Kinetics of anomalous transport and algebraic correlations in a long-range interacting system.

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Abstract. We propose a kinetic description of the Hamiltonian Mean Field model, which is paradigmatic for dynamical systems with long-range interactions. We predict algebraic tails for the momentum autocorrelations and anomalous diffusion for the angles. We derive analytically the corresponding laws in the limit of a large number of particles. We argue that the mechanism for such an anomalous transport does not depend on some complex structure of the phase space: indeed, the transport is anomalous for out-of-equilibrium distributions but also for the equilibrium microcanonical distribution.

1. Introduction

Since Boltzmann's seminal work on the "Kinetic theory of Gases", kinetic physics aims at bridging the gap between the microscopic dynamics of physical systems and their macroscopic behavior. The kinetic approach is also important for the study of transport properties. In this paper, we consider a kinetic approach to study anomalous transport in an Hamiltonian system with a large number of particle: the Hamiltonian Mean Field (HMF) model.

Recently, a new light was shed on long-range interacting systems [1]. The first reason is the broad spectrum of applications: self-gravitating and Coulomb systems, vortices in two-dimensional fluid mechanics, wave-particles interaction, trapped charged particles,... We consider here the HMF model, presented in Sec. 2, which is considered as the paradigmatic dynamical model for long-range interacting systems. Using classical kinetic theory, we then predict and characterize anomalous transport in this model.

Transport characteristics of many systems correspond to a self similar evolution of the probability distribution function, explaining the fundamental role of the heat equation in such a domain. The moment of order n of the distribution scales thus like $t^{n/2}$ at large time t. Such a transport is called *normal*. However, interest has also been concentrated on *anomalous* transport [2, 3, 4], where moments do not scale as in the diffusive case. It has indeed been observed that anomalous transport occurs in a large class of systems. In stochastic models, for instance, one of the mechanism leading to anomalous transport is linked to the non-validity of the law of large numbers. In continuous time random walks (Levy walks), such a behavior may be linked to distributions with large tails [5]. Another possible mechanism for the appearance of anomalous transport can be the lack of stationarity of the corresponding stochastic process [6] (acceleration).

For deterministic chaotic systems, it would interesting to link the transport properties to some characteristics of the dynamics. To reach this goal, the kinetic approach necessitates some probabilistic assumptions, which would take into account the essential features of the dynamics. The information kept from the dynamics determines indeed the characteristics of the kinetics. For instance, for Hamiltonian chaos, the detailed study of the structure of the phase space for several systems has led to the concept of dynamical barriers. Complex phase space structures (island, stochastic sea, stochastic web) may be responsible for the non uniformity in the phase space. Associated to this picture, anomalous transport may result from flights, stickiness... Such an anomalous transport has been described using fractional kinetics, in which the characteristics of the phase space structure should be linked to the properties of the kinetic equation (See Ref. [7] for a review).

For Hamiltonian systems, most of the studies have been performed for systems with a few degrees of freedom N. For a system with a large number of degrees of freedom ($N \gg 1$), one usually assumes that the effect of non uniformity in phase space is negligible. This is presumably a reasonable assumption, even if such an hypothesis is difficult to characterize and to quantify.

In this paper, we describe anomalous transport obtained by a mechanism which is not linked to non uniformity in the phase space. We consider the HMF model in the limit $N \to \infty$, and use standard methods in kinetic theory of plasmas. Using a perturbative approach valid for large N, we show that the Lenard-Balescu equation identically vanishes in such a system. In addition, we derive a Fokker-Planck equation which describes the stochastic process of the particle momentum. This classical kinetic approach is detailed in Sec. 3. If, at first sight, this seems to correspond to normal transport process, we however show that, because of a very strong decrease of the diffusion coefficient for large values of the momentum, this Fokker-Planck equation leads to anomalous transport. At the end of Sec. 3, we map the latter equation with a variable diffusion coefficient, to a Fokker-Planck equation with a logarithmic potential and a constant diffusion coefficient.

In Sec. 4, using perturbative methods, we analytically study the large time behavior of this latter Fokker-Planck equation. Let us emphasize that this study is independent on the HMF model, and gives a detailed analysis of Fokker-Planck equations leading to anomalous transport. We analytically compute the long-range temporal momentum autocorrelation laws, and we in particular explicitly characterize anomalous transport by the relevant exponents.

Finally, in Sec. 5, we use these results for the HMF model, and we analytically compute the anomalous diffusion coefficient for angles. Anomalous diffusion phenomena, indicating non standard diffusions in the asymptotic time-limit, were previously found numerically in this system and highly debated [8, 9]. The main achievement of this work is the theoretical prediction and analytical characterization of such anomalous transport from the microscopic Hamiltonian dynamics, and for a large class of initial distributions. We show how to characterize initial distributions leading to anomalous diffusion. The result that anomalous diffusion also occurs at equilibrium proves that the mechanism involved is independent of any lack of uniformity in phase space. One thus gives a mechanism to obtain anomalous transport, complementary to the one described in Ref. [7].

Moreover, the recent discovery of non gaussian distributions [10] in the HMF model led to an intense and productive debate on the applicability of usual Boltzmann-Gibbs statistical mechanics to long-range interacting systems (see Tsallis *et al.* in Ref. [1]). Such non Gaussian distributions have been fitted using Tsallis' distributions [10]. The striking algebraic large time behaviors for momentum autocorrelations have also been fitted using q-exponential functions [10, 11] derived from non extensive statistical mechanics. The analytical results of this paper, revisiting this question, explain these numerical findings, by relying only on usual statistical mechanics. We refer to Ref. [12] to further discussions of this point.

2. The Hamiltonian Mean Field Model

The Hamiltonian Mean Field model

$$H_N = \frac{1}{2}\sum_{j=1}^{N} p_j^2 - \frac{1}{2N}\sum_{i,j=1}^{N} \cos(\theta_i - \theta_j), \tag{1}$$

is nowadays thought to be the simplest model to study dynamical and thermodynamical properties of system with long-range interactions [1]. In addition to its pedagogical properties, it corresponds to a simplification of one-dimensional gravitational interactions and is an excellent first step before the Colson-Bonifacio's model for free-electron Lasers [13]. Note that the factor $1/N$ is the appropriate and classical *mean field scaling* relevant for long-range interacting systems [14], since the physically interesting limit for such systems amounts to let the number of particles go to infinity at fixed volume, by contrast with the usual thermodynamical limit. In the present work, we consider the case of an attractive potential. This model has been introduced independently in several contexts [15, 16, 17, 18]. In this section, we briefly review properties of this system, but see Ref. [19] for a complete presentation of the model. More recent results are cited when appropriate.

The corresponding equations of motion are

$$\dot{\theta}_i = p_i \quad \text{and} \quad \dot{p}_i = -\frac{1}{N}\sum_{j=1}^{N} \sin(\theta_i - \theta_j). \tag{2}$$

The state of the N-particles system can be described by the *discrete* single particle time-dependent density function

$$f_d\left(t, \theta, p\right) = \frac{1}{N}\sum_{j=1}^{N} \delta\left(\theta - \theta_j\left(t\right)\right)\delta\left(p - p_j\left(t\right)\right), \tag{3}$$

where δ is the Dirac function, (θ, p) the Eulerian coordinates of the $\mu-$space (projection of the phase space onto the one particle phase space) and (θ_i, p_i) the coordinates of the particles which verify Eqs. (2). The time evolution of the density f_d is governed by the Klimontovitch equation and is equivalent to the Hamiltonian evolution (2).

2.1. Microcanonical and canonical equilibria

Let us define the magnetization

$$\overrightarrow{M} = (M_x, M_y) = \frac{1}{N}\sum_{j=1}^{N}(\cos\theta_j, \sin\theta_j) \tag{4}$$

and its associated modulus $M = |\overrightarrow{M}|$. The statistical mechanics study has been performed both in the microcanonical [20, 21] and canonical ensembles [18]. Both ensembles are equivalent and the system exhibits a second order phase transition between a homogenous state ($\langle M\rangle = M_{eq} = 0$) at high energy, and a clustered phase ($\langle M\rangle = M_{eq} > 0$) at low energy (the bracket $\langle\cdot\rangle$ denotes phase space averages using either the microcanonical or the canonical measures). In both ensembles, the average density is given by $\langle f_d\rangle = f_{eq}(\theta, p) = A\exp[-\beta(p^2/2 - M_{eq}\cos\theta)]$, where A is a normalization constant and β is either the microcanonical or the canonical inverse temperature.

In the remainder of the text, we mostly consider out-of-equilibrium states.

2.2. Short time evolution. Vlasov dynamics

The evolution of the discrete single particle distribution f_d is equivalent to the Hamiltonian evolution. However, it is important to realize that we are not interested in the exact motion of all particles, far too precise for usual physical quantities of interest.

When N is large, it is natural to approximate f_d by the continuous function $f_0\,(t,\theta,p)$. Considering an ensemble of microscopic initial conditions close to the same initial macroscopic state, one defines their statistical average $\langle f_d\,(t,\theta,p)\rangle = f_0(t,\theta,p)$. Initially, the distribution fluctuations are of order $1/\sqrt{N}$, so that one can write

$$f_d(t=0,\theta,p) = f_0(t=0,\theta,p) + \frac{1}{\sqrt{N}}\,\delta f(t=0,\theta,p). \tag{5}$$

For systems with long-range interactions, it is well known that the short-time dynamics of the averaged distribution f_0 are well approximated by the Vlasov dynamics

$$\frac{\partial f_0}{\partial t} + p\frac{\partial f_0}{\partial \theta} - \frac{\mathrm{d}V}{\mathrm{d}\theta}\frac{\partial f_0}{\partial p} = 0, \tag{6}$$

where the potential V that affects all particles is

$$V(t,\theta) \equiv -\int_0^{2\pi}\mathrm{d}\alpha \int_{-\infty}^{+\infty}\mathrm{d}p\;\cos(\theta-\alpha)\,f_0(t,\alpha,p). \tag{7}$$

If one starts from an out-of-equilibrium distribution f_0, the latter relax towards an equilibrium distribution for the Vlasov equation. The time scale for this Vlasov relaxation does not depends on N. In a second stage of the dynamics, equilibrium distributions evolve due to corrections to the Vlasov equation towards the equilibrium distribution. The time of validity of the Vlasov approximation diverges with N, and is linked to the stability of the Vlasov solution f_0 [14]. For this reason, in the large N-limit, the time scale for Vlasov relaxation and relaxation towards equilibrium are well separated.

We are here interested in the second stage of the dynamics. We thus consider ensembles of initial distributions $f_0(p)$ corresponding to any homogeneous stable stationary solutions of the Vlasov equation. When f_0 is stable for the Vlasov equation, the evolution f_0 is expected to occur on times scaling at least as N, when N is large. Such an evolution is discussed in terms of kinetics equation in Sec. 3.

The linear stability of the Vlasov equation has been studied in Ref. [12], where the importance of the dielectric permittivity

$$\varepsilon(\omega,k) = 1 + \pi k\,(\delta_{k,1}+\delta_{k,-1})\int_{-\infty}^{+\infty}\mathrm{d}p\;\frac{\dfrac{\partial f_0}{\partial p}}{pk-\omega} \tag{8}$$

has been emphasized. For a given type of initial distribution, the condition for the neutral mode $\varepsilon(0,k)=0$ gives the stability threshold. In a recent paper, we have reported in terms of the density an explicit criterion for the study of the formal stability of the Vlasov equation [22]. In the HMF model, the formal stability and the linear stability criteria for the Vlasov equation turn out to be equivalent.

The huge quantity of such stationary stable solutions [22] explains the generic existence of out-of-equilibrium distributions. These quasi-stationary states do not evolve on time scales much smaller than N. This explains the extremely slow relaxation of the system toward the statistical equilibrium. In the following, we study transport properties of particles for ensembles of initial conditions close to some stable solution of the Vlasov equation.

3. Kinetic description

In this section, we are interested, on the one hand, in the evolution of the density f_0 on time scales of order N and, on the other hand, in the stochastic process describing the dynamics of a single particle in contact with an ensemble of particles characterized by their density f_0. Both of these problems are classical problems in kinetic theory of plasma physics. The approach developed for this system with long-range interactions is thus directly inspired from these technics.

Such a kinetic theory could be studied by perturbative expansions of

- the Liouville equation around the free particle motion and performing appropriate resummations [23, 24],
- the Liouville equation around the Vlasov evolution [25],
- the Klimontovich equation [26, 27].

In all cases, the small parameter of the expansion turns out to be $1/N$, and these three methods give completely equivalent results. This is why, for the sake of simplicity, we here follow the third alternative.

3.1. The Lenard Balescu equation

We recall that $f_0(p)$ is any stable homogeneous stationary solution of the Vlasov equation. The discrete time-dependent density function (3) can thus be rewritten as

$$f_d(t, \theta, p) = f_0(\theta, p) + \frac{1}{\sqrt{N}}\, \delta f(t, \theta, p), \tag{9}$$

where the fluctuation δf is of zero average. This decomposition is the self-consistent ansatz of the perturbative expansion, and makes sense only for perturbations around *stable* stationary solutions. To any fluctuation of the density δf, we associate a fluctuation of the potential (7) so that $V(t, \theta) = \langle V \rangle + \delta V(t, \theta)/\sqrt{N}$.

The leading order correction to the Vlasov equation is then given by

$$\frac{\partial f_0}{\partial t} = \frac{1}{N} \left\langle \frac{\mathrm{d}\delta V}{\mathrm{d}\theta} \frac{\partial \delta f}{\partial p} \right\rangle. \tag{10}$$

Let us remark that the Vlasov part of the evolution vanishes because f_0 is stationary.

At leading order, the fluctuation δf evolves, on the contrary, according to the linearized Vlasov equation, which can be solved explicitly using a Fourier-Laplace transform. It is therefore possible to compute the correlations; we refer to Ref. [27] for such lengthy but classical calculations. One obtains, for instance, the potential autocorrelation

$$\langle \delta V(t_1, \pm 1) \delta V(t_2, \mp 1) \rangle = \frac{\pi}{2} \int_{\mathcal{C}} \mathrm{d}\omega\; e^{-i\omega(t_1 - t_2)}\; \frac{f_0(\omega)}{|\varepsilon(\omega, 1)|^2}\;, \tag{11}$$

where the dielectric permittivity ε is defined in Eq. (8).

Following the same procedure, one can compute the right-hand-side of Eq. (10). In plasma physics, this lead to the Lenard-Balescu equation. In the case of the HMF model, it turns out that the left-hand-side of Eq. (10) identically vanishes at leading order. Indeed, in the expression of the Lenard-Balescu operator, a resonance condition appears, which can never be satisfied for a one dimensional model.

We thus conclude from this analysis that the Lenard-Balescu operator identically vanishes for the HMF model. As a consequence, the average density f_0 does not evolve on time-scales of order N or smaller. This is in agreement with the evolution of the distribution on times scaling as $N^{1.7}$, as it has been numerically found and reported in Refs. [9, 22].

3.2. Fokker-Planck equation for the stochastic process of a single particle
Let us now consider the relaxation properties of a test-particle, indexed by 1, surrounded by a background system of $(N-1)$ particles with a homogeneous distribution $f_0(p)$. Taking into account of the known position of the particle 1, the fluctuations of the potential are

$$\delta V(t,\theta) \equiv -\frac{N-1}{N}\int_0^{2\pi} \mathrm{d}\alpha \int_{-\infty}^{+\infty} \mathrm{d}p \;\cos(\theta-\alpha)\,\delta f(t,\alpha,p) - \frac{1}{\sqrt{N}}\cos\left(\theta-\theta_1\right). \qquad (12)$$

In the large N-limit, one can easily forget the prefactor $(N-1)/N$, unimportant at the order we consider. We note that the averaged potential $\langle V\rangle$ vanishes for a homogeneous distribution, so that the particle only feels the fluctuations of the potential. Its momentum thus evolves on time scales of order $\sqrt{N}$. Using the equations of motion of the test particle

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = p_1 \qquad \text{and} \qquad \frac{\mathrm{d}p_1}{\mathrm{d}t} = -\frac{\mathrm{d}\delta V(t,\theta_1)}{\mathrm{d}\theta} = \mathrm{o}\left(\frac{1}{\sqrt{N}}\right), \qquad (13)$$

and omitting the index 1 for the sake of simplicity, one obtains

$$\theta(t) = \theta(0) + p(0)\,t - \frac{1}{\sqrt{N}}\int_0^t \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}u_2 \frac{\mathrm{d}\delta V}{\mathrm{d}\theta}(u_2,\theta(u_2)) \qquad (14)$$

$$p(t) = p(0) - \frac{1}{\sqrt{N}}\int_0^t \mathrm{d}u \frac{\mathrm{d}\delta V}{\mathrm{d}\theta}(u,\theta(u)). \qquad (15)$$

The key point of this approach is that we do not limit the study to the usual ballistic approximation, in order to have an expansion exact at order $1/N$. This is of paramount importance here to treat accurately the essential *collective effects*.

By introducing iteratively the expression for the variable θ in the right-hand-side and by expanding the derivatives of the potential, one gets the result at order $1/N$

$$\begin{aligned} p(t) &= p(0) - \frac{1}{\sqrt{N}}\int_0^t \mathrm{d}u \frac{\mathrm{d}\delta V}{\mathrm{d}\theta}\left(u,\theta(0)+p(0)u\right) \\ &+\frac{1}{N}\int_0^t \mathrm{d}u \frac{\mathrm{d}^2\delta V}{\mathrm{d}\theta^2}\left(u,\theta(0)+p(0)u\right)\int_0^u \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}u_2 \frac{\mathrm{d}\delta V}{\mathrm{d}\theta}(u_2,\theta(0)+p(0)u_2). \end{aligned} \qquad (16)$$

As the changes in the impulsion are small, since of order $1/\sqrt{N}$, the description of the impulsion stochastic process by a Fokker-Planck equation is valid [28]. This latter equation is then characterized by the time behavior of the first two moments $\langle p(t)-p(0)\rangle$ and $\langle (p(t)-p(0))^2\rangle$. Using the generalization of formula (11) when the effect of the test particle is taken into account, one obtains in the large t-limit $(1 \ll t \ll N)$

$$\langle (p(t)-p(0))\rangle \sim \frac{t}{N}\left(\frac{\mathrm{d}D}{\mathrm{d}p}(p) + \frac{1}{f_0}\frac{\partial f_0}{\partial p}D(p)\right) \qquad (17)$$

$$\langle (p(t)-p(0))^2\rangle \sim \frac{2t}{N}D(p), \qquad (18)$$

where the diffusion coefficient $D(p)$ is defined by

$$D(p) = 2\,\mathrm{Re}\int_0^{+\infty} \mathrm{d}t\; e^{ipt}\;\langle \delta V(t,1)\delta V(0,-1)\rangle = \frac{\pi^2 f_0(p)}{|\varepsilon(p,1)|^2}. \qquad (19)$$

Equations (17) and (18) show that the momentum distribution of particle 1 evolve on timescales of order N. Let us introduce the rescaled time $\tau = t/N$, and the distribution for the dyed particle $f_1(\tau, p)$. Equations (17) and (18), valid for $1 \ll t \ll N$, prove that f_1 evolves according to the Fokker-Planck equation

$$\frac{\partial f_1}{\partial \tau} = \frac{\partial}{\partial p}\left[D(p)\left(\frac{\partial f_1}{\partial p} - \frac{1}{f_0}\frac{\partial f_0}{\partial p} f_1\right)\right], \tag{20}$$

valid for time $\tau \ll N$.

In the limit $\tau \to \infty$ provided the condition $1 \ll \tau \ll N$ is still fulfilled, the bracket vanishes: the pdf f_1 of the test particle converges towards the quasi-stationary distribution f_0 of the surrounding bath. It thus does not converge towards the equilibrium Gaussian distribution, in complete agreement with the result that f_0 is stationary for times scales of order N.

In order to illustrate the behavior of the diffusion coefficient, let us carry on by explicitly evaluating the diffusion coefficient for a homogenous Gaussian distribution function

$$f_{\mathrm{g}}(\theta, p) = \frac{1}{2\pi}\sqrt{\frac{\beta}{2\pi}}\; e^{-\beta p^2/2}. \tag{21}$$

In that case, after straightforward calculations, one gets the expression derived in Ref. [29] for the diffusion coefficient of a test particle in a equilibrium bath. The diffusion coefficient has gaussian-like tails (see Fig. 1), given by the asymptotic expression

$$D(p) \sim \sqrt{\pi\beta/2}\; e^{-\beta p^2/2}. \tag{22}$$

Interestingly, the method presented here can be used for *any Vlasov-stable out-of-equilibrium distributions.* For instance, in Fig. 1, we present the result for the waterbag distribution.

$$f_{\mathrm{wb}}(p) = \begin{cases} \dfrac{1}{4\pi p_0} & \text{if } |p| \le p_0 \\ 0 & \text{if } |p| > p_0 \end{cases} . \tag{23}$$

Let us remark that the existence of zeroes for the dielectric permittivity $\varepsilon(\omega, 1) = 1 - 1/[2(p_0^2 - \omega^2)]$ for $p = \pm\sqrt{p_0^2 - 0.5}$ is a peculiarity of the water bag distribution, since $\varepsilon(\omega, 1)$ cannot have zeroes on the real axis, for any even distributions *strictly* decreasing for $p > 0$. The diffusion coefficient is plotted in Fig. 1.

The knowledge of the Fokker-Planck Eq. (20) allows, for instance, to compute the momenta autocorrelation $\langle p(\tau)p(0)\rangle$, fitted numerically [11, 30, 9] with power laws, stretched-exponentials or q-exponentials. The time dependence of the momentum autocorrelation function scales with N as

$$\langle p(t)p(0)\rangle = C(t/N), \tag{24}$$

where C is a function to be determined.

Let us first present the particular but very important case of a test particle in contact with a *gaussian distribution* f_0 (equilibrium bath). Fig. 2 shows the momenta autocorrelation $\langle p(\tau)p(0)\rangle$, numerically computed from the Fokker-Planck Eq. (20); it presents an unexpected very slow relaxation which can, numerically, hardly be distinguished from a $1/\tau$-law.

As shown below, the very fast decrease of the diffusion coefficient shown in Fig. 1 is actually the key point in these interesting and unusual properties of the momenta autocorrelation. The qualitative explanation is that particles that have a large momentum p relax very slowly to typical values because of the extremely small value of the diffusion coefficient. This results in long-range temporal correlations and the associated long flight may lead to anomalous diffusion for the angles θ.

We would like to explain such a very interesting algebraic behavior for large time (long-range temporal correlation) by studying theoretically the Fokker-Planck equation (20). In the following subsection, we thus first map this Eq. (20) to a constant diffusion Fokker-Planck equation, before studying it in Sec. 4.

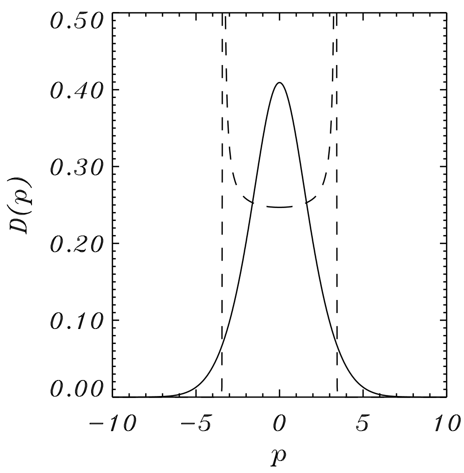

Figure 1. Diffusion coefficient $D(p)$ in the case $H_N/N = 2$ for a Boltzmann thermal bath (solid line) and a waterbag distribution (dashed line).

3.3. Mapping to a logarithmic potential Fokker-Planck equation

By introducing the appropriate change of variable $x = x(p)$, defined by $\mathrm{d}x/\mathrm{d}p = 1/\sqrt{D(p)}$, and the associated distribution function $\widehat{f}_1$, defined by $\widehat{f}_1(\tau, x)\mathrm{d}x = f_1(\tau, p)\mathrm{d}p$, one can map the Fokker-Planck equation (20) to the constant diffusion coefficient Fokker-Planck equation

$$\frac{\partial \widehat{f}_1}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\partial \widehat{f}_1}{\partial x} + \frac{\partial \psi}{\partial x}\widehat{f}_1\right), \tag{25}$$

where

$$\psi(x) = -\ln\left(\sqrt{D(p)}f_0(p)\right) \overset{p\to\pm\infty}{\sim} -\frac{3}{2}\ln\left(f_0(p)\right). \tag{26}$$

Let us note that we have used the property $\varepsilon(p, 1) \overset{|p|\to\infty}{\sim} 1$, and Eq. (19) to show that $D(p) \overset{|p|\to\infty}{\sim} \pi^2 f_0(p)$.

Let us now prove that for large classes of distribution functions f_0, the potential $\psi(x)$ is asymptotically equivalent to a logarithm. More precisely

$$\psi(x) \overset{x\to\pm\infty}{\sim} \alpha \ln|x|, \tag{27}$$

with $\alpha = 3$ if $f_0(p)$ decreases to zero more rapidly than algebraically for large p, and $\alpha < 3$ if $f_0(p)$ decreases to zero algebraically. To illustrate this point, we evaluate the asymptotic behavior explicitly in two cases: stretched exponential and algebraic tails.

3.4. Gaussian, exponential and stretched exponential tails for the bath distribution f_0

Let us first consider distribution functions so that

$$f_0(p) \overset{|p|\to\infty}{\sim} C\exp(-\gamma p^\delta), \tag{28}$$

which includes not only the gaussian ($\delta = 2$) and exponential tails ($\delta = 1$), but also stretched-exponential ones with $\delta > 0$.

From $\mathrm{d}x/\mathrm{d}p = 1/\sqrt{D(p)}$, asymptotic analysis leads to

$$p(x) = \left(\frac{2}{\gamma}\ln(|x|)\right)^{1/\delta} + o(1) \tag{29}$$

whereas $\psi(x) =_{x\to\pm\infty} 3\ln|x| - 3\gamma(1-\delta)/(2\delta)\ln\left(\ln|x|\right) + \mathcal{O}(\prime)$.

3.5. Algebraic tails for the bath distribution f_0

Let us now consider distribution function $f_0(p)$ with algebraic tails

$$f_0(p) \overset{|p|\to\infty}{\sim} Cp^{-\nu}, \tag{30}$$

where $\nu > 3$. The criteria $\nu > 3$ ensures that the second moment of the distribution f_0 exists. This is essential here since the second moment is linked to the kinetic energy.

Using $\mathrm{d}x/\mathrm{d}p = 1/\sqrt{D(p)}$, one can prove that Eq. (27) is still valid for distribution functions with algebraic tails. However, one obtains $\alpha = 3\nu/(2+\nu)$, so that for $\nu > 3$, one has $9/5 < \alpha < 3$. We also have

$$p(x) \overset{|x|\to\infty}{\sim} C'x^{\eta} \quad \text{with} \quad \eta = \frac{2}{2+\nu}. \tag{31}$$

4. Analysis of the logarithmic potential Fokker-Planck equation

Let us study now the long time behavior of a Fokker-Planck equation with constant diffusion coefficient and with a potential having an asymptotic logarithmic behavior. This will be used in Sec. 5 to predict anomalous diffusion for the HMF model.

4.1. The logarithmic potential Fokker-Planck equation

Let us consider the Fokker-Planck equation

$$\frac{\partial \widehat{f}_1}{\partial \tau} = \frac{\partial}{\partial x}\left(\frac{\partial \widehat{f}_1}{\partial x} + \frac{\partial \psi}{\partial x}\widehat{f}_1\right) \quad \text{with} \quad \psi(x) \overset{x\to\pm\infty}{\sim} \alpha \ln|x| \quad \text{and} \quad \alpha > 1. \tag{32}$$

Its ground state solution is

$$\varphi_0(x) = C\exp(-\psi(x)), \tag{33}$$

where C is a normalization constant. As $\ln(\varphi_0(x)) \overset{|x|\to\pm\infty}{\sim} \alpha \ln|x|$, the ground state φ_0 is normalizable provided $\alpha > 1$.

Let us show that this Fokker-Planck equation has long-range temporal correlations. For instance, for any odd function p_η such that $p_\eta(x) \overset{x\to+\infty}{\sim} x^\eta$, we have

$$\langle p_\eta(x(\tau))p_\eta(x(0))\rangle \overset{\tau\to+\infty}{\propto} \tau^{\eta-(\alpha-1)/2}, \tag{34}$$

provided $\eta < (\alpha-1)/2$, as it is necessary for the autocorrelations to be defined.

For a weakly confining potential $\psi(x)$, Eq. (32) would have a non-normalizable ground state. The heat equation, for example, which corresponds to $\psi(x) = 0$, describes a diffusive process leading to an asymptotic self-similar evolution. In such a case, the spectrum of the Fokker-Planck equation is purely continuous. By contrast, a strongly confining potential $\psi(x)$ (for instance, the Ornstein-Ulhenbeck process with a quadratic potential) would lead to exponentially decreasing distributions and autocorrelation functions, linked to the existence in the spectrum of a gap above the ground state.

The logarithmic potential (27) is a limiting case between these both behaviors. The normalizable ground state φ_0 is unique and coincides with the bottom of the continuum. The absence of gap forbids *a priori* any exponential relaxation.

4.2. Matched asymptotics expansion of the eigenfunctions

Let us define the eigenfunctions φ_λ and associated eigenvalues λ of the Fokker-Planck Eq. (32) by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\varphi_\lambda}{\mathrm{d}x} + \frac{\mathrm{d}\psi}{\mathrm{d}x}\varphi_\lambda\right) = -\lambda\varphi_\lambda \tag{35}$$

and by the normalization condition

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\varphi_\lambda(x)\varphi_{\lambda'}(x)}{\varphi_0(x)} = \delta(\lambda - \lambda') \; , \tag{36}$$

where the first eigenfunction is the ground state φ_0.

As we are interested in the asymptotic large-τ limit, we restrict the analysis to the small-λ values and determine φ_λ by matched asymptotic expansions. For $|x| > \ell(\lambda)$, the large-x asymptotic expansion of ψ can be used, whereas for $|x| < \ell(\lambda)$, we perform an expansion in power of λ. Part of this analysis is inspired from Refs. [31, 32].

In the first domain, $|x| > \ell(\lambda)$, we evaluate the leading order correction by introducing in Eq. (35), the asymptotic estimate $\psi(x) \overset{|x|\to\pm\infty}{\sim} \alpha \ln |x|$. Defining $z = \sqrt{\lambda}\, x$ and $g_\lambda(z) = z\varphi_\lambda(z/\sqrt{\lambda})$, one ends up with

$$z^2 \frac{d^2 g_\lambda}{dz^2} + \frac{dg_\lambda}{dz}(\alpha - 2)z + g_\lambda(2 - 2\alpha + z^2) = 0. \tag{37}$$

The solutions can be expressed in terms of Bessel functions of order ν, J_ν and Y_ν, as

$$g_\lambda(z) = A_\lambda \, z^{(3-\alpha)/2} J_{(\alpha+1)/2}(z) + B_\lambda \, z^{(3-\alpha)/2} Y_{(\alpha+1)/2}(z). \tag{38}$$

In the domain $|x| < \ell(\lambda)$, at leading order, one neglects the term proportional to the vanishing eigenvalue λ. Eq. (35) can thus be reduced to

$$\frac{d^2\varphi_\lambda}{dx^2} + \frac{d}{dx}\left(\varphi_\lambda \frac{d\psi}{dx}\right) = 0, \tag{39}$$

whose solutions are

$$\varphi_\lambda(x) = D_\lambda \, e^{-\psi(x)} + C_\lambda \, e^{-\psi(x)} \int_0^x \mathrm{d}u \, e^{\psi(x)}. \tag{40}$$

In order to compute the momenta autocorrelation function, noting that $p_\eta(x)$ is an odd function of x, we focus on odd eigenfunctions by considering $D_\lambda = 0$. Both asymptotic expansion can be matched in the domain $1 \ll \ell(\lambda) \ll \lambda^{-1/2}$. Using the leading order matching relations in this domain, and taking care of the normalization condition (36), one ends up with the scaling

$$A_\lambda \overset{\lambda\to 0}{\propto} \lambda^{\frac{\alpha-1}{4}} \tag{41}$$

$$C_\lambda \overset{\lambda\to 0}{\propto} \lambda^{\frac{\alpha+1}{4}} \tag{42}$$

$$B_\lambda \overset{\lambda\to 0}{\propto} \lambda^{\frac{3\alpha+1}{4}}, \tag{43}$$

where multiplicative constants independent on λ are not reported. This determines the eigenfunctions φ_λ.

4.3. Large time behavior of the auto-correlations

All these results are finally useful to derive the auto-correlations. Indeed, let us consider $f_p(x,t)$ the solution of the Fokker-Planck equation (32) corresponding to the initial condition $f_p(x,0) \equiv p(x)\varphi_0(x)$, where p is an odd continuous function of x. The autocorrelation $\langle p(x(\tau))p(x(0))\rangle$ can be rewritten as

$$\langle p(x(\tau))p(x(0))\rangle \;\; = \;\; \int_{-\infty}^{+\infty} \mathrm{d}x \, p(x) f_p(x,\tau). \tag{44}$$

In order to evaluate this integral, we decompose $f_p(x,0)$ on the eigenfunctions $\{\varphi_\lambda\}$

$$f_p(x,0) = \int_0^{+\infty} \mathrm{d}\lambda\, \mu_p(\lambda)\varphi_\lambda(x) \quad \text{with} \quad \mu_p(\lambda) \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x)\varphi_\lambda(x). \tag{45}$$

The Fokker-Planck evolution leads therefore to

$$\langle p(x(\tau))p(x(0))\rangle = \int_0^{+\infty} \mathrm{d}\lambda\, \mu_p(\lambda)^2 e^{-\lambda\tau}. \tag{46}$$

The limiting behavior of $\langle p(x(\tau))p(x(0))\rangle$ in the $\tau \to \infty$ limit is thus given by the behavior of $\mu(\lambda)$ when $\lambda \to 0$, which is itself determined from (45) by the large $|x|$ behavior of $p(x)$.

Let us be more specific in several important cases. Let us consider an odd function $p_\eta(x)$ such that $p_\eta(x) \overset{x\to+\infty}{\sim} x^\eta$, with $\eta < (\alpha-1)/2$. We evaluate $\mu_p(\lambda)$, given by Eq. (45), for small λ values by using the expansion for φ_λ obtained previously. The leading contribution

$$\mu_p(\lambda) \overset{\lambda\to 0}{\propto} \lambda^{(\alpha-2\eta-3)/4} \tag{47}$$

leads to the limiting behavior

$$\langle x^\eta(\tau)x^\eta(0)\rangle \quad = \quad \int_0^{+\infty} \mathrm{d}\lambda\, e^{-\lambda\tau}\lambda^{\frac{\alpha-2\eta-3}{2}} \overset{\tau\to+\infty}{\propto} \tau^{\eta-(\alpha-1)/2}. \tag{48}$$

as anticipated in equation (34).

Let us now make the choice $p(x) \propto (\ln x)^{1/\delta}$. Similar computations lead to

$$\langle p(x(\tau))p(x(0))\rangle \overset{\tau\to+\infty}{\propto} \frac{(\ln\tau)^{2/\delta}}{\tau}, \tag{49}$$

which does not depend on α.

5. Long-range correlations and anomalous diffusion in the HMF model

Let us now apply results derived in Sec. 4 to the HMF model, to prove in particular the occurrence of anomalous diffusion. It is important to emphasize that this latter behavior depends crucially on the exponent α of the logarithmic potential (25). As discussed in Sec. 3, if the distributions $f_0(p)$ decrease faster than any algebraic power law, one obtains $\alpha = 3$, whereas $\alpha < 3$ if the decay is algebraic. We discuss both cases in the remainder of this section.

5.1. Strong anomalous diffusion for $f_0(p)$ with algebraic tails

For algebraic distributions $f_0(p) \overset{|p|\to\infty}{\sim} Cp^{-\nu}$ where $\nu > 3$, we proved in Sec. 3 that the prefactor of the logarithmic potential is $\alpha = 3\nu/(2+\nu)$. Moreover, we have shown that $p(x)$ is asymptotically algebraic (see formula (31)) with $\eta = 2/(2+\nu)$. Introducing these expressions for α and η in Eq. (34), we thus obtain

$$\langle p(\tau)p(0)\rangle \overset{\tau\to+\infty}{\propto} \tau^{(3-\nu)/(2+\nu)}. \tag{50}$$

This characterizes the algebraic asymptotic behavior of the momentum autocorrelation for the HMF model.

From the momenta autocorrelations, one usually derives the angle diffusion

$$\langle(\theta(\tau)-\theta(0))^2\rangle = 2D_\theta\,\tau \tag{51}$$

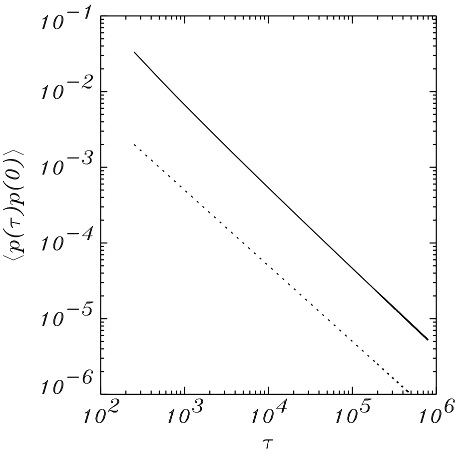

Figure 2. The solid line represents the time evolution of the momentum autocorrelations obtained using the numerical integration of the Fokker-Planck equation for a test particle in a surrounding gaussian reservoir ($\beta = 0.25$). Its slope can hardly be distinguished from a $1/\tau$ law (dotted line).

where D_θ is defined via the Kubo formula

$$D_\theta = \int_0^{+\infty} \mathrm{d}\tau \; \langle p(\tau)p(0)\rangle. \tag{52}$$

However, since the exponent $(\nu - 3)/(2 + \nu) = -1 + 5/(2 + \nu)$ is larger than -1, the asymptotic result (50) shows that integral (52) diverges, leading to anomalous diffusion.

Relying on a direct generalization of the Kubo formula, we thus predict the anomalous diffusion law

$$\langle(\theta(\tau) - \theta(0))^2\rangle \overset{\tau\to+\infty}{\propto} \tau^{1+\frac{5}{2+\nu}}. \tag{53}$$

This is the first prediction of a superdiffusive behavior at large times for the HMF model.

5.2. Neutrally anomalous diffusion for stretched exponential $f_0(p)$

For stretched exponential distributions $f_0(p)$ similar to (28), we derived in Sec. 3 that $\alpha = 3\nu/(2 + \nu)$ and that $p(x)$ exhibits asymptotically the logarithmic behavior (29). Using result (49), we thus obtain

$$\langle p(\tau)p(0)\rangle \overset{\tau\to+\infty}{\propto} \frac{(\ln\tau)^{2/\delta}}{\tau}. \tag{54}$$

This proves the existence of long-range temporal momentum autocorrelation for stretched exponential distributions $f_0(p)$. Figure 2 shows a numerical evidence of such a behavior for a Gaussian distribution function, obtained by a numerical integration of the Fokker-Planck equation (20)

As in the case of distributions with algebraic tails, the diffusion of angles is anomalous since the Kubo integral (52) diverges. Nevertheless, the extremely small anomaly (logarithmic) for distribution functions with stretched exponential tails explains the difficulty, and consequently the debate, to detect numerically anomalous diffusion [33, 9].

Let us emphasize that the algebraic behavior of the momentum autocorrelation and the anomalous diffusion for angles also occur for the Gaussian distribution which corresponds to the special case $\delta = 2$. As gaussian distributions correspond to equilibrium distributions in the microcanonical and the canonical ensembles, it is important to realize that above results are valid both for equilibrium and out-of-equilibrium initial conditions.

6. Conclusion

In this paper, we have analytically exhibited algebraic decay of the momentum autocorrelations and anomalous diffusion of the angles, by considering the microscopic dynamics of the HMF model. We have in particular shown that the coefficient for the algebraic laws depends on the tails of the distributions functions $f_0(p)$ for the bath of particles. We have computed explicitly these exponents for algebraic and stretched exponential tails.

Numerical results reported in Ref. [11] have also shown algebraic momenta correlations. Here we have given an explanation of such a behavior without invoking non extensive statistical mechanics. Besides the prediction of algebraic behavior, it would be of course very interesting to compare the exponents to these numerical simulations. However a direct comparison is unfortunately not possible, as the bath distribution is not reported in Ref. [11].

In the general case, a comparison of the predictions of anomalous diffusion for angles with direct numerical computation of the HMF dynamics is a tough task because of the scaling with N of the time dependance of the autocorrelation function. However, work along this line is in progress.

Finally, as we have obtained a typical example of anomalous diffusion by characterizing analytically the second moment directly from the microscopic dynamics, it would be very interesting to pursue this study for the higher moments $\langle (\theta(\tau) - \theta(0))^{2n} \rangle$. The evolution of the exponent of the algebraic time evolution as a function of the index n would be a very interesting quantity to characterize [4] the dynamics.

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References

[1] T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens (Eds), *Dynamics and Thermodynamics in Systems with Long-range Interactions*, Lecture Notes in Physics 602, Springer (2002).
[2] J. P. Bouchaud, A. Georges, *Physics Reports*, **195**, 127 (1990).
[3] A. J. Majda, P. R. Kramer, *Physics Reports*, **314**, 238 (1999).
[4] P. Castiglione , A. Mazzino, P. Muratore-Ginanneschi, A. Vulpiani, *Physica D* 134, 75-93 (1999).
[5] K. H. Andersen, P. Castiglione, A. Mazzino, A. Vulpiani, *Eur. Phys. J. B* **18**, 447 (2000).
[6] F. Bouchet, F. Cecconi, A. Vulpiani, *Phys. Rev. Lett.* **92**, 040601 (2004).
[7] G. M. Zaslavsky, *Physics Reports* **371**, 461-580 (2002).
[8] V. Latora, A. Rapisarda, S. Ruffo, *Phys. Rev. Lett.* **83**, 2104 (1999).
[9] Y. Y. Yamaguchi, *Phys. Rev. E* **68**, 066210 (2003).
[10] V. Latora, A. Rapisarda, C. Tsallis, *Phys. Rev. E* **64**, 056134 (2001).
[11] A. Pluchino, V. Latora, A. Rapisarda, *Physica A* **69**, 056113 (2004).
[12] F. Bouchet, T. Dauxois, cond-mat/0407703.
[13] J. Barré, T. Dauxois, G. De Ninno, D. Fanelli, S. Ruffo *Phys. Rev. E* **69**, 045501 (2004).
[14] J. Messer, H. Spohn, *J. Stat. Phys.* **29**, 561-578 (1982).
[15] G. M. Zaslavsky, V. F. Shabanov, K. S. Aleksandrov, I. P. Aleksandrov, *Sov. Phys. JETP* **45**, 315-321 (1977).
[16] T. Konishi, K. Kaneko, *J. Phys. A* **25**, 6283, (1992).
[17] C. Pichon, PhD thesis, Cambridge (1994).
[18] M. Antoni, S. Ruffo, *Phys. Rev. E* **52,** 2361 (1995).
[19] T. Dauxois, V. Latora, A. Rapisarda, S. Ruffo, A. Torcini in Ref. [1]; also cond-mat/0208456.
[20] L. Velazquez, R. Sospedra, J. C. Castro, F. Guzman, *cond-mat/0302456*.

[21] J. Barré, F. Bouchet, T. Dauxois, S. Ruffo, submitted to *Journal of Statistical Physics*; also cond-mat/0406358.
[22] Y. Y. Yamaguchi, J Barré, F. Bouchet, T. Dauxois, S. Ruffo, *Physica A* **337**, 36-66 (2004).
[23] N. N. Bogoliubov, *Problems of a Dynamical Theory in Statistical Physics*, State Technical Press (1946).
M. Born, H. S. Green, *A General Kinetic Theory of Liquids* Cambridge University Press (1949).
J. G. Kirkwood, *J. Chem. Phys.* **14,** 180 (1946); *J. Chem. Phys.* **15,** 72 (1947).
J. Yvon, *La Théorie des Fluides et l'Équation d'État*, Hermann (1935).
[24] A. Lenard, *Annals of Physics* **10,** 390 (1960).
R. Balescu, *Physics of Fluids* **3,** 52 (1960).
[25] D. R. Nicholson *Introduction to Plasma Theory*, John Wiley (1983).
[26] Yu. L. Klimontovich, *The Statistical Theory of Non-equilibrium Processes in a Plasma*, MIT Press (1967).
[27] E. M. Lifshitz, L. P. Pitaevsii, *Physical Kinetics*, Pergamon Press, Oxford (1981).
[28] N. G. Van Kampen, *Stochastic processes in physics and chemistry*, North-Holland Publishing Company (1981).
[29] F. Bouchet, *Phys. Rev. E* **70**, 036113 (2004).
[30] A. Pluchino, V. Latora, A. Rapisarda, *Physica A* **340**, 187 (2004); *Physica D* **193**, 315-328 (2004).
D. Zanette, M. Montemurro, *Phys. Rev. E* **67**, 031105 (2003).
[31] J. Farago, *Europhysics Letters* **52**, 379 (2000).
[32] F. Lillo *et al*, submitted to *Phys. Rev. Letters* (2002); also cond-mat/0203442.
[33] V. Latora, A. Rapisarda, S. Ruffo *Physica A* **280**, 81 (2000).