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Pulse solutions of the fractional effective models of the Fermi–Pasta–Ulam lattice with long-range interactions

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Abstract. We study the analytical solutions of the fractional Boussinesq equation (FBE), which is an effective model for the Fermi–Pasta–Ulam one-dimensional lattice with long-range couplings. The couplings decay as a power-law with exponent s, with $1 < s < 3$, so that the energy density is finite, but s is small enough to observe genuine long-range effects. The analytic solutions are

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obtained by introducing an ansatz for the dependence of the field on space and time. This allows the FBE be reduced to an ordinary differential equation, which can be explicitly solved. The solutions are initially localized and they delocalize progressively as time evolves. Depending on the value of s, the solution is either a pulse (meaning a bump) or an anti-pulse (i.e. a hole) on a constant field for $1 < s < 2$ and $2 < s < 3$, respectively.

Keywords: Fermi-Pasta-Ulam (FPU) model, long-range interactions (LRI), fractional Boussinesq equation (FBE), non-Galilean invariance

Contents

1. Introduction

The celebrated Fermi–Pasta–Ulam (FPU) model [1, 2] consists of a one-dimensional lattice of identical particles interacting through a nearest neighbour nonlinear potential. The displacements of the particles with respect to their equilibrium positions are described by the function of time $u_i(t)$, with i the label of the lattice site. The problem under investigation was a central one in the foundations of statistical mechanics: relaxation to microcanonical equilibrium. The lack of relaxation turned the problem into a 'paradox', which led to a long series of investigations that were summarized in 50th anniversary volumes [3, 4]. The FPU paper also proved to be seminal in nonlinear dynamics, leading to the pioneering work on *solitons* in the Korterveg–de Vries equation by Kruskal and Zabusky [5] and to the study of integrability of nonlinear lattices [6, 7].

Long-range interactions are ubiquitous in nature, the most common examples being gravitational and Coulomb interactions. Recently, long-range interacting systems have been extensively studied for their, often intriguing, statistical behaviour [8–11]. In fact, these systems exhibit inequivalence of ensembles, breakdown of ergodicity, long-lived quasi-stationary states and suppression of chaos, to name just a few anomalous phenomena. Due to the paradigmatic importance of the FPU model, it is interesting to explore the effect of long-range interactions in this context. Given the presence in this Special Issue of several papers on non-equilibrium statistical mechanics, we think that the FPU model provides an appropriate arena to explore the behaviour of systems that persist out-of-equilibrium for long time depending on the features of the interactions.

In this paper we focus on the specific role played by long-range interactions by analyzing the dynamical properties of the one-dimensional FPU model appropriately modified to include long-range couplings among particles [12, 13]. Particles interact through couplings that decay with a power-law of the distance among them. The exponent of the power-law is denoted as s. This model has been also recently studied in connection with the problem of anomalous heat conduction in one-dimensional lattices [14–16].

We have shown in a previous paper [17] that, in the continuum limit, the model can be *effectively* described by a fractional Boussinesq equation (FBE) in terms of the space-time field $u(x, t)$ for small variations of the amplitude. In fact, the presence of long-range couplings determines the appearance of nonlocal terms in space in the continuum equation, where the non-locality is mathematically represented by fractional derivatives. Fractional differential calculus, i.e. non-integer derivatives and integrals [18–20], is playing a very important role in various fields of science and technology, such as elasticity, water waves, quantum mechanics, signal analysis, control theory and finance.

We here deal with the problem of determining the solutions of the FBE introduced in our previous paper [17]. In order to achieve this target, we introduce an ansatz to express the solution in terms of a single variable $\eta = |x|^\rho - vt$, where ρ is related to the power-law decay exponent s. This ansatz generalizes the usual change of frame of the Galilean transformation which takes the usual form $\eta = \mathrm{x} - \mathrm{vt}$.

We are able to derive an ordinary differential equation in η which can be explicitly solved, providing solutions of the FBE which depend on the parameters of the model, in particular on the exponent s. Two classes of solutions emerge depending on the value of s: *(i) pulse* solutions displaying a 'bump' over a constant field in the range $1 < s < 2$; *(ii) anti-pulse* solutions showing a 'hole' in a constant field in the range $2 < s < 3$.

The paper is organized as follows. In section 2 we introduce the α-FPU model with long-range interactions and its continuum limit. In section 3 we provide details on the study of the fractional derivatives involved in the FBE. The ansatz used to determine the solutions is presented in section 4. We determine the exact solutions of the FBE in section 5. Concluding remarks are given in section 6.

2. The FPU model with long-range interactions and its continuum limit

The FPU model we consider is a one-dimensional lattice of anharmonic oscillators, each with mass m, displacement u_i and momentum p_i (i denoting the lattice site) and coupled by long-range interactions. The Hamiltonian for the long-range FPU model reads

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$$\mathcal{H}\left(u_i,\, p_i\right) = \sum_i \frac{p_i^2}{2m} + \frac{\chi}{2}\sum_{j<i}\frac{\left(u_i - u_j\right)^2}{\left|a\left(i-j\right)\right|^s} + \frac{\gamma}{3}\sum_{j<i}\frac{\left(u_i - u_j\right)^3}{\left|a\left(i-j\right)\right|^s}, \qquad (1)$$

where a is the lattice spacing. We can observe that the two-body coupling is assumed to be translationally invariant since it acts between pairs of sites at distance $a|i-j|$. More importantly for our purposes, the coupling decays for large distances as a power-law with an exponent s. In order for the energy to be extensive, we consider s larger than the dimension d where the system is embedded, in this case $d=1$. The lattice sites i and j take all possible negative and positive integer values: $i, j = 0, \pm 1, \pm 2, \cdots$. We consider $1 < s < 3$ since at $s=3$ the system is supposed to have a typical short-range long-time behaviour [17]. The fact that a value of s exists, denoted by s^*, such that for $s > s^*$ one retrieves the critical properties of the short-range case ($s \to \infty$) is well studied in Ising and $O(n)$ models [21, 22]. Notice that the range $d < s < s^*$ is often referred to as the weak long-range region [23]. In equation (1) we assume also that both the harmonic and anharmonic couplings decay with the same exponent. One could study as well the case in which the two power-law decay exponents are different [17]. Finally, we observe that, since the anharmonic term is characterized by the power 3, the resulting FPU model is called α-FPU [24], so that equation (1) corresponds to the Hamiltonian of the long-range α-FPU model.

The interaction potential energy has two parts: the second term of the rhs in equation (1) is the long-range harmonic interaction, whereas the third term corresponds to the long-range anharmonic interaction. The parameters χ and γ measure the strength of the linearity and nonlinearity, respectively, and they can be viewed as the linear and nonlinear spring coefficients, respectively. In the following we consider the case of positive parameters $\chi, \gamma > 0$. The parameter s sets then the range of the interaction. For $s \to \infty$, the second and the third terms reduce to nearest neighbour, short-range, harmonic and anharmonic interactions, respectively, and we return to the standard short-range FPU model.

In order to study the dynamical properties, we move to the effective model describing the continuum limit of the FPU Hamiltonian (1), where $u_i(t) \to u(x,t)$. This effective model was derived in [17] and the corresponding equation for the field $u(x,t)$ is a fractional differential equation, the FBE, which generalizes to the weak long-range region the Bousinessq equation found for the short-range FPU model [25]. We refer to [17] for a discussion of the details needed to perform the continuum limit and its range of validity, which essentially requires to consider a field u, which is slowly varying. We also point out that the long-range FPU model has the merit to allow for a microscopic derivation of the fractional differential equation, which is not—as done in other cases—assumed on the basis of physical considerations.

The FBE for the long-range α-FPU model reads [17]

$$\frac{\partial^2 u\left(x,t\right)}{\partial t^2} - g_{s-1}\frac{\partial^{s-1} u\left(x,t\right)}{\partial\left|x\right|^{s-1}} - h_{s-1} u\left(x,t\right)\left[D_{x^-}^{s-1} - D_{x^+}^{s-1}\right] u\left(x,t\right) = 0, \qquad (2)$$

where the constants g_{s-1} and h_{s-1} are given by

$$g_{s-1} = \frac{\chi\pi}{\Gamma\left(s\right)\sin\left(\frac{s-1}{2}\pi\right)}, \quad h_{s-1} = -\frac{\gamma\pi}{\Gamma\left(s\right)\sin\left(s\pi\right)} \tag{3}$$

where Γ stands for the Euler-gamma function, and we have set $a = m = 1$). In the following we are going to take $s \neq 2$.

Equation (2) is the equation for which analytical solutions are needed to be found, which is the task we deal with in the next sections. The analytical solution then provides a guide to the numerical solution of the FPU lattice model, studied in the regime in which the continuum limit is a reasonable approximation. Of course, such a comparison is a benchmark for the assessment of the validity of the effective model. Moreover, as we point out in the following, the solutions of the effective model may reveal new class of solutions, possibly not previously known, and motivate further numerical studies of the lattice equations.

3. The fractional chain rule

In order to construct exact solutions, we need to give some details on the action of the fractional derivatives appearing in equation (2).

We first observe that the chain rule can be written as

$$\frac{\partial^\alpha w}{\partial x^\alpha} = \sigma_\alpha \frac{\partial^\alpha \zeta}{\partial x^\alpha}\frac{\partial w}{\partial \zeta}, \tag{4}$$

in general valid for analytic functions of the form $w\left(x\right) = \sum_{m=0}^{\infty} a_m\, x^m$, where a_m are constants. σ_α is sometimes called fractal index [26] and it has to be further determined. It is usually determined in terms of Euler-gamma functions. The reason for studying relation (4) is that it can be used to transform the FBE into an ordinary differential equation.

To show the validity of (4), we may use the function $w(\zeta) = \zeta^\beta$ with $\zeta(x) = x^\gamma$ where $\beta,\ \gamma > 0$, and the definition of the fractional derivative of non-integer order which reads

$$\frac{\partial^\alpha}{\partial x_+^\alpha}x^\beta \equiv \frac{\partial^\alpha}{\partial x^\alpha}x^\beta = \frac{\Gamma\left(1+\beta\right)}{\Gamma\left(1+\beta-\alpha\right)}x^{\beta-\alpha} \qquad x>0,\ \ \textit{and} \quad \alpha>0. \tag{5}$$

One gets

$$D_x^\alpha\, w(\zeta(x)) = D_x^\alpha x^{\beta\gamma} = \frac{\Gamma\left(1+\beta\gamma\right)}{\Gamma\left(1+\beta\gamma-\alpha\right)}\; x^{\beta\gamma-\alpha}, \tag{6}$$

$$D_x^\alpha \zeta(x) \equiv \frac{\partial^\alpha \zeta}{\partial x^\alpha} = \frac{\partial^\alpha x^\gamma}{\partial x^\alpha} = \frac{\Gamma\left(1+\gamma\right)}{\Gamma\left(1+\gamma-\alpha\right)}\; x^{\gamma-\alpha}, \tag{7}$$

and

$$\frac{\partial w}{\partial \zeta} = \left(\beta\, \zeta^{\beta-1}\right)|_{\zeta=x^\gamma} = \beta\, x^{(\beta-1)\gamma}. \tag{8}$$

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Substitution of equations (6)–(8) into equation (4) gives

$$\left(\frac{\Gamma\left(1+\beta\gamma\right)}{\Gamma\left(1+\beta\gamma-\alpha\right)}-\sigma_{\alpha}\,\frac{\beta\,\Gamma\left(1+\gamma\right)}{\Gamma\left(1+\gamma-\alpha\right)}\right)x^{\beta\gamma-\alpha}=0,\qquad x>0. \tag{9}$$

As a result, we have the condition

$$\frac{\Gamma\left(1+\beta\gamma\right)}{\Gamma\left(1+\beta\gamma-\alpha\right)}-\sigma_{\alpha}\,\frac{\beta\,\Gamma\left(1+\gamma\right)}{\Gamma\left(1+\gamma-\alpha\right)}=0,\qquad \alpha,\beta,\gamma>0. \tag{10}$$

Assuming

$$\sigma_{\alpha}=\frac{\Gamma\left(1+\beta\gamma\right)\Gamma\left(1+\gamma-\alpha\right)}{\beta\,\Gamma\left(1+\beta\gamma-\alpha\right)\Gamma\left(1+\gamma\right)}, \tag{11}$$

one immediately sees that equation (4) holds. We then conclude that the modified chain rule given by equation (4) holds for the correct choice of the fractal index σ_{α}. We emphasize that the fractal index has to be determined separately in each problem.

Since

$$\frac{\partial^{\alpha}}{\partial\left|x\right|^{\alpha}}\equiv-\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}\left[D_{x^{-}}^{\alpha}+D_{x^{+}}^{\alpha}\right],\qquad i.e.\qquad D_{x^{-}}^{\alpha}\equiv-2\cos\left(\frac{\alpha\pi}{2}\right)\frac{\partial^{\alpha}}{\partial\left|x\right|^{\alpha}}-D_{x^{+}}^{\alpha}. \tag{12}$$

Equation (2) can be rewritten as follows:

$$\frac{\partial^{2}u\left(x,t\right)}{\partial t^{2}}-g_{s-1}\frac{\partial^{s-1}u\left(x,t\right)}{\partial\left|x\right|^{s-1}}-q_{s-1}u\left(x,t\right)\frac{\partial^{s-1}u\left(x,t\right)}{\partial\left|x\right|^{s-1}}+h_{s-1}u\left(x,t\right)\frac{\partial^{s-1}u\left(x,t\right)}{\partial x^{s-1}}=0, \tag{13}$$

where

$$q_{s-1}=\frac{\gamma\pi}{\Gamma\left(s\right)\cos\left(\frac{s}{2}\pi\right)}. \tag{14}$$

4. Ansatz for the dynamics

A possible way to construct solutions of equation (13), the one we explore in this paper, is to assume that the field $u(x,t)$ depends not on x and t separately, but via a combination of them defining a new variable η. That is typical when looking for solutions having solitonic form, and in that case one puts $\eta=\mathrm{x}-\mathrm{vt}$, as routinely done in studying soliton solutions of nonlinear differential equations [27]. However, despite one can certainly investigate such dependence, putting $\eta=\mathrm{x}-\mathrm{vt}$ does not reduce the FBE to an ordinary differential equation. By inspection one sees that this a direct consequence of the fractional derivative, and that in turn it is ultimately caused by the presence of the long-range couplings. Moreover, one has a derivative with respect to the modulus of the variable x, also in the linear case ($\gamma=0$). Therefore, one is lead to consider the following ansatz:

$$G\left(\eta\right)\equiv u\left(x,t\right),\qquad\text{where}\qquad\eta=\left|x\right|^{\rho}-v\,t, \tag{15}$$

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with v a parameter having the dimension $[v] = L^{\rho} T^{-1}$ and ρ to be fixed. Ansatz (15) is often referred to as the fractional complex transform in the context of fractional differential equations [29, 30]. Notice that with the ansatz (15) the cone-like structure in the $(x - t)$ space defined by $\eta = 0$ is no longer a straight line, and this corresponds to accelerating or decelerating fields u according the value of ρ. One readily observes that to reduce the FBE to an ordinary differential equation one needs to put

$$\rho \equiv s - 1. \tag{16}$$

We pause here to comment that the ansatz (15) is less innocent that it may at first sight appears. First, it breaks the Galilean invariance, which is broken evidently by the initial condition. Second, it involves the space variable x in the form $|x|$, so that this implies that care has to be put on choosing the matching conditions in $x = 0$. Moreover, to compare with numerical simulations on the lattice one has to choose even in x initial conditions for the u. Third, and more important, by considering the ansatz (15) one is also assuming that if the $u(x, t = 0)$ at the initial time is not vanishing, also its velocity $\dot{u}(x, t = 0)$ is not vanishing. In other words, the ansatz (15) corresponds to a class of solutions with certain conditions to be satisfied at the initial time $t = 0$.

To further proceed, we observe that it exists a duality between left and right fractional derivatives [28]:

$$D^{\alpha}_{x+} f\left(x\right) = D^{\alpha}_{x-} f^{*}\left(-x\right) \quad with \quad f^{*}\left(x\right) = f\left(-x\right).$$

Then it follows

$$\frac{\partial^{\alpha}}{\partial x_{-}^{\alpha}}(-x)^{\beta} \equiv \frac{\partial^{\alpha}}{\partial (-x)^{\alpha}}(-x)^{\beta} = \frac{\Gamma\left(1+\beta\right)}{\Gamma\left(1+\beta-\alpha\right)}(-x)^{\beta-\alpha} \qquad x < 0, \tag{17}$$

and

$$\frac{\partial^{\alpha}}{\partial \left|x\right|^{\alpha}}\left|x\right|^{\beta} = -\frac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta+1-\alpha\right)\cos\left(\frac{\alpha}{2}\pi\right)}\left|x\right|^{\beta-\alpha}, \qquad \alpha \neq 2k+1, \quad k \in \mathbb{R}. \tag{18}$$

In the case $\rho \to 2$, corresponding to $s \to 3$, we retrieve the well known integer classical derivative. Using equations (4), (5), (11), (15) and (18) into equation (13) we obtain the following ODE:

$$v^2 \frac{\mathrm{d}^2 G(\eta)}{\mathrm{d}\eta^2} + \tilde{g}_s \frac{\mathrm{d}G(\eta)}{\mathrm{d}\eta} + \tilde{h}_s G(\eta) \frac{\mathrm{d}G(\eta)}{\mathrm{d}\eta} = 0, \tag{19}$$

where

$$\tilde{g}_s \equiv -\frac{2\chi\pi}{\Gamma^2\left(s\right)\sin\left(s\pi\right)}$$

and

$$\tilde{h}_s \equiv \frac{\gamma\pi}{\Gamma^2\left(s\right)\sin\left(s\pi\right)}.$$

Integrating equation (19) with respect to η yields

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$$v^2\,\frac{\mathrm{d}G(\eta)}{\mathrm{d}\eta}\;+\;\tilde{g}_s\,G(\eta)\;+\;\frac{\tilde{h}_s}{2}\,G^2(\eta)\;+\;\delta_0\;=0, \tag{20}$$

where δ_0 is a constant to be later determined using appropriate boundary conditions. From equation (20) we can write

$$\int\frac{\mathrm{d}G}{\delta_0\;+\;\tilde{g}_s\,G\;+\;\frac{\tilde{h}_s}{2}\,G^2}=-\frac{1}{v^2}\;\int\mathrm{d}\,\eta. \tag{21}$$

The integral on the lhs can be explicitly done

$$\int\frac{\mathrm{d}X}{a+b\,X+c\,X^2}=\begin{cases}-\frac{2}{\sqrt{-\Delta}}arc\tanh\left(\frac{b+2\,c\,X}{\sqrt{-\Delta}}\right) & \text{for}\quad \Delta<0,\\ -\frac{2}{b+c\,X} & \text{for}\quad \Delta=0,\\ \frac{2}{\sqrt{\Delta}}arc\tan\left(\frac{b+2\,c\,X}{\sqrt{\Delta}}\right) & \text{for}\quad \Delta>0,\end{cases} \tag{22}$$

with $\Delta\equiv 4\,a\,c-b^2$ (see equation (2.172) in section 2.17 of [31]).

5. Solutions of the fractional Bousinessq equation

Using the results of the previous section, via the identification $a=\delta_0$, $b=\tilde{g}_s$ and $c=\tilde{h}_s/2$, one finds that the considered solution of equation (19) reads

$$G\,(\eta)=\begin{cases}2\,\frac{\chi}{\gamma}\;+\;\frac{\sqrt{-\Delta}}{\tilde{h}_s}\;\tanh\left[\frac{\sqrt{-\Delta}}{2\,v^2}\;(\eta-\eta_0)\right] & \Delta<0,\\ -\frac{4\,\chi}{\gamma}\;+\;\frac{4\,v^2}{\tilde{h}_s\,(\eta-\eta_0)} & \Delta=0,\\ 2\,\frac{\chi}{\gamma}\;-\;\frac{\sqrt{\Delta}}{\tilde{h}_s}\;\tan\left[\frac{\sqrt{\Delta}}{2\,v^2}\;(\eta-\eta_0)\right] & \Delta>0,\end{cases} \tag{23}$$

where

$$\Delta=2\delta_0\tilde{h}_s-\tilde{g}_s^2=\frac{2\,\pi\,\gamma}{\Gamma^2\,(s)\sin\,(s\pi)}\left[\delta_0-\frac{2\,(\chi^2/\gamma)\pi}{\Gamma^2\,(s)\sin\,(s\pi)}\right]. \tag{24}$$

Without any loss of generality we can make the choice $\eta_0=0$.

We have now to fix the boundary conditions. We look for the solutions that satisfy the following boundary conditions:

$$\begin{cases}G\,(\eta=0)=A,\\ G\,(\eta\to+\infty)=C,\end{cases} \tag{25}$$

where A and C are two constants to be later specified. Combining equations (23) and (25), it turns out that the solution reads

$$G\left(\eta\right)=2\,\frac{\chi}{\gamma}\;+\;\frac{\sqrt{-\Delta}}{\tilde{h}_s}\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta\right). \tag{26}$$

Equation (26) can be written also as

$$u\left(x,t\right)=2\,\frac{\chi}{\gamma}\;+\;\frac{\sqrt{-\Delta}}{\tilde{h}_s}\tanh\left[\frac{\sqrt{-\Delta}}{2\,v^2}\left(\left|x\right|^{s-1}-v\,t\right)\right], \tag{27}$$

where we have to specify the parameter δ_0 in terms of A and C. One sees that

$$A\equiv 2\,\frac{\chi}{\gamma},$$

with the constant A to be taken positive, in agreement with our choice of considering χ and γ positive parameters. Moreover it has to be

$$C\neq A.$$

The point essential for the subsequent discussion is that

$$C=A+\frac{\sqrt{-\Delta}}{\tilde{h}_s}.$$

Since for $1<s<2$ one has $\tilde{h}_s<0$ and for $2<s<3$ one has $\tilde{h}_s>0$, it follows that $C<A$ for $1<s<2$ and $C>A$ for $2<s<3$

In terms of C, Δ is given by

$$\Delta=-\tilde{h}_s^2\left(C-A\right)^2<0,$$

so that the choice in the top equation of (23) is justified. From this expression for Δ one gets the following value for δ_0:

$$\delta_0=-\frac{\tilde{h}_s^2\left(C-A\right)^2-\tilde{g}_s^2}{2\tilde{h}_s}. \tag{28}$$

The behaviour of the function G and therefore of $u(x,t)$ is summarized in figure 1. Figure 1(a) and (b) refer to a value of s between 2 and 3. The $G(\eta)$ is increasing for η passing from negative to positive values. In this case A and C have to be both positive, with $C>A$. The initial condition $u(x,t=0)$ has a 'hole', to which we may refer as an anti-pulse (notice that G has a typical kink-like form). As soon as that t increases the field $u(x,t)$ tends to reach the value $G(\eta\to-\infty)$, i.e. $A-|C-A|$, showing the delocalization of the initial condition. It is interesting to observe that the values of x and t, denoted by $\bar{x}$ and $\bar{t}$ such that $u(\bar{x},\bar{t})=A$ are defined by the relation $\bar{\eta}=0$, i.e. $|\bar{x}|^{s-1}=v\bar{t}$. Since $\mathrm{d}\bar{x}/\mathrm{d}\bar{t}=[v^{1/(s-1)}/(s-1)]\bar{t}^{(1/(s-1)-1)}$, for $2<s<3$ one sees that the velocity decreases and goes to zero for large time, so that the propagation of the hole *decelerates*. At variance figure 1(c) and (d) refer to a value of s between 1 and 2. The $G(\eta)$ is now decreasing for η passing from negative to positive values. In this case $A>0$ and $C<A$. The initial condition $u(x,t=0)$ has a 'bump', to which we may refer as a pulse (or an anti-kink for the G). When t increases, the field $u(x,t)$ tends to reach the value $G(\eta\to+\infty)$, i.e. $A+|C-A|$, showing also now the delocalization of the initial condition. Since $\mathrm{d}\bar{x}/\mathrm{d}\bar{t}\propto\bar{t}^{(1/(s-1)-1)}$, for $1<s<2$ one sees that the velocity increases so

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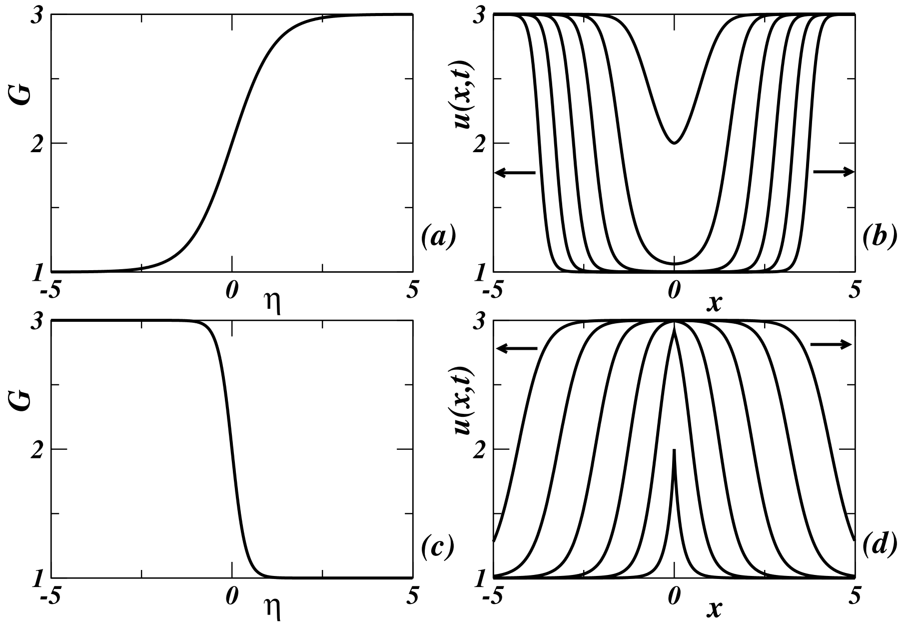

Figure 1. Panels (a)–(b): $s = 2.75$, with $G(\eta)$ versus η in (a) and $u(x,t)$ for different times $t = 0, 2, 4, 6, 8, 10$ in (b). The values of A and $|C - A|$ are 2 and 1, respectively. Panels (c)–(d): $s = 1.75$, with $G(\eta)$ versus η in (c) and $u(x,t)$ for different times $t = 0, 0.6, 1.2, 1.8, 2.4, 3$ (again with $A = 2$ and $|C - A| = 1$) in (d). v, χ and γ are constants of order unity.

that the propagation of the bump *accelerates* (reaching infinite velocity for $s = 1$, which represents the mean-field limit).

To conclude, we have to show that the results above do not qualitatively depend on the choice of the boundary condition (25). Suppose indeed that one fixes the function G in two points η_1, η_2, with the boundary conditions (25) corresponding to $\eta_2 = 0$ and $\eta_1 \to \infty$. Using (26) one has

$$G(\eta_1) - G(\eta_2) = \frac{\sqrt{-\Delta}}{\tilde{h}_s}\left[\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_1\right) - \tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_2\right)\right]. \tag{29}$$

Since $\tanh(a - b) = \frac{\tanh(a) - \tanh(b)}{1 - \tanh(a)\tanh(b)}$, one can rewrite equation (29) as follows:

$$G(\eta_1) - G(\eta_2) = \frac{\sqrt{-\Delta}}{\tilde{h}_s}\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,(\eta_1 - \eta_2)\right)\left[1 - \tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_1\right)\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_2\right)\right]. \tag{30}$$

Assuming $\eta_1 > \eta_2 > 0$ one has $0 < \tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_1\right)\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_2\right) \leqslant 1$, i.e.

$$1 - \tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_1\right)\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,\eta_2\right) \geqslant 0. \tag{31}$$

Equation (31) simply implies that the sign of $G(\eta_1) - G(\eta_2)$ depends of the sign of $\frac{\sqrt{-\Delta}}{\tilde{h}_s}\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,(\eta_1 - \eta_2)\right)$. With $\eta_1 > \eta_2 > 0$ as we are assuming, one has

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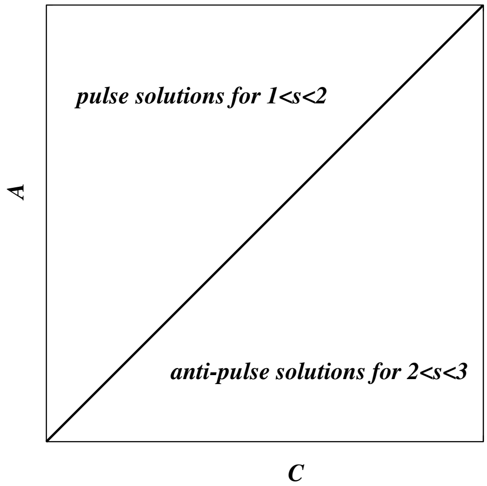

Figure 2. Representation of the phase diagram in the (A, C) plane with: *(i)* for $1 < s < 2$ anti-kink solutions for the G, the corresponding counterpart for the field u being pulse solutions; *(ii)* for $2 < s < 3$ kink solutions for the G, corresponding to anti-pulse solutions for the field u.

$$\tanh\left(\frac{\sqrt{-\Delta}}{2\,v^2}\,(\eta_1-\eta_2)\right) > 0. \tag{32}$$

One then sees that the sign of $G\,(\eta_1) - G\,(\eta_2)$ depends only on the sign of $\tilde{h}_s$: since it is $\tilde{h}_s < 0$ for $1 < s < 2$ and $\tilde{h}_s > 0$ for $2 < s < 3$ we conclude that indeed

$$\begin{cases} G\,(\eta_1) - G\,(\eta_2) < 0, \quad \text{for} \quad 1 < s < 2, \quad \Rightarrow \text{pulse solutions}, \\ G\,(\eta_1) - G\,(\eta_2) > 0, \quad \text{for} \quad 2 < s < 3, \quad \Rightarrow \text{anti}-\text{pulse solutions}. \end{cases} \tag{33}$$

The corresponding behaviour of the solutions of the FBE is represented in figure 2, where we confine ourself to the case $C > 0$ with $A > 0$ (for $C < 0$, since $A > 0$ one has anti-pulse solutions for $1 < s < 2$). The results are summarized by concluding that for any value of s in the range $1 < s < 2$ one obtains pulse solutions, while in the range $2 < s < 3$, one obtains anti-pulse solutions.

6. Conclusions

We have introduced a method which, relying on an appropriate ansatz, allows us to obtain exact solutions of the fractional differential equation derived in the continuum limit from the FPU lattice with long-range interactions. We are able to provide solutions of the effective fractional Bousinessq equation (FBE) which depend on the parameters of the model, in particular on the exponent s. Two classes of solutions emerge depending on the value of s: *(i) pulse* solutions displaying a 'bump' over a constant field in the range $1 < s < 2$; *(ii) anti-pulse* solutions showing a 'hole' in a constant field in the range $2 < s < 3$. As time progresses, the analytical solution reveals that both the 'bump' and the 'hole' damp down and, asymptotically at infinite time, the field becomes constant in space x, showing the delocalization of the initially localized condition. We have focused our analysis on the α-FPU model, but the method we present could be extended to the β-FPU model as well.

On one hand this is a specific property of a certain class of solutions, but on the other this could be an indication of the fact that the solutions of the FBE tend in general to delocalize in presence of long-range interactions. This is at variance with the short-range case where it is well known that a wide class of initial conditions leads to the creation of trains of solitons [5, 32].

Our results crucially depend on the ansatz presented in section 4. Numerical solutions of the lattice FPU model with long-range interactions would be needed in order to confirm the analytical predictions. In particular, we can already expect that localized solutions become progressively more unstable as soon as s decreases. Preliminary numerical investigation indeed do confirm that choosing a solution that is initially localized without initial velocity, one obtains stable solutions as soon as $s > 3$ [33]. However, the ansatz (15) requires that a suitable initial velocity is considered and to perform a proper comparison with analytical results one should appropriately choose the initial velocities $\dot{u}_i$ at time $t = 0$. Work in progress on this point is currently on-going in the region $1 < s < 3$, where large enough sizes of the lattice have to be considered.

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References

[1] Fermi E, Pasta J and Ulam S 1993 *Los Alamos Scientific Laboratory* Report No. LA-1940 (1955), reprinted in *The Many-Body Problem: an Encyclopedia of Exactly Solved Models in One Dimension* ed D C Mattis (Singapore: World Scientific)
[2] Dauxois T 2008 It is well known that there is a fourth author of the paper, Mary Tsingou Menzel *Phys. Today* **61** 55
[3] 2005 Special issue on the Fermi–Pasta–Ulam problem *Chaos* 15
[4] Gallavotti G (ed) 2008 *The Fermi–Pasta–Ulam Problem: a Status Report* (*Lecture Notes in Physics* vol 728) (Berlin: Springer)
[5] Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240
[6] Hénon M 1974 *Phys. Rev.* B **9** 1921
[7] Flaschka H 1974 *Phys. Rev.* B **9** 1924
[8] Campa A, Dauxois T and Ruffo S 2009 *Phys. Rep.* **480** 57
[9] Campa A, Dauxois T, Fanelli D and Ruffo S 2014 *Physics of Long-Range Interacting Systems* (Oxford: Oxford University Press)
[10] Levin Y, Pakter R, Rizzato F B, Teles T N and Benetti F P C 2014 *Phys. Rep.* **535** 160
[11] Bouchet F, Gupta S and Mukamel D 2010 *Physica* A **389** 4389

[12] Miloshevich G, Nguenang J P, Dauxois T, Khomeriki R and Ruffo S 2015 *Phys. Rev.* E **91** 032927

[13] Miloshevich G, Nguenang J P, Dauxois T, Khomeriki R and Ruffo S 2017 *J. Phys. A: Math. Theor.* **50** 12LT02

[14] Olivares C and Anteneodo C 2016 *Phys. Rev.* E **94** 042117

[15] Iubini S, Di Cintio P, Lepri S, Livi R and Casetti L 2018 *Phys. Rev.* E **97** 032102

[16] Di Cintio P, Iubini S, Lepri S and Livi R 2019 *J. Phys. A: Math. Theor.* **52** 274001

[17] Chendjou G N B, Nguenang J P, Trombettoni A, Dauxois T, Khomeriki R and Ruffo S 2018 *Commun. Nonlinear Sci. Numer. Simul.* **60** 115

Chendjou G N B, Nguenang J P, Trombettoni A, Dauxois T, Khomeriki R and Ruffo S 2019 *Commun. Nonlinear Sci. Numer. Simul.* **74** 282

[18] Miller K S and Ross B 1993 *An Introduction to the Fractional Calculus and Fractional Differential Equations* (New York: Wiley)

[19] Podlubny I and Thimann K V (ed) 1999 *Fractional Differential Equations* (San Diego, CA: Academic)

[20] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (Amsterdam: Elsevier)

[21] Defenu N, Trombettoni A and Codello A 2015 *Phys. Rev.* E **92** 052113

[22] Defenu N, Trombettoni A and Ruffo S 2016 *Phys. Rev.* B **94** 224411

[23] Defenu N, Codello A, Ruffo S and Trombettoni A 2019 (arXiv:1908.05158)

[24] Livi R, Pettini M, Ruffo S and Vulpiani A 1985 *Phys. Rev.* A **31** 2740

[25] Davydov A S 1980 *Theory of Solids* (Moscow: Nauka)

[26] Tarasov V E 2016 *Commun. Nonlinear Sci. Numer. Simul.* **30** 1

[27] Novikov S P, Pitaevskiĭ L P, Zakharov V E and Manakov S V 1984 *Theory of Solitons: the Inverse Scattering Method* (New York: Consultants Bureau)

[28] Caputo M C and Torres D F M 2015 *Signal Process.* **107** 265

[29] Li Z B and He J H 2010 *Math. Comput. Appl.* **15** 973

[30] Tarasov V E 2013 *Commun. Nonlinear Sci. Numer. Simul.* **18** 2945

[31] Gradshteyn I S and Ryzhik I M 2007 *Table of Integrals, Series, and Products* (New York: Academic) p 79

[32] Pace S D and Campbell D K 2019 *Chaos* **29** 023132

[33] Pham J 2019 *Master 1 Thesis Report*, *ENS of Lyon and University of Geneva*

J. Stat. Mech. (2019) 104015