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Research paper

Fermi-Pasta-Ulam chains with harmonic and anharmonic long-range interactions[☆]Gervais Nazaire Beukam Chendjou^{a,b,c,d}, Jean Pierre Nguenang^{a,b},
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ABSTRACT

We study the dynamics of Fermi-Pasta-Ulam (FPU) chains with both harmonic and anharmonic power-law long-range interactions. We show that the dynamics is described in the continuum limit by a Generalized Fractional Boussinesq differential Equation (GFBE), whose derivation is performed in full detail. We also discuss a version of the model where couplings are alternating in sign.

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1. Introduction

It is nowadays well recognized that the study of the Fermi-Pasta-Ulam (FPU) model, first performed six decades ago, leads to deep insights into the behavior of discrete nonlinear systems and into fundamental problems of statistical mechanics [1,2]. The FPU model describes the dynamics of an oscillator chain with nearest neighbor nonlinear couplings among masses. At low energies, it exhibits the phenomenon of mode recurrence [3], where the energy, initially fed into high wavelength modes, recursively returns to those modes. This remarkable effect was historically tackled by considering the continuum limit, pioneered in the classical paper by Zabusky and Kruskal [4]. In that limit one finds a Boussinesq equation [5] and, after a change of variables, a Korteweg-de Vries (KdV) equation [4] which gives an accurate description of waves and localized solutions of the FPU model. Performing in a controlled way the continuum limit for generalized FPU models is of paramount

[☆] Deriving fractional equations from physical models is an interesting research topic. We here show how to derive a class of fractional Boussinesq equations from several different Fermi-Pasta-Ulam models with long-range interactions. This is the first step in finding solutions in the continuum limit.

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importance, since it would allow one to construct solitonic solutions and to determine the low-energy properties of these models.

Our goal here is to derive the effective equations describing the dynamics of FPU models characterized by both harmonic and anharmonic Long-Range Interactions (LRI). LRI have been intensively studied in the last decades for a variety of physical systems [6] and considerable interest has been devoted to the study of power-law LRI. Pioneering work by Dyson [7] has revealed that the one-dimensional Ising ferromagnet with power-law couplings among the spins displays a non trivial phase transition for values of the power exponent s in the range $1 < s \leq 2$.

Coupled oscillators with power-law LRI were also studied [8,9,10–16]. The attention was mainly focused on long-range interactions in DNA [8,10], the existence of standing localized solutions like breathers [9], deriving the continuum counterpart of the discrete long-range models, which implies the use of fractional derivatives [11–13], weakly chaotic [14] and thermalization [15] properties caused by the long-range character of the interactions and the existence of solitons in a long-range extension of the quartic FPU chain [16].

In this paper, we will show that fractional differential equations describe the continuum limit of the FPU model with both harmonic and anharmonic power-law LRI. Fractional calculus has gained considerable interest and importance as an extension of differential equations with integer order derivatives. These techniques are used for the investigation of various problems in physics, engineering, life sciences and economy [17–21]. The use of fractional derivatives may lead to an elegant and more compact way of treating dynamical systems with non-local interactions and/or couplings. Typical examples are fractional diffusion equations derived from the dynamics of oscillator chains using a hydrodynamic approach [22,23]. Also physical phenomena with long memory [24,25] and random displacements with space jumps of arbitrary length [26] have been considered.

Fractional derivatives are also used in condensed matter physics. We can mention the recent studies of classical spin systems [27,28] and of fermionic quantum chains [29,30] with long-range couplings, which can be described by effective field theories with a dispersion relation associated in real space to a kinetic term with fractional derivatives. Spatial fractional diffusion was also observed in experimental cold atom physics [31] through a mechanism induced by the interaction of atoms with the laser fields.

We here derive the Generalized Fractional Boussinesq differential Equation (GFBE), which describes the dynamics of the FPU model with LRI in both the harmonic and anharmonic terms in the continuum limit.

The paper is organized as follows. In Section 2 we discuss the derivation of the GFBE for the α -FPU model. The situation in which the couplings alternate in sign is presented in Section 3. In Section 4, we repeat all the derivations for the β -FPU model. We discuss our results and perspectives in Section 5. In Appendix A we briefly review the basic definitions of fractional calculus. In Appendices B–D some derivations of lengthy formulas needed for the continuum approximation of long-range FPU models are presented. Appendix E is devoted to the case in which cubic and quartic terms are both present.

2. The α -FPU model with power-law long-range interactions

We consider in this Section a Hamiltonian where both the couplings in the harmonic and the anharmonic terms of the α -FPU model have a power-law behavior with different exponents s_1 and s_2 , respectively

$$H(u_n, \dot{u}_n) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \dot{u}_n^2 + \frac{\chi}{2} \sum_{n, m=-\infty}^{+\infty} \frac{[u_n - u_m]^2}{|a(n-m)|^{s_1}} + \frac{\gamma}{3} \sum_{n, m=-\infty}^{+\infty} \frac{[u_n - u_m]^3}{|a(n-m)|^{s_2}}, \quad (1)$$

where χ and γ are positive constants giving the strength of the quadratic and cubic potentials, n and m stand for the indices of the lattice sites and a is the lattice spacing.

We choose the power-law decay exponents in the range $1 < s_1, s_2 < 3$. For smaller values the Hamiltonian diverges, while, when the powers tend to infinity, we get back the conventional short-range α -FPU model. When both powers are finite and above 3, the system becomes effectively short-range. When s_1 (respectively s_2) is in the range (1, 3) and s_2 (resp. s_1) is above 3, then the effective continuum equation displays a long-range behavior in the anharmonic (harmonic) terms. The power $s_1 = 2$ (with $s_2 \rightarrow \infty$) has been considered for crack front propagation along disordered planes between solid blocks [32,33] and for contact lines of liquid spreading on solid surfaces [34].

From Eq. (1), we get the following equations of motion

$$\ddot{u}_n + \chi \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} + \gamma \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^2}{|a(n-m)|^{s_2}} f_{n,m} = 0 \quad (2)$$

where

$$f_{n,m} \equiv \begin{cases} +1, & m < n \\ -1, & m > n \end{cases}$$

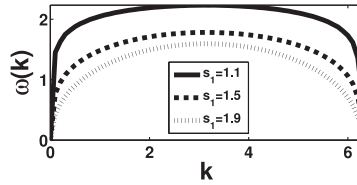


Fig. 1. Dispersion relation (4) for $s_1 = 1.1, 1.5, 1.9$.

To understand the qualitative property of the wave equation, we study the dispersion relation. We consider Eq. (2) in which we ignore the anharmonic terms. Thus, Eq. (2) admits the plane wave solution

$$u_n(t) = u_0 e^{i(\omega t - kx_n)}, \tag{3}$$

where k is the wavenumber and $x_n = na$. The corresponding linear dispersion relation is

$$\omega^2(k) = 2\chi \sum_{\ell=1}^{\infty} \frac{1 - \cos(k\ell)}{\ell^{s_1}}, \tag{4}$$

where from now on we put for simplicity $a \equiv 1$ (except when it is necessary for the analytical treatment). Eq. (4) is plotted in Fig. 1 for three specific values of s_1 . It is straightforward to observe that the dispersion relation diverges for $s_1 < 1$.

For small wavenumbers, $k \rightarrow 0$, $\omega(k) \propto |k|$ for $s_1 > 3$ and the phase and group velocities are constant. If instead $1 < s_1 < 3$, $\omega(k) \propto |k|^{(s_1-1)/2}$ and the phase and group velocities are given by $v_{ph} \propto |k|^{(s_1-3)/2}$ and $v_g \propto |k|^{(s_1-3)/2} (s_1 - 1)/2 = v_{ph} (s_1 - 1)/2$. They both diverge in the limit $k \rightarrow 0$.

Now we are ready for deriving the fractional partial differential equation which describes Hamiltonian (1) in the long wavelength, $k \rightarrow 0$, limit. Basic properties and definitions of fractional calculus are given for convenience in Appendix A. Let us start by defining the Fourier transform of $u_n(t)$ as

$$u_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikn} \hat{u}(k, t) \tag{5}$$

where

$$\hat{u}(k, t) = \sum_{n=-\infty}^{+\infty} e^{-ikn} u_n(t). \tag{6}$$

Substituting Eq. (5) into Eq. (2), one gets separately for the sums in the latter the following expressions

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n - m)|^{s_1}} \approx \frac{\pi}{\Gamma(s) \sin \frac{s_1-1}{2}\pi} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp e^{ipx} |p|^{s_1-1} \hat{u}(p, t), \tag{7}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n - m)|^{s_2}} f_{n,m} \approx \frac{u(x, t)}{2\pi} \int_{-\infty}^{+\infty} dp e^{ipx} \frac{2(2 - 2^{s_2-1})\pi}{\Gamma(s_2) \sin(s_2\pi)} ((-ip)^{s_2-1} - (ip)^{s_2-1}) \tilde{u}(p, t), \tag{8}$$

where $p \equiv k/a$ and $u_n(t) \rightarrow u(x, t)$ as $a \rightarrow 0$. Further details of the derivation of Eqs. (7) and (8) are given in Appendix B.

In the continuum limit, the Fourier sum become the Fourier transform, hence

$$\hat{u}(p, t) = \int_{-\infty}^{+\infty} dx e^{-ipx} u(x, t), \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp e^{ipx} \hat{u}(p, t). \tag{9}$$

We now introduce the Fourier transformation of fractional differentiation of order α via the relations

$$D_{x^+}^\alpha u(x, t) \equiv \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp (+ip)^\alpha e^{ipx} \hat{u}(p, t) \tag{10}$$

and

$$D_{x^-}^\alpha u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp (-ip)^\alpha e^{ipx} \hat{u}(p, t). \tag{11}$$

The Fourier transform involving the absolute value of momentum $|p|^\alpha$ is expressed by a Riesz derivative in real space as

$$-\frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp |p|^\alpha \hat{u}(p, t) e^{ipx}, \tag{12}$$

and, using Eqs. (10) and (11), one gets

$$\frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) = -\frac{1}{2\cos \frac{\alpha\pi}{2}} [D_{x^+}^\alpha + D_{x^-}^\alpha] u(x, t). \tag{13}$$

Combining Eqs. (7) and (8) with Eqs. (10), (11) and (13), one ends up with

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \approx -\frac{\pi}{\Gamma(s_1) \sin \frac{s_1-1}{2}\pi} \frac{\partial^{s_1-1}}{\partial |x|^{s_1-1}} u(x, t), \tag{14}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{2(2-2^{s_2-1})\pi}{\Gamma(s_2) \sin(s_2\pi)} u(x, t) [D_{x^-}^{s_2-1} - D_{x^+}^{s_2-1}] u(x, t). \tag{15}$$

Finally, Eqs. (15), (14), (11), (10) and (2) combine into the following Generalized Fractional Boussinesq Equation (GFBE)

$$\frac{\partial^2 u(x, t)}{\partial t^2} - g_{s_1-1} \frac{\partial^{s_1-1} u(x, t)}{\partial |x|^{s_1-1}} - h_{s_2-1} u(x, t) [D_{x^-}^{s_2-1} - D_{x^+}^{s_2-1}] u(x, t) = 0, \tag{16}$$

where the constants g_{s_1-1} and h_{s_2-1} are given by

$$g_{s_1-1} = \frac{\chi\pi}{\Gamma(s_1) \sin \left(\frac{s_1-1}{2}\pi\right)}, \quad h_{s_2-1} = \frac{2(2-2^{s_2-1})\gamma\pi}{\Gamma(s_2) \cos \frac{s_2\pi}{2}}. \tag{17}$$

Eq. (16) is the main result of the paper.

3. α -FPU model with alternating masses and interactions

The main goal of the paper is to establish that general FPU chains with LRI are mapped onto fractional equations of motion and we aim at illustrating it in a variety of models. To investigate how general is this mapping, we consider in this Section an α -FPU model with alternating signs in kinetic and interacting terms. Despite the physical realization of both alternating masses and interaction terms is certainly not easily implementable, alternating/varying interactions are rather common and one can think to implement alternating in sign hopping coefficients in ultracold chains using a variation of well-known shaking techniques [35]. The corresponding Hamiltonian is written as

$$H(u_n, \dot{u}_n) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \dot{u}_n^2 (-1)^n + \frac{\chi}{2} \sum_{\substack{n, m=-\infty \\ m < n}}^{+\infty} \frac{[u_n - u_m]^2 (-1)^n}{|a(n-m)|^s} + \frac{\gamma}{3} \sum_{\substack{n, m=-\infty \\ m < n}}^{+\infty} \frac{[u_n - u_m]^3 (-1)^n}{|a(n-m)|^s}, \tag{18}$$

where χ and γ are positive constants. Here again the exponent is chosen in the range of $1 < s < 3$. Applying the Hamiltonian formalism to Eq. (18), one gets the following equation on the lattice

$$\ddot{u}_n + \chi \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^s} + \gamma \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^2}{|a(n-m)|^s} = 0. \tag{19}$$

We can obtain a plane wave solution as

$$u_n(t) = u_0 e^{i(\omega t - kx_n)}, \tag{20}$$

with the dispersion relation of the fractional wave equation given in Eq. (4).

To derive the continuum equation describing the system while using the long wavelength limit for the corresponding lattice-field model, we use the expressions of the Fourier series defined in Eqs. (5) and (6). The third term of Eq. (19) is therefore transformed (see Appendix C) into

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^s} \approx \frac{(2-2^{s-1})\pi}{\Gamma(s) \sin \left(\frac{s-1}{2}\pi\right)} \frac{u(x, t)}{2\pi} \int_{-\infty}^{+\infty} dp |p|^{s-1} e^{ipx} \tilde{u}(p, t). \tag{21}$$

Using Riesz fractional derivative, definitions Eqs. (12) and (13), we obtain

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^s} \approx -\frac{(2-2^{s-1})\pi}{\Gamma(s) \sin \left(\frac{s-1}{2}\pi\right)} u(x, t) \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x, t). \tag{22}$$

Finally, Eqs. (14), (22) and (19) can be combined into the following GFBE

$$\frac{\partial^2}{\partial t^2} u(x, t) - g_{s-1} \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x, t) - k_{s-1} u(x, t) \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x, t) = 0, \tag{23}$$

with

$$k_{s-1} = \frac{(2 - 2^{s-1})\gamma\pi}{\Gamma(s) \sin \frac{s-1}{2}\pi}. \tag{24}$$

Despite the presence of the alternating terms $(-1)^n$ in Eq. (18), one finds again a fractional differential equation, even simpler than Eq. (1). One can also write Eq. (23) as

$$u_{tt} - g_{s-1} D_{|x|}^\alpha u - k_{s-1} u D_{|x|}^\alpha u = 0 \tag{25}$$

where $\alpha = s - 1$, $g_{s-1} \neq 0$ and $k_{s-1} \neq 0$ are non-zero constants.

4. The β -FPU model with power-law long-range interactions

In this Section we study the effect of LRI both in the harmonic and anharmonic terms of an extended β -FPU Hamiltonian model. The main difference with the previous Sections is that now the interaction is quartic instead of cubic - for the rest our goal is to parallel the results presented in the previous Sections. The model reads

$$H(u_n, \dot{u}_n) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \dot{u}_n^2 + \frac{\chi}{4} \sum_{\substack{n, m = -\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^2}{|a(n - m)|^{s_1}} + \frac{\gamma}{8} \sum_{\substack{n, m = -\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^4}{|a(n - m)|^{s_2}} \tag{26}$$

where χ and γ are positive constants. Hamiltonian (26) is referred to as the extended β -FPU model. Again we choose the parameters describing the order of the fractional space derivative in the range $1 < s_1, s_2 < 3$. The equations of motion read

$$\ddot{u}_n + \chi \sum_{\substack{m = -\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n - m)|^{s_1}} + \gamma \sum_{\substack{m = -\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^3}{|a(n - m)|^{s_2}} = 0 \tag{27}$$

Doing the same analytical calculation illustrated in detail in the previous Sections and Appendices B and C, we can derive the continuum equation describing the macroscopic system within the long-wavelength limit framework. The third term of Eq. (27) can be rewritten as (see for more details Appendix D)

$$\sum_{\substack{m = -\infty \\ m \neq n}}^{m = +\infty} \frac{(u_n - u_m)^3}{|a(n - m)|^{s_2}} \approx \frac{3(1 - 2^{s_2-1} - 3^{s_2-2})\pi}{\Gamma(s_2) \sin(\frac{s_2-1}{2}\pi)} \frac{u^2(x, t)}{2\pi} \int_{-\infty}^{+\infty} dp |p|^{s_2-1} e^{ipx} \tilde{u}(p, t). \tag{28}$$

Using Riesz fractional derivative definitions, Eqs. (12) and (13), we obtain

$$\sum_{\substack{m = -\infty \\ m \neq n}}^{m = +\infty} \frac{(u_n - u_m)^3}{|a(n - m)|^s} \approx - \frac{3(1 - 2^{s_2-1} - 3^{s_2-2})\pi}{\Gamma(s_2) \sin(\frac{s_2-1}{2}\pi)} u^2(x, t) \frac{\partial^{s_2-1}}{\partial |x|^{s_2-1}} u(x, t). \tag{29}$$

Substituting Eqs. (14) and (29) into Eq. (27) it follows that Eq. (27) is reduced to the following GFBE

$$\frac{\partial^2}{\partial t^2} u(x, t) - g_{s_1-1} \frac{\partial^{s_1-1}}{\partial |x|^{s_1-1}} u(x, t) - j_{s_2-1} u^2(x, t) \frac{\partial^{s_2-1}}{\partial |x|^{s_2-1}} u(x, t) = 0, \tag{30}$$

with

$$j_{s_2-1} = 3\gamma\pi \frac{1 - 2^{s_2-1} - 3^{s_2-2}}{\Gamma(s_2) \sin \frac{s_2-1}{2}\pi}, \tag{31}$$

Finally, we report in Appendix E the results for the $\alpha + \beta$ -FPU model in which cubic and quartic terms appear simultaneously.

5. Conclusions and perspectives

Starting from the lattice dynamics of the FPU model with both harmonic and anharmonic long-range power law couplings, we have derived in the continuum limit a Generalized Fractional Boussinesq Equation (GFBE) as it is usually done for short-range model [4,5]. We have performed the analytical derivations by two different methods: (i) using Riesz derivative

and Hurwitz formula of fractional calculus, (ii) performing a direct analysis of the Fourier spectrum in the $k \rightarrow 0$ limit. We also dealt with a variant of the model where masses and couplings are alternating in sign.

In general, the presence of long-range couplings in the FPU model is reflected into the appearance of nonlocal terms in the continuum equations, the nonlocality being mathematically represented by fractional derivatives. When the power of the couplings tends to infinity, the interactions become short-range, nonlocality is thus removed and fractional derivatives convert into ordinary partial derivatives. In this paper, the fractional derivatives are mainly considered in the Riesz sense.

Our systematic formulation of mechanics based on fractional derivatives can be used to develop models of biological systems in which fractional power-law interaction are essential elements of biological phenomena (for instance anomalous diffusion in cell biology). In the area of physics, fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle spreads at a rate inconsistent with classical Brownian motion [26]. In particular, the Riesz fractional derivative includes a left Riemann–Liouville derivative and a right Riemann–Liouville derivative that allows one to model flow regimes where impacts occur from either side of the domain [24].

In the future work, we plan to study solutions of the GFBE, Eqs. (16), (23), (30) and (E.3). Extended wave solutions could be derived using the homotopy analysis method [40] or the rotating wave approximation [41]. Also localized solutions may exist, but for what we could preliminarily find numerically [42], they seem to be unstable for the lattice equations. All these issues of correspondence between solutions of GFBE and FPU lattice equations will be the subject of forthcoming studies.

The effect of different boundary conditions should also be analyzed in much detail due to its importance for models with long-range interactions. Another challenge for the future would be to find the correct transformation of space and time variable which would allow one to derive from the GFBE the generalized KdV equation valid for long-range interactions.

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Appendix A. Basic definitions of fractional calculus

In this Appendix, we briefly review basic notions of fractional differentiation and integration. Fractional differentiation is an extension of the order of differentiation from an integer to a real number. We also introduce Riesz derivatives, which are the most useful representation applicable to a model with long-range interaction [36]. In addition, we present fractional integration, which is the inverse operation of fractional differentiation.

Definition 1. A real function $u(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbf{R}$ if there exists a real number $p > \mu$, such that $u(x) = x^p u_1(x)$, where $u_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $d^n u(x)/dx^n \in C_\mu$, $n \in \mathbf{N}$ [17].

Definition 2. The definition of a fractional derivative of order α ($\alpha > 0$) for $u(x) \in C_\mu$ ($\mu \geq -1$) with respect to x is formulated in the following two ways

$$D_{x^+}^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_{-\infty}^x dy u(y) (x-y)^{n-\alpha-1}, \quad (\text{A.1})$$

$$D_{x^-}^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx} \right)^n \int_x^{+\infty} dy u(y) (y-x)^{n-\alpha-1}. \quad (\text{A.2})$$

In this formula, the integer n is chosen for a given real number α such that $n-1 \leq \alpha < n$, and $\Gamma(n-\alpha)$ denotes Euler's gamma function. When an order α is an integer n , a fractional derivative Eqs. (A.1) and (A.2) is reduced to standard derivatives of integer order

$$D_{x^+}^n = \left(\frac{d}{dx} \right)^n, \quad D_{x^-}^n = \left(-\frac{d}{dx} \right)^n. \quad (\text{A.3})$$

Therefore, $D_{x^+}^n = (-1)^n D_{x^-}^n$, but in general $D_{x^+}^\alpha \neq (-1)^\alpha D_{x^-}^\alpha$; see Ref. [36] and

$$D_{x^+}^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \quad \text{for } x > 0. \quad (\text{A.4})$$

The formulation of fractional differentiation [17] includes integration of order α for $u(x)$, as defined in the following two ways

$$I_{x^+}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x dy u(y) (x-y)^{\alpha-1}, \quad (\text{A.5})$$

$$I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} dy u(y) (y - x)^{\alpha-1}. \tag{A.6}$$

$$I^0 u(x) = u(x). \tag{A.7}$$

Fractional differentiation of order α is the inverse operation of fractional integration of order α

$$D_{\mu^+}^\alpha I_{\mu^+}^\alpha u(x) = u(x), \quad D_{\mu^-}^\alpha I_{\mu^-}^\alpha u(x) = u(x). \tag{A.8}$$

If we assume $u(x)$ to be a regular function in $-\infty < x < \infty$, an integer derivative and a fractional derivative are commutable as follows

$$D_{\mu^+}^\alpha (\partial_v u(x)) = \partial_v (D_{\mu^+}^\alpha u(x)), \quad D_{\mu^-}^\alpha (\partial_v u(x)) = \partial_v (D_{\mu^-}^\alpha u(x)). \tag{A.9}$$

However, the commutation relation between two fractional derivatives is not satisfied in general

$$D_{\mu^+}^\alpha (D_{\mu^-}^\beta u) \neq D_{\mu^-}^\beta (D_{\mu^+}^\alpha u). \tag{A.10}$$

Definition 3. The Caputo fractional derivative D^α of $u(x)$ is defined [17] as

$$D^\alpha u(x) = \frac{1}{\Gamma(n - \alpha)} \int_0^x (x - \xi)^{n-\alpha-1} u^{(n)}(\xi) d\xi \quad (\alpha > 0), \tag{A.11}$$

for $n - 1 < \alpha \leq n$, $n \in \mathbf{N}$, $x > 0$, $u \in C_{-1}^n$. The following are two basic properties of the Caputo fractional derivative:

- (1) Let $u \in C_{-1}^n$, $n \in \mathbf{N}$. Then $D^\alpha u$, $0 \leq \alpha \leq n$ is well defined and $D^\alpha u \in C_{-1}$.
- (2) Let $n - 1 < \alpha \leq n$, $n \in \mathbf{N}$ and $u \in C_\lambda^n$, $\lambda \geq -1$.

Then one has

$$(I^\alpha D^\alpha)u(x) = u(x) - \sum_{k=0}^{n-1} u^{(k)}(0^+) \frac{x^k}{k!}, \quad (x > 0). \tag{A.12}$$

In this paper, only real and positive values of α have been considered. Similar to integer-order differentiation, the Caputo fractional differentiation is a linear operation

$$D^\alpha (\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x), \tag{A.13}$$

where λ, μ are constants.

Definition 4. A generalization of the classical Leibniz rule

$$D^n (fg) = \sum_{k=0}^{\infty} \binom{n}{k} (f^{n-k}) g^k \tag{A.14}$$

from integer n to fractional α contains an infinite series

$$D^\alpha (fg)(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D_x^{\alpha-k} f)(x) D_x^k g, \tag{A.15}$$

with $f(\xi)$ continuous in $[0, x]$ and $g(\xi)$ having $(n + 1)$ continuous derivatives in $[0, x]$. The sum is infinite and contains integrals of fractional order (for $k > [\alpha] + 1$).

Definition 5. Let n be the smallest integer that exceeds α , then the Caputo fractional derivative operator of order $\alpha > 0$ with respect to x_μ , is formulated [17] as

$$D_{x_\mu^+}^\alpha u(x) = \frac{1}{\Gamma(n - \alpha)} \int_{-\infty}^{x_\mu} dy \frac{\partial^n}{\partial y^n} u(x_1, \dots, y, x_{\mu+1}, \dots) (x_\mu - y)^{n-\alpha-1} \tag{A.16}$$

and

$$D_{x_\mu^-}^\alpha u(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_{x_\mu}^{+\infty} dy \frac{\partial^n}{\partial y^n} u(x_1, \dots, y, x_{\mu+1}, \dots) (y - x_\mu)^{n-\alpha-1} \tag{A.17}$$

if $n - 1 < \alpha < n$. Of course $D_x^\alpha u(x) = \frac{\partial^n u(x)}{\partial x^n}$ if $\alpha \in \mathbf{N}$. The corresponding Fourier transformation of order α is defined as [36]

$$D_{x^+}^\alpha u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp (+ip)^\alpha \hat{u}(p, t) e^{ipx} \tag{A.18}$$

$$D_{x^-}^\alpha u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp (-ip)^\alpha \hat{u}(p, t) e^{ipx}, \tag{A.19}$$

where

$$\hat{u}(p, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp u(x, t) e^{-ipx}. \tag{A.20}$$

Riesz derivatives can be defined as follows, except when α is an odd number [37]

$$\frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) = -\frac{1}{2\cos \frac{\alpha\pi}{2}} [D_{x^+}^\alpha + D_{x^-}^\alpha] u(x, t). \tag{A.21}$$

The above derivative is singular at $\alpha = 1, 3, 5, \dots$. Riesz derivatives have symmetry with respect to the transformation $x \rightarrow -x$. When $u(x)$ is a regular function in $-\infty < x < \infty$, the two formulations provided by Riemann–Liouville and Caputo are equivalent. We note that the commutation relation between the two fractional derivatives defined by Caputo is the same as that of the Riemann–Liouville derivatives.

Definition 6. The relationship between left and right-derivatives in real space reads

$$D_{x^-}^\alpha u(x) = D_{(-x)^+}^\alpha u(x) \tag{A.22}$$

if $u(x)$ is a real and even function, and

$$D_{x^-}^\alpha u(x) = -D_{(-x)^+}^\alpha u(x) \tag{A.23}$$

if $u(x)$ is a pure imaginary and odd function. Notice that the above transformation is valid only for $x < 0$.

Definition 7. The modified Riemann–Liouville derivative reads

$$D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (u(\xi) - u(0)) d\xi, \tag{A.24}$$

where u is a continuous (but not necessarily differentiable) function. However, there exists a non-commutative property

$$D^{\alpha+\beta} \neq D^\alpha D^\beta \neq D^\beta D^\alpha. \tag{A.25}$$

The Riemann–Liouville fractional derivative has some notable disadvantages in applications such as the nonzero value of the fractional derivative of constants,

$$D_t^\alpha C = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} C. \tag{A.26}$$

The Caputo fractional differentiation of a constant results in zero. Therefore, in order to define the usual initial value problem

$$u(t_0) = t_0, \quad (D_t^k f)(t_0) = C_k, \quad k = 1, \dots, n, \tag{A.27}$$

one is lead to the application of Caputo fractional derivatives instead of the Riemann–Liouville derivative.

Finally, we should clearly emphasize that many usual properties of the ordinary derivative D^n are not realized for fractional derivative operators D^α . For example, the Leibniz rule, chain rule, semi-group property ($D_x^\alpha D_x^\alpha \neq D_x^{2\alpha}$) have strongly complicated analogs for the operators D^α . We refer to [17,37,38] for more informations on the mathematical properties of fractional derivatives and integrals.

Appendix B

Substituting Eq. (5) into Eq. (2), one gets

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \sum_{\substack{m=-\infty \\ m-n \neq 0}}^{+\infty} \frac{1}{|a(n-m)|^{s_1}} e^{ikn} \hat{u}(k, t) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \sum_{\substack{m=-\infty \\ m-n \neq 0}}^{+\infty} \frac{1}{|a(n-m)|^{s_1}} e^{ikm} \hat{u}(k, t), \tag{B.1}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \sum_{\substack{m=-\infty \\ m-n \neq 0}}^0 \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} - \sum_{\substack{m=1 \\ m-n \neq 0}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}}. \tag{B.2}$$

Denoting $n' = m - n$, one obtains for the two sums on the r.h.s. of Eq. (B.2)

$$\sum_{\substack{m=-\infty \\ m-n \neq 0}}^0 \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} = \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \left(\sum_{n'=1}^{+\infty} \frac{1 - 2e^{-ikn'} + e^{-i(k+k')n'}}{|an'|^{s_2}} \right) e^{ikn} \tilde{u}(k, t), \tag{B.3}$$

$$\sum_{\substack{m=1 \\ m-n \neq 0}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} = \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \left(\sum_{n'=1}^{+\infty} \frac{1 - 2e^{ikn'} + e^{i(k+k')n'}}{|an'|^{s_2}} \right) e^{ikn} \tilde{u}(k, t). \tag{B.4}$$

Hence,

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \left(\sum_{\substack{n'=-\infty \\ n' \neq 0}}^{+\infty} \frac{1}{|an'|^{s_1}} \right) e^{ikn} \hat{u}(k, t) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \left(\sum_{\substack{n'=-\infty \\ n' \neq 0}}^{+\infty} \frac{e^{ikn'}}{|an'|^{s_1}} \right) e^{ikn} \hat{u}(k, t), \tag{B.5}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \sum_{n'=1}^{+\infty} \left(\frac{(1 - 2e^{-ikn'} + e^{-i(k+k')n'}) - (1 - 2e^{ikn'} + e^{i(k+k')n'})}{|an'|^{s_2}} \right) e^{ikn} \tilde{u}(k, t) \tag{B.6}$$

$$= \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \sum_{n'=1}^{+\infty} \left(2 \left(\frac{e^{ikn'} - e^{-ikn'}}{|an'|^{s_2}} \right) - \left(\frac{e^{i(k+k')n'} - e^{-i(k+k')n'}}{|an'|^{s_2}} \right) \right) e^{ikn} \tilde{u}(k, t). \tag{B.7}$$

We can rewrite Eqs. (B.5) and (B.7) as

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikn} \hat{u}(k, t) [\tilde{f}_1(0) - \tilde{f}_1(k)], \tag{B.8}$$

and

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} = \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikn} \tilde{u}(k, t) [2\tilde{f}_2(k) - \tilde{f}_2(k+k')], \tag{B.9}$$

where

$$\tilde{f}_1(k) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{e^{ikn}}{|an|^{s_1}} \quad \text{and} \quad \tilde{f}_2(k) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{e^{ikn}}{|an|^{s_2}}. \tag{B.10}$$

One has $\sum_{n=1}^{\infty} \frac{e^{+ikn}}{|n|^s} = L_{i,s}(e^{+ikn})$, where $L_{i,s}$ is the polylogarithmic function

$$L_{i,s}(e^{\mu}) = \Gamma(1-s)(-\mu)^{s-1} + \sum_{n=0}^{\infty} \frac{\zeta(s-n)}{n!} (\mu)^n. \tag{B.11}$$

Using the Hurwitz formula [39], one can rewrite $\tilde{f}_1(k)$ as

$$\tilde{f}_1(k) = \sum_{n=1}^{+\infty} \frac{e^{-ikn} + e^{ikn}}{|an|^{s_1}} = a^{-s_1} \left[\Gamma(1-s_1)((-ik)^{s_1-1} + (ik)^{s_1-1}) + \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!} ((-ik)^{2n} + (ik)^{2n}) \right]. \tag{B.12}$$

Since

$$(ik)^\alpha + (-ik)^\alpha = 2|k|^\alpha \cos\left(\frac{\alpha\pi}{2}\right), \tag{B.13}$$

then

$$(ik)^{2n} + (-ik)^{2n} = 2(-1)^n |k|^{2n}. \tag{B.14}$$

Therefore

$$\tilde{f}_1(k) = 2a^{-s_1} \left[\Gamma(1-s_1) \cos\left(\frac{s_1-1}{2}\pi\right) |k|^{s_1-1} + \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!} (-1)^n |k|^{2n} \right]. \tag{B.15}$$

Using the relation

$$\Gamma(1-s) \cos\left(\frac{s-1}{2}\pi\right) = -\frac{\pi}{2\Gamma(s) \sin\left(\frac{s-1}{2}\pi\right)},$$

one gets

$$\tilde{f}_1(k) = 2a^{-s_1} \left[-\frac{\pi}{2\Gamma(s_1) \sin\left(\frac{s_1-1}{2}\pi\right)} |k|^{s_1-1} + \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!} (-1)^n |k|^{2n} \right], \tag{B.16}$$

and

$$\tilde{f}_1(0) - \tilde{f}_1(k) = \frac{\pi a^{-s_1}}{\Gamma(s_1) \sin\left(\frac{s_1-1}{2}\pi\right)} |k|^{s_1-1} - 2a^{-s_1} \sum_{n=0}^{\infty} \frac{\zeta(s_1-2n)}{(2n)!} (-1)^n |k|^{2n} + 2a^{-s_1} \zeta(s_1). \tag{B.17}$$

It is easy to observe that $\tilde{f}_1(0) - \tilde{f}_1(k) \sim \omega^2(k)$. In the same way, i.e. combining the polylogarithmic function and the Hurwitz formula, we obtain

$$2\tilde{f}_2(k) - \tilde{f}_2(k+k') = 2a^{-s_2} (2-2^{s_2-1}) \Gamma(1-s_2) [(-ik)^{s_2-1} - (ik)^{s_2-1}] + 2a^{-s_2} \sum_{n=0}^{+\infty} \frac{(2-2^{s_2-1}) \zeta(s_2-n)}{n!} [(-ik)^n - (ik)^n]. \tag{B.18}$$

In Eqs. (B.17) and (B.18), ζ is the Riemann zeta function. In the continuum limit ($k \rightarrow 0$) the long-wavelength modes are singled out and the leading terms of Eqs. (B.8) and (B.9) are derived from the first terms of the r.h.s. of Eqs. (B.17) and (B.18). Therefore,

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \left(\frac{\pi a^{-s_1}}{\Gamma(s_1) \sin\left(\frac{s_1-1}{2}\pi\right)} |k|^{s_1-1} \right) e^{ikn} \hat{u}(k, t). \tag{B.19}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \left\{ 2a^{-s_2} \Gamma(1-s_2) (2-2^{s_2-1}) [(-ik)^{s_2-1} - (ik)^{s_2-1}] \right\} e^{ikn} \tilde{u}(k, t). \tag{B.20}$$

Taking into account that $n = x_n/a$, in the continuum limit $a \rightarrow 0$ one can replace n by x/a , where x now labels a point in the real line. In this limit $u_n(t) = u(x, t)$, obtaining

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a} e^{ik\frac{x}{a}} \left(\frac{\pi}{\Gamma(s_1) \sin\left(\frac{s_1-1}{2}\pi\right)} \right) \left| \frac{k}{a} \right|^{s_1-1} \hat{u}(k, t). \tag{B.21}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a} \left\{ \frac{2(2-2^{s_2-1})\pi}{\Gamma(s_2) \sin(s_2\pi)} \left(\left(-i\frac{k}{a} \right)^{s_2-1} - \left(i\frac{k}{a} \right)^{s_2-1} \right) \right\} e^{ikn} \tilde{u}(k, t). \tag{B.22}$$

Setting $k/a = p$, it follows

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^{s_1}} \approx \frac{\pi}{\Gamma(s) \sin\frac{s_1-1}{2}\pi} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp e^{ipx} |p|^{s_1-1} \hat{u}(p, t), \tag{B.23}$$

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^{s_2}} f_{n,m} \approx \frac{u(x, t)}{2\pi} \int_{-\infty}^{+\infty} dp e^{ipx} \frac{2(2-2^{s_2-1})\pi}{\Gamma(s_2) \sin(s_2\pi)} [(-ip)^{s_2-1} - (ip)^{s_2-1}] \tilde{u}(p, t), \tag{B.24}$$

which are the correct expressions in the continuum limit.

Appendix C

One has

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^s} = \sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{u_n^2 - 2u_n u_m - u_m^2}{|a(n-m)|^s} \tag{C.1}$$

$$= \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk \sum_{\substack{n'=-\infty \\ n' \neq 0}}^{n'+\infty} \left(\frac{1}{|an'|^s} - 2 \frac{e^{ikn'}}{|an'|^s} + \frac{e^{i(k+k')n'}}{|an'|^s} \right) e^{ikn} \tilde{u}(k, t) \tag{C.2}$$

$$= \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikn} \tilde{u}(k, t) [\tilde{f}(0) - 2\tilde{f}(k) + \tilde{f}(k+k')], \tag{C.3}$$

where $\tilde{f}(k) = \sum_{n \neq 0} \frac{e^{ikn}}{|an|^s}$. Using again the Hurwitz formula, $\tilde{f}(0) - 2\tilde{f}(k) + \tilde{f}(k+k')$ is rewritten as

$$\begin{aligned} & \tilde{f}(0) - 2\tilde{f}(k) + \tilde{f}(k+k') \\ &= \frac{\pi a^{-s}}{\Gamma(s) \sin(\frac{s-1}{2}\pi)} \left(2|k|^{s-1} - |k+k'|^{s-1} \right) + 2a^{-s} \sum_{n=0}^{n=+\infty} \frac{\zeta(s-2n)}{(2n)!} (-1)^n \left(2|k|^{2n} - |k+k'|^{2n} \right) + 2a^{-s} \zeta(s). \end{aligned} \tag{C.4}$$

In the continuum limit ($k \rightarrow 0$), the leading term of Eq. (C.3) is derived from the first term of the r.h.s. of Eq. (C.4):

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^2}{|a(n-m)|^s} \approx \frac{u_n}{2\pi} \int_{-\infty}^{+\infty} dk e^{i\frac{k}{a}x} \left\{ \frac{\pi a^{-s}}{\Gamma(s) \sin(\frac{s-1}{2}\pi)} \left(2|k|^{s-1} - |k+k'|^{s-1} \right) \right\} \tilde{u}(k, t) \tag{C.5}$$

$$= \frac{u(x, t)}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a} e^{i\frac{k}{a}x} \left\{ \frac{\pi a^{-1}}{\Gamma(s) \sin(\frac{s-1}{2}\pi)} \left(2 \left| \frac{k}{a} \right|^{s-1} - \left| \frac{k+k'}{a} \right|^{s-1} \right) \right\} \tilde{u}(k, t) \tag{C.6}$$

$$= \frac{u(x, t)}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a} e^{i\frac{k}{a}x} \left(\frac{(2-2^{s-1})\pi}{\Gamma(s) \sin(\frac{s-1}{2}\pi)} \right) \left| \frac{k}{a} \right|^{s-1} \tilde{u}(k, t), \tag{C.7}$$

Appendix D

We start by observing

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^3}{|a(n-m)|^s} = \sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{u_n^3 - 3u_n^2u_m + 3u_nu_m^2 - u_m^3}{|a(n-m)|^s} \tag{D.1}$$

$$= \frac{u_n^2}{2\pi} \int_{-\infty}^{+\infty} dk \sum_{\substack{n'=-\infty \\ n' \neq 0}}^{n'+\infty} \left(\frac{1}{|an'|^{s_2}} - 3 \frac{e^{ikn'}}{|an'|^{s_2}} + 3 \frac{e^{i(k+k')n'}}{|an'|^{s_2}} - \frac{e^{i(k+k'+k'')n'}}{|an'|^{s_2}} \right) e^{ikn} \tilde{u}(k, t) \tag{D.2}$$

$$= \frac{u_n^2}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikn} \tilde{u}(k, t) [\tilde{f}_2(0) - 3\tilde{f}_2(k) + 3\tilde{f}_2(k+k') - \tilde{f}_2(k+k'+k'')]. \tag{D.3}$$

Once again, combining the polylogarithmic function and the Hurwitz formula, we obtain

$$\begin{aligned} & \tilde{f}_2(0) - 3\tilde{f}_2(k) + 3\tilde{f}_2(k+k') - \tilde{f}_2(k+k'+k'') = \frac{\pi a^{-s_2}}{\Gamma(s_2) \sin(\frac{s_2-1}{2}\pi)} \left(3|k|^{s_2-1} - 3|k+k'|^{s_2-1} - |k+k'+k''|^{s_2-1} \right) \\ & + 2a^{-s_2} \sum_{n=0}^{n=+\infty} \frac{\zeta(s_2-2n)}{(2n)!} (-1)^n \left(3|k|^{2n} + 3|k+k'|^{2n} + |k+k'+k''|^{2n} \right) + 2a^{-s_2} \zeta(s_2). \end{aligned} \tag{D.4}$$

Therefore

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^3}{|a(n-m)|^{s_2}} \approx \frac{u_n^2}{2\pi} \int_{-\infty}^{+\infty} dk e^{i\frac{k}{a}x} \left\{ \frac{\pi a^{-s}}{\Gamma(s) \sin(\frac{s-1}{2}\pi)} \left(3|k|^{s-1} - 3|k+k'|^{s-1} - |k+k'+k''|^{s-1} \right) \right\} \tilde{u}(k, t). \tag{D.5}$$

In the continuum limit

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{m=+\infty} \frac{(u_n - u_m)^3}{|a(n-m)|^{s_2}}$$

$$\approx \frac{u^2(x, t)}{2\pi} \int_{-\infty}^{+\infty} dk e^{i\frac{k}{a}x} \left\{ \frac{\pi a^{-1}}{\Gamma(s_2) \sin\left(\frac{s_2-1}{2}\pi\right)} \left(3 \left| \frac{k}{a} \right|^{s_2-1} - 3 \left| \frac{k+k'}{a} \right|^{s_2-1} - \left| \frac{k+k'+k''}{a} \right|^{s_2-1} \right) \right\} \tilde{u}(k, t) \tag{D.6}$$

$$= \frac{u^2(x, t)}{2\pi} \int_{-\infty}^{+\infty} \frac{dk}{a} e^{i\frac{k}{a}x} \left(\frac{(3 - 3 \cdot 2^{s_2-1} - 3^{s_2-1})\pi}{\Gamma(s_2) \sin\left(\frac{s_2-1}{2}\pi\right)} \right) \left| \frac{k}{a} \right|^{s_2-1} \tilde{u}(k, t). \tag{D.7}$$

Appendix E. The $(\alpha + \beta)$ -FPU model with power-law long-range interactions

For the sake of completeness, we report in this Appendix results for the $\alpha + \beta$ -FPU model in which cubic and quartic terms are both present. The Hamiltonian of the model reads

$$H(u_n, \dot{u}_n) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \dot{u}_n^2 + \frac{\chi}{4} \sum_{\substack{n, m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^2}{|a(n-m)|^s} + \frac{\gamma}{3} \sum_{\substack{n, m=-\infty \\ m < n}}^{+\infty} \frac{[u_n - u_m]^3}{|a(n-m)|^s} + \frac{\lambda}{8} \sum_{\substack{n, m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^4}{|a(n-m)|^s} \tag{E.1}$$

One gets

$$\ddot{u}_n + \chi \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{u_n - u_m}{|a(n-m)|^s} + \gamma \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^2}{|a(n-m)|^s} f_{n,m} + \lambda \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{[u_n - u_m]^3}{|a(n-m)|^s} = 0, \tag{E.2}$$

where again

$$f_{n,m} = \begin{cases} +1, & m < n \\ -1, & m > n. \end{cases}$$

In the continuum limit for a lattice field model, introducing Eqs. (14), (15) and (29) into Eq. (E.2), we get

$$\frac{\partial^2}{\partial t^2} u(x, t) - g_{s-1} \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x, t) - h_{s-1} u(x, t) [D_{x^-}^{s-1} - D_{x^+}^{s-1}] u(x, t) - r_{s-1} u^2(x, t) \frac{\partial^{s-1}}{\partial |x|^{s-1}} u(x, t) = 0, \tag{E.3}$$

where the constants g_{s-1} and k_{s-1} are given by Eq. (17) and Eq. (24), respectively (when $s_2 = s_1 = s$). $r_{s-1} = j_{s-1} (\gamma = \lambda)$ and j_{s-1} is given by Eq. (31).

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