition from classical gas to BEC affect CARL?

(i) Newtonian atomic motion equations are replaced with a Schrodinger equation for the single particle wavefunction $\Psi(\theta,\tau)$

$$\frac{d\theta_j}{d\tau} = \bar{p}_j \qquad \frac{d\bar{p}_j}{d\tau} = -(Ae^{i\theta_j} + c.c.) \quad \Longrightarrow \quad \frac{\partial\Psi(\theta,\tau)}{\partial\tau} = \frac{i}{\rho}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\rho}{2}\left[Ae^{i\theta} - c.c.\right]\Psi$$

(ii) Average in wave equation becomes QM average

$$\frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j} \quad \Longrightarrow \quad \int_0^{2\pi} d\theta\, |\Psi|^2 e^{-i\theta}$$

Maxwell-Schrodinger Equations

$$\frac{\partial\Psi(\theta,\tau)}{\partial\tau} = \frac{i}{\rho}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\rho}{2}\left[Ae^{i\theta} - c.c.\right]\Psi$$

$$\frac{dA(\tau)}{d\tau} = \int_0^{2\pi} d\theta\, |\Psi|^2 e^{-i\theta} - \kappa A$$

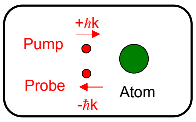
See : G. Preparata, PRA (1988)
N. Piovella et al.
Optics Comm, **194**, 167 (2001)

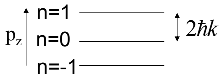
We assume uniform BEC density with L >> λ/2, so Ψ is periodic with period λ/2

$$\Psi(\theta,\tau) \propto \sum_{n=-\infty}^{\infty} c_n(\bar{t})e^{in\theta}$$

Momentum exchange no longer continuous.
Only discrete values of momentum exchange are possible :

$p_z = n\,(2\hbar k)$, n=0,±1,..





Dynamical regime is determined by the CARL parameter, ρ

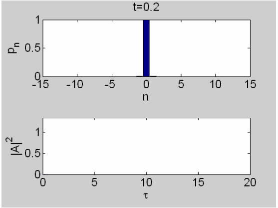
$$\frac{\partial\Psi(\theta,\tau)}{\partial\tau} = \frac{i}{\rho}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\rho}{2}\left[Ae^{i\theta} - c.c.\right]\Psi$$

$$\frac{dA(\tau)}{d\tau} = \int_0^{2\pi} d\theta\, |\Psi|^2 e^{-i\theta} - \kappa A$$

$$\rho \propto \frac{(I_{pump} n)^{1/3}}{\Delta_a^{2/3}}$$

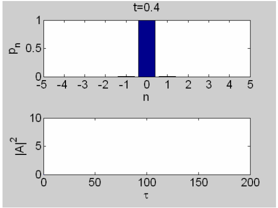
ρ can be interpreted as ~ number of photons scattered per atom

Classical CARL (ρ>>1)

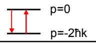


Many momentum states occupied
Field evolves as in particle model

Quantum CARL (ρ<1)

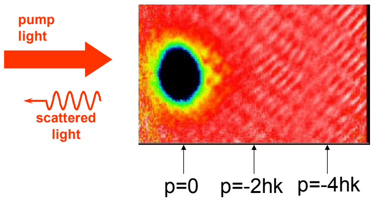


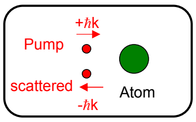
Only 2 momentum states occupied



Evidence for quantum regime observed at LENS (Florence) by Inguscio group (& in Tubingen & at MIT by Ketterle group)

Animation shows momentum distribution of ^{87}Rb BEC illuminated by pump laser





L. Fallani et al., **PRA 71**, 033612 (2005).
G. Robb, N. Piovella & R.Bonifacio, J. Optics B **7**, 93 (2005).

4.1 Atom-atom Interactions

In the case of an interacting condensate, the Maxwell-Schrodinger equations now become the Maxwell-Gross-Pitaevskii equations.

$$\frac{\partial \Psi}{\partial \tau} = \frac{i}{\rho}\frac{\partial^2 \Psi}{\partial \theta^2} - \frac{\rho}{2}\left[Ae^{i\theta} - c.c.\right]\Psi + \beta|\Psi|^2\Psi$$

$$\frac{dA}{d\tau} = \int d\theta |\Psi|^2 e^{-i\theta}$$

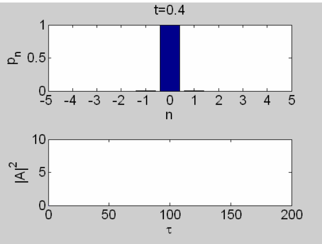
$\beta = \frac{8\pi\hbar kaN}{m\Sigma}$ is the scaled scattering length of the BEC

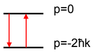
We have investigated the evolution of the CARL dynamics in the quantum limit ($\rho<1$) including the effect of atomic collisions, both in the linear and nonlinear regimes.

Good cavity limit ($\kappa=0$) is assumed

4.1. No atom-atom interaction ($\beta=0$)

$\rho=0.2$

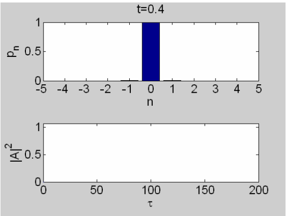




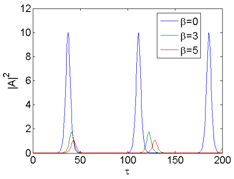
- Only 2 momentum states occupied
- Field evolution is Maxwell-Bloch evolution of an optical field interacting with a two-level system.

4.1 Atom-atom interactions – Repulsive condensate ($\beta>0$)

$\rho=0.2$, $\beta=5$

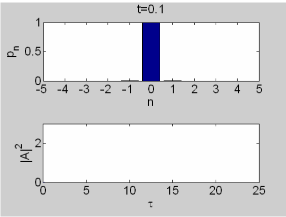


- Evolution is similar to collision-free case (hyperbolic secant pulses)
- Repulsive nature of condensate resists bunching
- Effect of condensate repulsion is to reduce peak intensity of scattered field



4.1 Atom-atom interactions – Attractive condensate ($\beta<0$)

$\rho=0.2$, $\beta=-5$



- Evolution now involves more momentum states (~|β|) with n>0 and n<0
- Initial field growth is much faster than in collision-free case but hyperbolic secant behaviour of field evolution is lost
- Atoms are bunching, but in “wrong” positions for amplification of probe
- Peak intensity of scattered field again reduced with respect to collision-free case

Conclusions

Collective light-matter interactions are of interest for both :

Practical optical applications :

- Tunable coherent light sources
- New optical nonlinearities

Cross-disciplinary interest:

- Physical example of nonlinear coupled oscillator dynamics (e.g. Kuramoto, mean field Hamiltonian)

- Versatile testing ground for models- allows control/study of e.g.
  - noise effects
  - chaos
  - coupling range (global →local)
  - quantum mechanical effects

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