Dynamics and Thermodynamics of Systems with Long-Range Interactions: An Introduction

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Abstract. We review theoretical results obtained recently in the framework of statistical mechanics to study systems with long-range forces. This fundamental and methodological study leads us to consider the different domains of applications in a trans-disciplinary perspective (astrophysics, nuclear physics, plasmas physics, metallic clusters, hydrodynamics,...) with a special emphasis on Bose–Einstein condensates.

The main issues discussed in this context are: non additivity, ensemble inequivalence, thermodynamic anomalies at phase transitions (e.g. negative specific heat), "convex intruders" in the entropy, non-extensive statistics and new entropies, coherent structures and self-consistent chaos, laser induced long-range interactions in cold atomic systems.

1 Introduction

Properties of systems with long-range interactions are to a large extent only poorly understood although they concern a wide range of problems in physics. Recently, the disclosure of new methodologies to approach the study of these systems has revealed its importance also in a trans-disciplinary perspective (astrophysics, nuclear physics, plasmas physics, Bose–Einstein condensates, atomic clusters, hydrodynamics,...). The main challenge is represented by the construction of a thermodynamic treatment of systems with long-range forces and by the understanding of analogies and differences among the numerous domains of applications.

Some promising results in this direction have been recently obtained in the attempt of combining tools developed in the framework of standard statistical mechanics with concepts and methods of dynamical systems. Particularly arduous, but very exciting, is the understanding of phase transitions for such systems which must be treated separately in the different statistical ensembles and reveal anomalies like negative specific heat and temperature jumps in the microcanonical ensemble. Important are also those aspects of non-equilibrium phenomena that involve the formation of chaotic coherent structures of extraordinary stability.

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This fundamental and methodological study should help us to detect the depth and the origin of the analogies found in the different domains mentioned above or on the contrary emphasize their specificities. In particular, we would like to put a special emphasis on Bose–Einstein Condensation (BEC), which could be the main field of applications, since experiments and theoretical ideas have reached an impressive quality in the last decade. In this domain, many inequivalences between ensembles have been reported and should be clarified. Moreover, long-range interactions in BEC have opened very exciting new perspectives to consider BEC as a model for other systems.

2 Why Systems with Long-Range Interactions Are Important?

2.1 The Problem of Additivity

The methods to describe a given system of N particles interacting via a gravitational potential in $1/r$ are dramatically dependent on the value N. If Newton showed the exact solution for $N = 2$, and one can expect to get a numerical solution in the range $N = 3 - 10^3$, the results are clearly out of reach for a larger number of particles. In addition, it is clear that the detailed knowledge of the evolution of the different trajectories is completely useless, since it is well known that these systems are chaotic as soon as N is greater than two. Therefore, one needs to get a *statistical* analysis, in order to get insights into the thermodynamical properties [1] of the system under study.

However, such statistical study leads immediately to unexpected behaviors for physicists used to neutral gases, plasmas or atomic lattices. The underlying reason is directly related to the long range of the interaction, and more precisely to the non additivity of the system.

To avoid misunderstandings, let us first clarify the definition of *extensivity* with respect to *additivity*. A thermodynamic variable, like the energy or the entropy, would be extensive, if it is proportional to the number of elements, once the intensive variables are kept constant. To be more precise, let us consider the mean-field Ising Hamiltonian,

$$H = -\frac{J}{N}\left(\sum_{i=1}^{N} S_i\right)^2 \quad , \tag{1}$$

where the spins $S_i = \pm 1$, $i = 1, \ldots, N$, are all coupled. Without the $1/N$ prefactor such a Hamiltonian would have an ill defined thermodynamic limit. This is correctly restored by applying the Kac prescription [2], within which the potential is rescaled by an appropriate volume dependent factor, here proportional to N: such a Hamiltonian is therefore extensive. Let us note in passing that this regularization is not always accepted. In cases with a kinetic energy term, such a regularization corresponds to a renormalization of the time scale. On the contrary, this Hamiltonian is not additive. Indeed, let us divide a system,

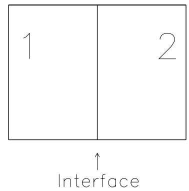

Fig. 1. Schematic picture of a system separated in two equal parts.

schematically pictured in Fig. 1, in two equal parts. In addition, one considers the particular case with all spins in the left part are equal to 1, whereas all spin in the right part are equal to -1. It is clear that the energy of the two different parts, will be $E_1 = E_2 = -\frac{J}{N}\left(\frac{N}{2}\right)^2 = -\frac{JN}{4}$. However, if one computes the total energy of the system, one gets $E = -\frac{J}{N}\left(\frac{N}{2} - \frac{N}{2}\right)^2 = 0$. It is therefore clear that such a system is not additive, since one cannot consider that $E_1 + E_2 = E$, even approximately. The energy of the interface, usually neglected, is clearly of the order of the energies of the two different parts: the system is not additive. The underlying reason is that Hamiltonian (1) is long (strictly speaking infinite) range, since every spin interact with all others: moreover, as the interaction is not dependent on the distance between spins, this is a mean-field model. This example is further elaborated in [4].

This non additivity has strong consequences in the construction of the canonical ensemble. Once the microcanonical ensemble has been defined, the usual construction of this ensemble is usually taught as follows. The probability that system 1 has an energy within $[E_1, E_1 + dE_1]$, given that the system 2 has an energy E_2, is proportional to $\Omega_1(E_1)\ \Omega_2(E_2)\ dE_1$, where the number of states of a system with a given energy E, $\Omega(E)$, is related to the entropy via the classical Boltzmann formula $S(E) = \ln E$ (we omit the k_B factor for the sake of simplicity). Using the additivity of the energy, and considering the case where system 1 is much smaller than system 2, one can expand the term $S_2(E - E_1)$, as shown in the following different steps

$$
\begin{aligned}
\Omega_1(E_1)\ \Omega_2(E_2)\ dE_1 &= \Omega_1(E_1)\ \Omega_2(E - E_1)\ dE_1 && (2)\\
&= \Omega_1(E_1)\ e^{S_2(E - E_1)}\ dE_1 && (3)\\
&= \Omega_1(E_1)\ e^{\left(S_2(E) - E_1 \frac{\partial S_2}{\partial E_2}_{|E} + ...\right)}\ dE_1 && (4)\\
&\propto \Omega_1(E_1)\ e^{-\beta E_1}\ dE_1 \quad , && (5)
\end{aligned}
$$

where $\beta = \frac{\partial S_2}{\partial E_2}_{|E}$. One ends up with the usual canonical distribution. It is clear that additivity is crucial to justify (2), which means that non additive systems will have a very peculiar behavior if there are in contact with a thermal reservoir. This is one of the topic discussed in this paper, and in numerous contributions in this book.

2.2 Definition of Long-Range Systems

To define now systems with long-range interactions, let us consider the potential energy for a given particle, situated in the center of a sphere of radius R, where mass or charge is homogenously distributed. We will omit at this stage the interaction of matter situated in a small neighborhood of radius $\varepsilon \ll R$ (see Fig. 2). The reason for excluding this neighborhood will be explained in the following subsection.

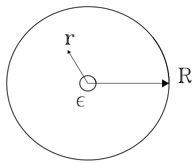

Fig. 2. Schematic picture of a particle interacting with all particles located in a homogeneous sphere of radius R, except the closest ones located in the sphere of radius ε.

If one considers that particles interact via a potential energy proportional to $1/r^\alpha$, where α is the key-parameter defining the range of interaction, we obtain in the three dimensional space

$$U = \int_\varepsilon^R 4\pi r^2 dr\ \rho\ \frac{1}{r^\alpha} = 4\pi\rho \int_\varepsilon^R r^{2-\alpha} dr \propto \left[r^{3-\alpha}\right]_\varepsilon^R \tag{6}$$

where ρ is the particle density. When increasing the radius R, the contribution due to the surface of the sphere, $R^{3-\alpha}$, could be neglected when $\alpha > 3$, but diverges if $\alpha < 3$. In the latter case, surface effects are important and therefore additivity is not fulfilled.

If one generalizes this definition to long-range systems in d dimensions, one easily shows that energy will not be additive if the potential energy behaves at long distance as

$$V(r) \sim \frac{1}{r^\alpha} \quad \text{with} \quad \frac{\alpha}{d} < 1 \quad . \tag{7}$$

Mean-field models, like Hamiltonian (1) correspond to the value $\alpha = 0$, since the interaction does not depend on the distance. They are therefore not additive as shown in Sect. 2.1. J. Barré et al consider [4] such a mean field model: the Blume-Emery-Griffith (BEG) model with infinite range interactions. The gravitational problem, which is at the origin of this study, and corresponds to $\alpha = 1$ in three dimensions, clearly belongs to this category, but presents also additional difficulties.

2.3 Difficulties with the Gravitational Problem

This problem is particularly tedious because, in addition to the non additivity due to long-range character, such a system needs a careful regularization at short distances to avoid collapse. To be more specific, let us consider the configurational partition function of a system of N particles

$$Z_U = \int_V d^{3N}\overrightarrow{r_i}\ e^{-\beta U(\overrightarrow{r_i})} \quad , \tag{8}$$

where one notes $U(\overrightarrow{r_i})$ the gravitational potential energy, β the inverse of the temperature and V the volume of the system. From the shape of the potential energy depicted in Fig. 3, one clearly see that Z_U will diverge if all particles collapse towards the same point. This difficulty arises because the potential energy is *not bounded from below* as for a Lennard-Jones or a Morse potential. This effect is of course physically forbidden by the Pauli principle. However, to avoid the use of Quantum Mechanics, the usual trick is to introduce an ad-hoc cut-off. The potential is therefore "regularized" by introducing the value $-C$, shown in Fig. 3. Thus, the inequality $U(\overrightarrow{r_i}) \geq -C$ allows easily to find a finite upper bound for the configurational partition function $Z_U \leq V\ e^{\beta C}$, where V is the volume of the system.

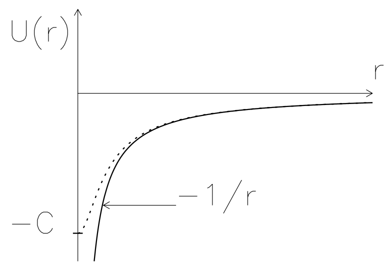

Fig. 3. The gravitational potential energy as a function of the distance r is represented by the solid curve, whereas the dotted one shows the regularized potential energy to avoid gravitational collapse.

However, there is a third difficulty in the case of gravitational interaction: the system is open, i.e. without boundary, strictly speaking. In the microcanonical ensemble, the number of states

$$\Omega(E) = \int \prod_i dp_i \int \prod_i dq_i\ \delta\left(E - H\left(q_i, p_i\right)\right) \tag{9}$$

will diverge if the system is not confined. This divergence is actually not restricted to the gravitational interaction but would also occur if one considers a perfect gas in an infinite volume. However, one considers of course always a gas in a finite domain, i.e. in a finite volume. This is not any more possible for the gravitational interaction where the system is clearly infinite.

Despite these additional difficulties, the astrophysics community has obtained an impressive quantity of results in this domain. Thanu Padmanabhan [5] describes several remarkable features, both for isolated gravitating systems as well as for systems undergoing nonlinear clustering in an expanding background cosmology. The emphasis is on general results and he brings out the interrelationships of this subject with topics in fluid mechanics, condensed matter and renormalization group theory.

Similarly, Pierre-Henri Chavanis [6] presents how the structure and the organization of stellar systems (globular clusters, elliptical galaxies,...) in astrophysics can be understood in terms of a statistical mechanics for a system of particles in gravitational interaction. Finally, Eddie Cohen and Iaroslav Ispolatov [7] consider the related gravitational-like collapse of particles with an attractive $1/r^\alpha$ potential. Using mean field continuous integral equation, they determine the saddle-point density profile that extremizes the entropy functional. For all $0 < \alpha < d = 3$, a critical energy is determined below which the entropy of the system exhibits a discontinuous jump.

2.4 Applications to Large Systems

A growing scientific community has recently begun to tackle the problem of long-range interactions with different viewpoints. One of the fascinating aspects of this problem is that, in addition to gravitating systems, it concerns a large variety of systems that we would like to discuss briefly in the following section.

Plasmas. Rarefied plasmas share many properties with collisionless stellar systems, and in particular the property that the mean field of the system is more important than the fields of individual nearby particles. Here again, the Coulomb force is of long-range character. However, there is a fundamental difference between plasmas and gravitation. Plasmas have both positive and negative charges, so that they are neutral on large scales and can form static homogeneous equilibria; on the contrary, gravitating systems can never form static homogeneous equilibria. This so-called Debye screening explains why many techniques of plasma physics can not be transferred immediately to the gravitational problem. Yves Elskens [8] and Diego Del Castillo Negrete [9] present some of their results in the framework of plasma physics.

2D Hydrodynamics. Two-dimensional incompressible hydrodynamics is another important case where the interaction is long-range. Indeed, the streamfunction ψ is related to the modulus of the vorticity ω, via the Poisson equation $\Delta\psi = \omega$. Using the Green's function technique, one easily finds that the solution is

$$\psi(\overrightarrow{r}) = -\frac{1}{2\pi} \int_D d^2\overrightarrow{r}'\, \omega(\overrightarrow{r}')\, G\left(\overrightarrow{r} - \overrightarrow{r}'\right) \quad , \tag{10}$$

where $G\left(\overrightarrow{r} - \overrightarrow{r}'\right)$ depends on D, but $G\left(\overrightarrow{r}\right) \sim |\ln \overrightarrow{r}|$, when $\overrightarrow{r} \to 0$. The kinetic energy being conserved by the Euler equation (dissipativeless), it is straightforward to compute it on the domain D, with boundary ∂D,

$$E = \int_D d^2\vec{r}\,\frac{1}{2}\left(\nabla\psi\right)^2 \tag{11}$$

$$= \oint_{\partial D} \vec{n}\,dl\;\psi\nabla\psi + \frac{1}{2}\int_D d^2\vec{r}\;\omega(\vec{r})\psi(\vec{r}) \tag{12}$$

$$= -\frac{1}{4\pi}\int\int_D d^2\vec{r}\,d^2\vec{r}'\;\omega(\vec{r}')\omega(\vec{r})\ln|\vec{r}-\vec{r}'| \tag{13}$$

since $\psi = 0$ on ∂D. This emphasizes that one gets a logarithmic interaction. The analogy is even more clear if one approximates the vorticity field by point vortices $\omega(\vec{r}) = \sum_i \Gamma_i\delta(\vec{r}-\vec{r}_i)$, located at $\vec{r}_i$, with a given circulation Γ_i. The energy of the system reads now

$$E = \frac{1}{2}\sum_{i\neq j}\Gamma_i\Gamma_j\ln|\vec{r}_i - \vec{r}'_j| \quad . \tag{14}$$

The interaction among vortices has a logarithmic character, which corresponds to $\alpha = 0$.

Pierre-Henri Chavanis [6] studies carefully the analogy between the statistics of large-scale vortices in two-dimensional turbulence and self-gravitating systems. This analogy concerns not only the equilibrium states, i.e. the formation of large-scale structures, but also the relaxation towards equilibrium and the statistics of fluctuations. Diego Del Castillo Negrete [9] discusses also his results in the framework of hydrodynamics.

Dipolar Interactions. Dielectrics and diamagnets in an external electric or magnetic field exhibit a shape dependent thermodynamic limit [11]. This is due to the marginal decay of the potential energy $\alpha = d = 3$ for systems of dipoles. There is some approach to the solution of this problem only in zero field and in the absence of spontaneous ferromagnetism [12]. This is a border case for the long-range interactions, but it deserves a special attention.

Fracture. Let us examine analytical solutions for the plane stress and displacement fields around the tip of a slit-like plane crack in an ideal Hookean continuum solid. The classic approach to any linear elasticity problem of this sort involves the search for a suitable “stress function” that satisfies the so-called biharmonic equation $\nabla^2(\nabla^2\psi) = 0$ where ψ is the Airy stress function, in accordance with appropriate boundary conditions. The deformation energy density is then defined as $U \propto \sigma\varepsilon$ where σ is the fracture stress field around the tip, whereas ε is the deformation field. Considering a crack-width a in a two-dimensional material and using the exact Muskhelishvili’s solution [10], one obtains the elastic potential energy due to the crack

$$U \simeq \frac{\sigma_\infty^2(1-\nu)}{2E}\,\frac{a^2}{r^2} \quad , \tag{15}$$

where E is the Young modulus, σ_∞ the stress field at infinity, ν the Poisson coefficient and r the distance to the tip: the elasticity equation in the bulk of solids leads therefore, again, to a border case for the long-range interactions since $U \sim 1/r^2$ in two dimensions. It appears that, despite of its engineering applications, the dynamics of this non conservative system has been very little studied, presumably because of its long-range character. In addition, in such a two dimensional material, the presence of several fractures could exhibit very interesting screening effects.

Table 1. Table listing different applications where systems are governed by long-range interactions. *Large systems* where the interactions is truly long-range and *small systems* where the range of the interactions is of the order of the size of the system are separated.

Interactions	α	α/d	Comments
Large systems			
Gravity	1	1/3	long-range
Coulomb	1	1/3	long-range with Debye screening
Dipole	3	1	Limiting value
2D Hydrodynamics	0	0	Logarithmic interactions
Fracture	2	1	Stress field around the tip
Small systems			
atomic and molecular clusters			
nuclei			
BE Condensation			

2.5 Applications to Small Systems

In addition to large systems where the interactions are truly long range, one should consider small systems where the range of the interactions is of the order of the size. In these cases, the system would not be additive either, and many similarities with the long-range case will be encountered. Phase transitions are universal properties of interacting matter which have been widely studied in the thermodynamic limit of *infinite* systems. However, in many physical situations this limit is not attained and phase transitions should be considered from a more general point of view. This is for example the case of some microscopic or mesoscopic systems: atomic clusters can melt, small drops of quantum fluids may undergo a Bose–Einstein condensation or a super-fluid phase transition, dense hadronic matter is predicted to merge in a quark and gluon plasma phase while nuclei are expected to exhibit a liquid-gas phase transition. For all these systems the experimental issue is how to characterize a phase transition in a *finite* system.

Philippe Chomaz and Francesca Gulminelli [13] discuss results from nuclear physics as well as from clusters physics. In particular, they propose a definition of first order phase transitions in finite systems based on topology anomalies of the event distribution in the space of observations. This generalizes the definitions based on the curvature anomalies [14] of thermodynamical potentials and provides a natural definition of order parameters. The new definitions are constructed to be directly operational from the experimental point of view. Finally, they show why, without the thermodynamic limit or at phase-transitions, the systems do not have a single peaked distribution in phase space.

In a closely related contribution, Dieter Gross [15] makes the statement, that the microcanonical ensemble with Boltzmann's principle $S = k_B \ln \Omega$ is the only proper basis to describe the equilibrium of a closed "small" system. Phase-transitions are linked to convex (upwards bending) intruders of the entropy, where the canonical ensemble defined by the Laplace transform to the intensive variables becomes multi-modal, non-local, and violates the basic conservation laws. The one-to-one mapping of the Legendre transform being lost, Gross claims that it is all possible to define phase transitions without invoking the thermodynamic limit, extensivity, or concavity of the entropy.

3 Thermodynamics

3.1 Inequivalence of Statistical Ensembles

Following the example exhibited long time ago by Hertel and Thirring [16], it is striking that these systems could lead to inequivalences between microcanonical, canonical or grand canonical ensembles. In this book, the first example is given by Barré et al [4] who present the Blume-Emery-Griffiths (BEG) model which allows a deep understanding of the fundamental reason why this happens. They studied the spin-1 BEG model both in the canonical and in the microcanonical ensemble. The canonical phase diagram exhibits a first order and a continuous transition lines which join at a tricritical point. It is shown that in the region where the canonical transition is first order, the microcanonical ensemble yields a phase diagram which differs from the canonical one. In particular it is found that the microcanonical phase diagram exhibits energy ranges with negative specific heat and temperature jumps at the transition energies. The global phase diagrams in the two ensembles and their multicritical behavior are calculated and compared.

Pierre-Henri Chavanis [6] shows similar features in self-gravitating systems where canonical and microcanonical tricritical points do not coincide either, as shown in Fig. 4 in the framework of self-gravitating fermions. Let us emphasize that this property survives to the introduction of a finite cut-off instead of quantum degeneracy as discussed by Chavanis.

3.2 Negative Specific Heats

This fact produces striking phenomena in the microcanonical ensemble, since it may result in a negative specific heat, as was emphasized by Eddington in

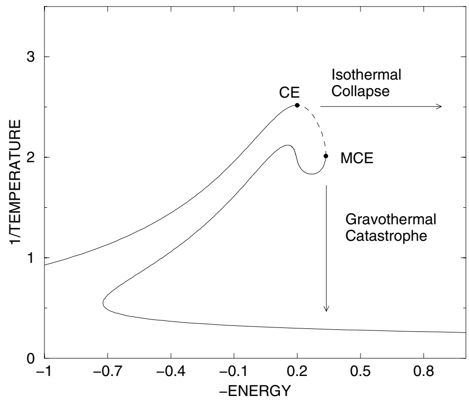

Fig. 4. Inverse temperature as a function of the energy for self-gravitating fermions without cut-off. CE (MCE) is the transition point in the canonical (microcanonica) ensemble. The dashed curve between CE and MCE has negative specific heat.

1926 [17] and then discussed by Lynden-Bell [18]. A first remark on the possibility of having negative specific heat in the microcanonical ensemble can even be found in the seminal paper on statistical mechanics by J.C. Maxwell [19]. Thirring [16] has finally clarified the point by showing that the paradox disappears if one realizes that only the microcanonical specific heat could be negative.

Indeed, in the canonical ensemble the mean value of the energy of a system with different energy levels E_i is

$$\langle E \rangle = \frac{\sum_i E_i \; e^{-\beta E_i}}{Z} = -\frac{\partial \ln Z}{\partial \beta} \tag{16}$$

where Z is the partition function. It is then straightforward to compute the specific heat

$$C_v = \frac{\partial \langle E \rangle}{\partial T} \propto \langle (E - \langle E \rangle)^2 \rangle > 0 \quad . \tag{17}$$

This clearly shows that the canonical specific heat is always positive. Notice also that this condition is true for systems of any size, regardless of whether a proper thermodynamic limit exists or not.

This is not the case if the energy is constant as shows the simplified following derivation for the example of interacting self-gravitating systems. Using the virial theorem for such particles

$$2\langle E_c \rangle + \langle E_{pot} \rangle = 0 \quad , \tag{18}$$

one gets that the total energy

$$E = \langle E_c \rangle + \langle E_{pot} \rangle = -\langle E_c \rangle \quad . \tag{19}$$

As the kinetic energy E_c is by definition proportional of the temperature one gets that

$$C_v = \frac{\partial E}{\partial T} \propto \frac{\partial E}{\partial \langle E_c \rangle} < 0 \tag{20}$$

Loosing its energy, the system is becoming hotter.

It is important at this stage to make a short comment on the Maxwell construction, usually taught in the framework of the Van der Waals liquid-gas transition. The existence of a negative specific heat region corresponds to a convex intruder in the entropy-energy curve, as shown in Fig. 5. When the interactions are short range, the system will phase separate in two parts, corresponding to the two phases 1 and 2 with a molar fraction x, so that the free energy $xF_1+(1-x)F_2$ is lower than the original free energy. This is clearly possible if the energy cost of the interface is proportional to the surface whereas the energy gain is proportional to the volume of the phase. However, this is not any more possible when the interactions are long-range since, on one hand, it is not straightforward to define a phase and, moreover, the system is not additive. The Maxwell construction has to be redefined in this new framework.

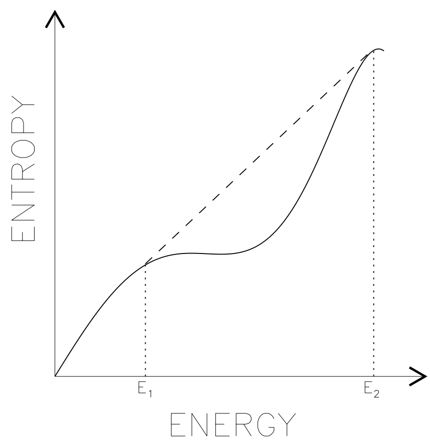

Fig. 5. Schematic shape of the entropy S as a function of the energy E with a convex intruder: the solid curve corresponds to the microcanonical result, whereas the dashed line to the canonical one.

Let us note that the microcanonical entropy as a function of the energy and of the order parameter generically leads to the landscape presented in Fig. 6. The projection for the critical points of the surface onto the entropy-energy plane produces the well known “swallowtail” catastrophe [20], depicted on the right of the figure. This corresponds to still another strange feature of the microcanonical ensemble, the presence of temperature jumps [4,6,7].

This concept of negative specific heat is now widely accepted in the astrophysical community, and was popularized in particular by Hawking [21] in 1974, with

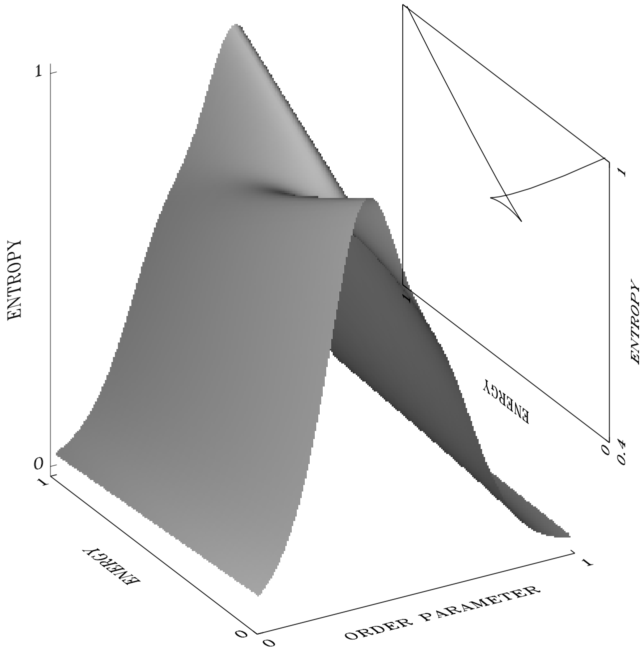

Fig. 6. A stylized microcanonical entropy as a function of the energy and of the order parameter mimicks an Alpine landscape where the workshop took place. The projection for the critical points of the surface onto the entropy-energy plane produces the well known “swallowtail” catastrophe.

some esoteric applications to black holes. The caloric curve of self-gravitating fermions derived by Chavanis and shown in Fig. 4 emphasizes such negative specific heat branch: the dotted branch is one example. Similarly one gets negative specific heat branch in the BEG model proposed by Barré et al [4]. In the canonical ensemble, they correspond to local maxima or saddle point of the corresponding free energy; it is the constraint of keeping the energy constant that stabilizes these canonical unstable states in the microcanonical ensemble.

Experimental groups have recently claimed signatures of negative specific heats in small systems. The first one corresponds to nuclear fragmentation [22], even if the authors use prudently the word “indication” of negative specific heat. The latter being inferred from the event by event study of energy fluctuations from Au + Au collisions. However, the signatures correspond to indirect measurements.

In the clusters community, two experimental groups have very recently reported negative specific heat. The first system [23] corresponds to atomic sodium clusters, namely Na^{+}_{147} and the negative microcanonical specific heat has been found near the solid to liquid transition. The cluster ion are produced in a gas aggregation source and then thermalized with Helium gas of controlled temperature. Accelerated thanks to the charge in a mass spectrometer, they are finally irradiated by a laser to determine the energy from the evaporation of several atoms after laser irradiation, also called photofragmentation. However, the control of equilibrium is as always the key point and therefore the evaluation of the temperature seems to be questionable, in particular since the temperature could not be constant during the motion of the ions.

In the Lyon's molecular cluster experiment [24], with H_{17}^+, the energy and the temperature are determined from the size distribution of fragments after collision of the cluster with a Helium projectile. To simplify the method, the larger the ratio of small fragments versus large ones, the larger is the temperature determined using the Bonasera et al procedure [25]. The reported caloric curve [24] shows a plateau. Work along this line is in progress and seems to show a negative specific heat region.

3.3 Non Extensive Statistics

Constantino Tsallis, Andrea Rapisarda, Vito Latora and Fulvio Baldovin [26] review the generalized non-extensive statistical mechanics formalism and its implications for different physical systems. The original very interesting idea is to generalize Boltzmann's entropy by defining

$$S_q = k_B \; \frac{1 - \sum_i {p_i}^q}{q-1} \tag{21}$$

where $\sum_i p_i = 1$. Using either the L'Hopital rule or a first order expansion of the term p_i^q in power of q, one immediately notices that

$$\lim_{q \to 1} S_q = -k_B \sum_i p_i \ln p_i \quad , \tag{22}$$

i.e. the well known Shannon entropy, known to be equivalent to the Boltzmann's one.

However, for q different from 1, this generalized entropy S_q is non additive, and one gets

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)\, S_q(A) S_q(B) \quad . \tag{23}$$

They illustrate in particular its application and the meaning of the entropic index q for conservative and dissipative low-dimensional maps. They also report on non Boltzmann-Gibbs behavior [26] and hindrance of relaxation for Hamiltonian systems with long-range interaction, where fingerprints of the generalized statistics have recently emerged.

This very interesting proposal [27] had however until now no strong foundations and many physicists were not ready to admit that the exponential Boltzmann distribution of states is at equilibrium only a particular case of a generalized distributions, with power tails. Dieter Gross [15] in particular makes different comments to this point. On the contrary, Tsallis et al emphasize also different situations were the Boltmann-Gibbs behavior is clearly not appropriate.

Recently, Beck and Cohen [28] showed that considering different statistics with large fluctuations, one can obtain generalized results, called superstatistics, with the Tsallis formalism being presumably so far the most relevant example. Moreover, Baldovin and Robledo worked out [29,26] exactly the q indices for the generalized largest Lyapunov exponent proposed by Tsallis for the logistic map. This an important step toward the derivation of a complete theory which, in particular, should help to understand the limits of its applications.

4 Dynamical Aspects

An essential peculiarity of these physical systems, and of some of their simplified models, is that a classical system of particles with long-range interactions will display strong non-equilibrium features. Dynamics is typically chaotic and self-consistent, since all particles give a contribution to the field acting on each of them: one calls this *self-consistent chaos*. Numerous physical systems fall in this category: galactic dynamics, dynamics of a plasma, vorticity dynamics,....

It is therefore essential to study the thermodynamic stability of these systems and in particular to understand the formation of structures trough *instabilities*. They should have logical similarities with the Jean's instability of self-gravitating systems, or with the modulational instability, leading to the formations of localized structures, as confirmed by preliminary results. Additional dynamical effects, like anomalous diffusion and Levy walks, which are reported in the negative specific heat regions, should be linked to these uncommon characteristics of thermodynamics [30].

In particular, Diego Del Castillo Negrete [9] discusses a mean-field single-wave model that describes the collective dynamics of marginally stable fluids and plasmas. He shows thus the role of self-consistent chaos in the formation and destruction of coherent structures, and presents a mechanism for violent relaxation of far from equilibrium initial conditions. The model bears many similarities with toy-models used in the study of systems with long range interactions in statistical mechanics, globally coupled oscillators, and gravitational systems.

One of these toy models is for example studied by Dauxois et al [31]. They consider the dynamics of the Hamiltonian Mean Field model which displays several interesting and new features. They show in particular the emergence of collective properties, i.e. the coherent (self-consistent) behavior of the particle motion. The space-time evolution of such coherent structures can sometimes be understood using the tools of statistical mechanics , otherwise can be a manifestation of the solutions of an associated Vlasov equation. Both cases in which the interaction among the particles is attractive and the one where it is repulsive are interesting to study: they offer different views to the process of cluster formation and to the development of the collective motion on different time-scales. The clustering transition can be first or second order, in the usual thermodynamical sense. In the former case, ensemble inequivalence naturally arises close to the transition. The behavior of the Lyapunov spectrum is also commented and the 'universal' features of the scaling laws that it involves.

Yves Elskens [8] shows that plasmas are a most common example of systems with long-range interactions, where the interplay between collective (wave) and individual (particle) degrees of freedom is well known to be central. This interplay being essentially non-dissipative, its prototype is described by a self-consistent Hamiltonian, which provides clear and intuitive pictures of fundamental processes such as the weak warm beam instability and Landau damping in their linear regimes. The description of the nonlinear regimes is more difficult. In the damping case, new insight is provided by a statistical mechanics approach, which identifies the distinction between a trapping behavior and linear Landau behavior in terms of a phase transition. In the unstable case, the model has shown that the commutation of long-time and large-N limits is not guaranteed.

Chavanis considers also dynamical aspects in the framework of stellar systems and two-dimensional vortices. He discusses in particular two possible relaxation scenarios: one due to collisions (or more precisely to discrete interactions) and the second one, called violent relaxation, really collisionless but due to the mean field effect and the long-range of the interaction.

Finally, the dynamical processes that give rise to power-law distributions and fractal structures have been studied extensively in the recent years. Ofer Biham and Ofer Malcai [32] describe recent studies of self-organized criticality in sandpile models as well as studies of multiplicative dynamics, giving rise to power-law distributions. Sandpile models turn out to exhibit universal behavior while in the multiplicative models the powers vary continuously as a function of the parameters. They consider the formation of a fractal object in the presence of a dynamical mechanism that generates a power-law distribution and present a model that demonstrates clustering when the probability of adding a particle decays with a power $\alpha > d$, so it has a short-range nature.

5 Bose–Einstein Condensation

Finally, we would like to put a special emphasis on Bose–Einstein Condensation (BEC), predicted by Bose and Einstein in 1924, which could be an important field of application. With the recent achievement [33] of Bose–Einstein condensation in atomic gases thanks to the evaporation cooling technique, it becomes possible to study these phenomena in an extremely diluted fluid, thus helping to bridge the gap between theoretical studies, only tractable in dilute systems, and experiments. In the BEC, atoms are trapped at such low temperatures that they tumble into the same quantum ground state creating an intriguing laboratory for testing our understanding of basic quantum phenomena.

First, Jean Dalibard [34] presents how coherence and superfluidity are hallmark properties of quantum fluids and encompass a whole class of fundamental phenomena. He reviews several experimental facts which reveal these two remarkable properties. Coherence appears in interference experiments, carried out either with a single condensate or with several condensates prepared independently. Superfluidity can be revealed by studying the response of the fluid to a rotating perturbation, which involves the nucleation of quantized vortices.

Second, Ennio Arimondo and Oliver Morsch [35] present the current investigations of Bose–Einstein condensates within optical lattices, where the long-range interactions are an essential part of the condensate stability. Previous work with laser cooled atomic gases is also briefly discussed.

On the theoretical side, the fluctuations of the number of particles in ideal Bose–Einstein condensates within the different statistical ensembles has shown interesting differences. Martin Holthaus explains [36] why the usually taught grand canonical ensemble is inappropriate for determining the fluctuation of the ground-state occupation number of a partially condensed ideal Bose gas: it predicts r.m.s.-fluctuations that are proportional to the total particle number at vanishing temperature. In contrast, both the canonical and the microcanonical ensemble yields fluctuations that vanish properly for the temperature going toward zero. It turns out that the difference between canonical and microcanonical fluctuations can be understood in close analogy to the familiar difference between the heat capacities at constant pressure and at constant volume. The detailed analysis of ideal Bose–Einstein condensates turns out to be very helpful for understanding the occupation number statistics of weakly interacting condensates.

Ulf Leohnardt [37] shows that Bose–Einstein condensates can serve as laboratory systems for tabletop astrophysics. In particular, artificial black holes can be made (sonic or optical black holes). A black hole represents a quantum catastrophe where an initial catastrophic event, for example the collapse of the hole, triggers a continuous emission of quantum radiation (Hawking radiation). The contribution summarizes three classes of quantum catastrophes, two known ones (black hole, Schwinger's pair creation) and a third new class that can be generated with slow light.

Finally, Gershon Kurizki presents [38] an exciting theoretical idea to induce long-range attractions between atoms that acts across the whole Bose–Einstein condensate. He shows that dipole-dipole interatomic forces induced by off-resonant lasers

$$V_{dd} = V_0 \left[\frac{2z^2 - x^2 - y^2}{r^3} (\cos qr + qr \sin qr) - \frac{2z^2 + x^2 + y^2}{r} q^2 \cos qr \right] \quad (24)$$

allow controllable drastic modifications of cold atomic media. "Sacrifying strength for beauty", Kurizki proposed [40] to average out the first term in $1/r^3$ of the dipole-dipole interaction by the different lasers, in order to keep only the last one with a $1/r$ interaction. The important point is that induced gravity-like force would be strong enough to see it acting among atoms in the BEC: i.e. that, having induced the gravity-like attraction in the BEC, one could switch off the trap used originally to create the BEC, and it will remain stable, holding itself together. Depending on the number of lasers, the resulting gravity-like force could be anisotropic for three lasers, or strictly identical to gravity with eighteen lasers ! If the last proposal is presumably too speculative and if the difficulties (the power of the laser required being really huge) facing the experimentalists are a real challenge, the ability to emulate gravitational interactions in the laboratory is of course fascinating. Indeed, these modifications may include the formation of self-gravitating "bosons stars" and their plasma-like oscillations, self-bound quasi-one-dimensional Bose condensates and their "supersolid" density modulation, giant Cooper pairs and quasibound molecules in optical lattices and anomalous scattering spectra in systems of interacting Bosons or Fermions. These novel regimes set the arena for the exploration of exotic astrophysical and condensed -matter objects, by studying their atomic analogs *in the laboratory.*

6 Conclusion

The dynamics and thermodynamics of long-range system is a rich and fascinating topic. We want to conclude with the following comments:

- long-range interactions are a rich laboratory for statistical physics. Let us only mention a few of the interesting phenomena and features: inequivalence of ensembles, negative specific heat, collisionless relaxation, role of coherent structures, nonadditivity, generalizations of entropy.
- This problem has also the nice property to be related to neighboring scientific disciplines. Let us mention mathematics, with the application of catastrophe theory [39] and large deviations theory [41], and computer science. In the latter, because of the long-range interactions, naive numerical codes are of order N^2, and the developments of efficient algorithms such as the heap based procedure [42] or local simulation algorithm for Coulomb interaction [43] is needed.
- This methodological and fundamental effort should provide a general approach to the problems arising in each specific domain which has motivated this study: astrophysical objects, plasmas, atomic and molecular clusters, fluid dynamics, fracture, Bose–Einstein condensation, ... in order to detect the depth and the origin of the observed analogies or, on the contrary, to emphasize their specificities.

Many of these different aspects are considered in this book but it is clear that, rather than closing the topic, it opens the pandora box.

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