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A stochastic model of long-range interacting particles

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Abstract. We introduce a model of long-range interacting particles evolving under a stochastic Monte Carlo dynamics, in which possible increase or decrease in the values of the dynamical variables is accepted with preassigned probabilities. For symmetric increments, the system at long times settles to the Gibbs equilibrium state, while for asymmetric updates, the steady state is out of equilibrium. For the associated Fokker–Planck dynamics in the thermodynamic limit, we compute exactly the phase space distribution in the nonequilibrium steady state, and find that it has a nontrivial form that reduces to the familiar Gibbsian measure in the equilibrium limit.

Keywords: stochastic particle dynamics (theory), stationary states

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Contents

1. Introduction

Nonequilibrium systems abound in nature, with examples encompassing different branches of science. Although there has been much recent progress in characterizing and understanding some features of nonequilibrium steady states [1], developing a general principle akin to the one due to Gibbs–Boltzmann for equilibrium has been one of the greatest challenges of modern statistical physics. In this respect, it is instructive to develop and analyze simple models in order to gain insights into features of nonequilibrium steady states that make them distinct from those in equilibrium. Often, even for simple models, the steady state distribution has been nontrivial to obtain [2], and in many cases has even remained elusive, thereby requiring one to resort to numerical simulations and approximation methods as the only possible tools to analyze the steady states [3].

Here, we develop a model of particles interacting via long-range interactions and evolving under a stochastic Monte Carlo dynamics, for which we could characterize exactly the steady state distribution. This is one of the first examples of engineering such a dynamics in the arena of long-range models interacting with an external heat bath, where all previous studies, to the best of our knowledge, have been based on Langevin equations with noise terms that mimic the effect of the heat bath (see, e.g., [4]). The model effectively simulates driven motion of particles in one dimension under the action of a mean field.

Long-range interactions have generated considerable interest in recent years, with examples ranging from plasma physics to gravitational systems [5]. These systems are characterized by an interparticle potential which in d dimensions decays at large separation, r, as $1/r^{\alpha}$, with $\alpha \leq d$. Long-range interacting systems are different from short-range ones in that they are generically non-additive, whereby thermodynamics quantities scale superlinearly with the system size. This latter feature manifests in properties, both static and dynamic, which are unusual for short-range systems [5].

In this work, we introduce a model of long-range interacting systems involving N globally coupled particles moving on a circle, in the presence of an external field acting individually on the particles, and in contact with an external heat bath at inverse temperature β. This inverse temperature will coincide with the steady state temperature of the system only in equilibrium. The system evolves under a stochastic Monte Carlo dynamics. Thus, a randomly chosen particle decides to move either to the left or to the right of its present position to take up a new location on the circle. The displacement from the initial position of the particle is a quenched random variable sampled independently from a common distribution for all the particles. The new location of the particle is accepted with a preassigned transition rate which is chosen in the following way. In the case where the particle jumps symmetrically to the left and to the right, the transition rate is such that the stationary state of the system is the Gibbs equilibrium state at the inverse temperature β. In the case of asymmetric particle jump to the left and to the right, the system at long times reaches a nonequilibrium stationary state. Considering the dynamics in the Fokker–Planck limit (i.e., only small jumps are allowed), we find that even with asymmetric jumps, if the external field is turned off and the jump distribution is a delta function so that all particles jump by the same amount, the steady state is in equilibrium in a suitable comoving frame obtained by performing a Galilean transformation. In all other cases, the steady state is out of equilibrium. In the thermodynamic limit $N \to \infty$, the system is characterized by a single-particle distribution, that we compute exactly in the nonequilibrium steady state. We find that the distribution has a nontrivial form that reduces to the usual Gibbsian distribution in the equilibrium limit. Our results show excellent agreement with N-particle Monte Carlo simulations of the dynamics.

The paper is organized as follows. In section 2, we give a precise definition of the model and discuss the master equation for the evolution of the N-particle phase space distribution. In section 3, we consider the Fokker–Planck limit of the dynamics and obtain the exact steady state single-particle distribution in the limit $N \to \infty$. In section 4, we compare N-particle Monte Carlo simulation results for the steady state single-particle distribution with our theoretical predictions in the Fokker–Planck limit, and demonstrate an excellent agreement between the two. The paper ends with conclusions.

2. Model

Consider a system of N interacting particles moving on a unit circle, with the particles labeled as $i = 1, 2, \ldots, N$. Let the angle θ_i denote the location of the ith particle on the circle. A microscopic configuration of the system is denoted by $\mathcal{C} = \{\theta_i; i = 1, 2, \ldots, N\}$. The particles interact via a long-range potential $\mathcal{V}(\mathcal{C}) = K/(2N)\sum_{i,j=1}^{N}[1 - \cos(\theta_i - \theta_j)]$, where K is the coupling constant; we consider $K = 1$ in the following. Application of an external field of strength h_i produces a potential $\mathcal{V}_{\rm ext}(\mathcal{C}) = \sum_{i=1}^{N} h_i \cos\theta_i$, so that the net potential energy is $V(\mathcal{C}) = \mathcal{V}(\mathcal{C}) + \mathcal{V}_{\rm ext}(\mathcal{C})$. The interaction $\mathcal{V}(\mathcal{C})$ has the same form as in the Hamiltonian mean-field (HMF) model, a paradigmatic example of systems with long-range interactions [5]. The fields h_is may be considered as quenched random variables with a common distribution $\mathscr{P}(h)$.

We now specify the dynamics of the system. We take hints from one of the first models devoted to studying characteristics of nonequilibrium steady states, the celebrated

Katz–Lebowitz–Spohn model [6]. The configuration $\mathcal{C}$ evolves according to a stochastic Monte Carlo dynamics. In discrete time, the dynamics in a small time Δt involves every particle attempting to hop to a new position on the circle. The ith particle attempts with probability p to move forward (in the counter-clockwise sense) by an amount f_i on the circle, $\theta_i \to \theta_i' = \theta_i + f_i$, while with probability $q = 1 - p$, it attempts to move backward by the amount f_i, so that $\theta_i \to \theta_i' = \theta_i - f_i$. In either case, the particle takes up the new position with probability $g(\Delta V(\mathcal{C}))\Delta t$. Here, f_i is a quenched random variable which for each particle is distributed according to a common distribution $\mathcal{P}(f)$, while the quantity $\Delta V(\mathcal{C})$ is the change in the potential energy due to the attempted hop from θ_i to θ_i': $\Delta V(\mathcal{C}) = (1/N)\sum_{j=1}^{N}[-\cos(\theta_i' - \theta_j) + \cos(\theta_i - \theta_j)] + h_i[\cos\theta_i' - \cos\theta_i]$. The function g is of the form $g(x) = (1/2)(1 - \tanh(\beta x/2))$, where β is the inverse temperature. The dynamics models the overdamped motion of the particles in contact with an external heat bath at inverse temperature β and in the presence of an external field. The case $p \neq q$ for which the particles move asymmetrically forward and backward mimics the action of an external drive that makes the particles move in one preferential direction along the circle. Note that in the dynamics, the initial ordering of particles on the circle is not conserved in time. Taking f_is as quenched random variables introduces in the dynamics a different source of noise than the one due to the Monte Carlo update scheme which is annealed in nature.

There are two sources of quenched randomness in the model through the presence of (i) jump lengths f_is, and (ii) field strengths h_is. Later, we will consider specifically the first of the two sources of randomness, and take the h_is to be the same for all particles. In the conclusions, we will comment on how our analytical approach may be easily adapted to consider the randomness due to the h_is.

Let $P = P(\{\theta_i\}; t)$ be the probability to observe the configuration $\mathcal{C} = \{\theta_i\}$ at time t. In the limit of continuous time, the evolution of P is given by the master equation, which may be derived by considering the change in P in a small time Δt according to the dynamical evolution rules given above, and then taking the limit $\Delta t \to 0$ while keeping f_is fixed and finite. Defining $\Delta\theta_{ij} = \theta_i - \theta_j$, we get the master equation as follows:

$$
\begin{aligned}
\frac{\partial P}{\partial t} = \sum_{i=1}^{N} \Bigg[& P(\ldots, \theta_i - f_i, \ldots; t) p \left\{ 1 - \tanh\frac{\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}[-\cos\Delta\theta_{ij} + \cos(\Delta\theta_{ij} - f_i)] \right.\right. \\
& + \left.\left. h_i[\cos\theta_i - \cos(\theta_i - f_i)] \right) \right\} + P(\ldots, \theta_i + f_i, \ldots; t) \\
& \times q \left\{ 1 - \tanh\frac{\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}[-\cos\Delta\theta_{ij} + \cos(\Delta\theta_{ij} + f_i)] \right.\right. \\
& + \left.\left. h_i[\cos\theta_i - \cos(\theta_i + f_i)] \right) \right\} - P(\ldots, \theta_i, \ldots; t) \\
& \times \left(p \left\{ 1 - \tanh\frac{\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}[-\cos(\Delta\theta_{ij} + f_i) + \cos\Delta\theta_{ij}] \right.\right.\right.
\end{aligned}
$$

$$+ \quad h_i[\cos(\theta_i + f_i) - \cos\theta_i]\Bigg)\Bigg\} + q\left\{1 - \tanh\frac{\beta}{2}\right.$$
$$\times \quad \left.\left.\left(\frac{1}{N}\sum_{j=1}^{N}[-\cos(\Delta\theta_{ij} - f_i) + \cos\Delta\theta_{ij}] + h_i[\cos(\theta_i - f_i) - \cos\theta_i]\right)\right\}\right)\Bigg]. \tag{1}$$

At long times, the system settles into a stationary state corresponding to the time-independent probability $P_{\rm st}(\{\theta_i\})$. For $p = 1/2$, the particles attempt to move forward and backward in a symmetric manner, and the system has an equilibrium stationary state in which the condition of detailed balance is satisfied with the measure $P_{\rm eq}(\{\theta_i\}) \propto {\rm e}^{-\beta V(\{\theta_i\})}$. On the other hand, for $p \neq 1/2$, the particles have a preferred direction to hop on the circle, and the system at long times settles into a nonequilibrium stationary state, characterized by a violation of detailed balance leading to nonzero probability currents in phase space. In the absence of the external field, the dynamics with $p = 1/2$ samples the equilibrium measure of the Brownian mean-field model of long-range interacting systems, developed as an extension of the microcanonical dynamics of the HMF model to a canonical dynamics that mimics the interaction of the system with an external heat bath [7].

3. Fokker–Planck limit

Here, we analyze the Fokker–Planck limit of the dynamics of our model. To this end, we first obtain the Fokker–Planck equation corresponding to the master equation (1) by making the assumption that $f_i \ll 1\ \forall\ i$, so that we may Taylor expand functions in powers of f_is [8]. As shown in appendix A, retaining terms to second order in f_is, we get the Fokker–Planck equation for $P(\{\theta_i\}; t)$ as follows:

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{N}\frac{\partial J_i}{\partial \theta_i}, \tag{2}$$

where the probability current J_i for the ith particle is given by

$$J_i = \left[(2p-1)f_i + \frac{f_i^2\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\sin\Delta\theta_{ji} + h_i\sin\theta_i\right)\right]P - \frac{f_i^2}{2}\frac{\partial P}{\partial \theta_i}. \tag{3}$$

The corresponding Langevin equation is easily written down as

$$\dot{\theta}_i = (2p-1)f_i + \frac{f_i^2\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + h_i\sin\theta_i\right) + f_i\eta_i(t), \tag{4}$$

where the dot denotes derivative with respect to time, and $\eta_i(t)$ is a random noise with

$$\langle \eta_i(t)\rangle = 0, \qquad \langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t - t'). \tag{5}$$

From the Fokker–Planck equation (2), it is evident that, as in the finite-f_i dynamics, the system for $p = 1/2$ settles into the equilibrium stationary state with $P_{\rm eq}(\{\theta_i\})$ which makes $J_i = 0$ individually for each i. On the other hand, for $p \neq 1/2$, the system at long times reaches a nonequilibrium stationary state. However, in the special case when the

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jump length is the same for all the particles and there is no external field ($f_i = f$ and $h_i = 0\ \forall\ i$), one may make a Galilean transformation, $\theta_i \to \theta_i + [(2p-1)f/2]t$, so that in the frame moving with the velocity $[(2p-1)f/2]$, the Langevin equation (4) takes a form identical to the one for $p = 1/2$, and the stationary state has the equilibrium measure $P_{\rm eq}(\{\theta_i\})$.

3.1. Limit $N \to \infty$ and constant field: single-particle distribution

In the thermodynamic limit $N \to \infty$, when the external field is the same for all the particles, $h_i = h$, let us define the single-particle distribution $\rho(\theta; f, t)$ such that $\rho(\theta; f, t)$ gives the density of particles with jump length f which are at location θ on the circle at time t. We have $\rho(\theta; f, t) = \rho(\theta + 2\pi; f, t)$, and also the normalization

$$\int_0^{2\pi} \mathrm{d}\theta\ \rho(\theta; f, t) = 1 \qquad \forall\, f. \tag{6}$$

In terms of $\rho(\theta; f, t)$, the Langevin equation (4) in the limit $N \to \infty$ for a particle with jump length f and at position θ reads

$$\dot{\theta} = (2p-1)f + \frac{f^2\beta}{2}(m_y \cos\theta - m_x \sin\theta + h\sin\theta) + f\eta(t), \tag{7}$$

where

$$(m_x, m_y) = \int \mathrm{d}\theta\, \mathrm{d}f\ (\cos\theta, \sin\theta)\rho(\theta; f, t)\mathcal{P}(f), \tag{8}$$

and

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t'). \tag{9}$$

Let us note that the dynamics (7) is similar with $\eta(t) = 0$ to that of the Kuramoto model of synchronization [9] and with $\eta(t) \neq 0$ to that of its extension considered in [10] that includes noise. However, in the latter case, a crucial difference is that in equation (7) the noise term and the drift term (the first term on the right hand side) contain the same factor f, and are therefore related, unlike the model in [10].

The single-particle Fokker–Planck equation satisfied by $\rho(\theta; f, t)$ may be obtained from the Langevin equation (7) as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial j}{\partial \theta}, \tag{10}$$

where the probability current j is given by

$$j = \left[(2p-1)f + \frac{f^2\beta}{2}\left(m_y\cos\theta - m_x\sin\theta + h\sin\theta\right)\right]\rho - \frac{f^2}{2}\frac{\partial\rho}{\partial\theta}. \tag{11}$$

The stationary solution $\rho_{\rm st}$ of the Fokker–Planck equation (10) is given by (see appendix B)

$$\rho_{\rm st}(\theta; f) = \rho(0; f)\mathrm{e}^{2(2p-1)\theta/f + \beta(m_x\cos\theta + m_y\sin\theta - h\cos\theta)} \\ \times \left[1 + (\mathrm{e}^{-4\pi(2p-1)/f} - 1)\frac{\int_0^\theta \mathrm{d}\theta'\ \mathrm{e}^{-2(2p-1)\theta'/f - \beta(m_x\cos\theta' + m_y\sin\theta' - h\cos\theta')}}{\int_0^{2\pi} \mathrm{d}\theta'\ \mathrm{e}^{-2(2p-1)\theta'/f - \beta(m_x\cos\theta' + m_y\sin\theta' - h\cos\theta')}}\right], \tag{12}$$

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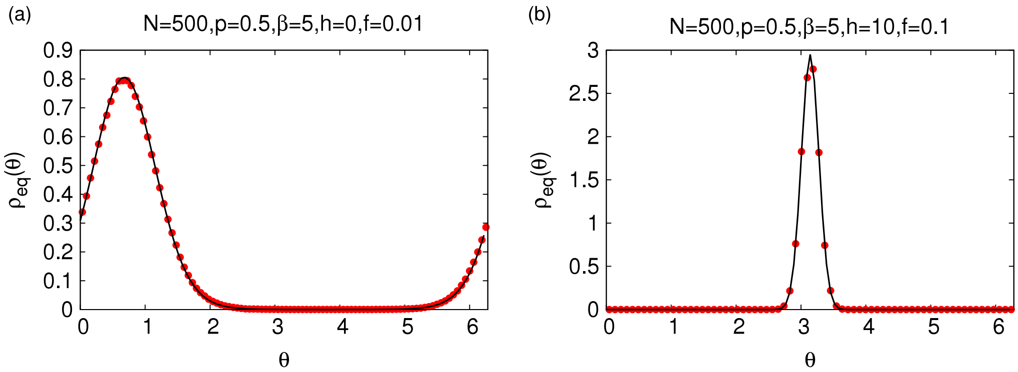

Figure 1. Equilibrium distribution $\rho_{\rm eq}(\theta)$: theory (continuous line) in the Fokker–Planck approximation and in the limit $N\to\infty$, given by equation (14), compared with Monte Carlo simulations (points), taking $f=0.01, h=0$ (panel (a)) and $f=0.1, h=10$ (panel (b)). We observe an excellent agreement between theory and simulations. In (a), when there is no field, the distribution is centered around a value of θ which is arbitrary; this corresponds to a spontaneous breaking of the $O(2)$ symmetry of the potential $\mathcal{V}(\mathcal{C})$.

where $(m_x,m_y)=\int {\rm d}\theta\,{\rm d}f(\cos\theta,\sin\theta)\rho_{\rm st}(\theta;f)\mathcal{P}(f)$, and the constant $\rho(0;f)$ is fixed by the normalization condition (6).

When the jump length is the same for all particles, $f_i=f$, we have

$$\rho_{\rm st}(\theta)=\rho(0){\rm e}^{2(2p-1)\theta/f+\beta(m_x\cos\theta+m_y\sin\theta-h\cos\theta)} \times\left[1+({\rm e}^{-4\pi(2p-1)/f}-1)\frac{\int_0^\theta{\rm d}\theta'\,{\rm e}^{-2(2p-1)\theta'/f-\beta(m_x\cos\theta'+m_y\sin\theta'-h\cos\theta')}}{\int_0^{2\pi}{\rm d}\theta'\,{\rm e}^{-2(2p-1)\theta'/f-\beta(m_x\cos\theta'+m_y\sin\theta'-h\cos\theta')}}\right], \quad (13)$$

where the constant $\rho(0)$ is fixed by normalization: $\int_0^{2\pi}{\rm d}\theta\;\rho_{\rm st}(\theta)=1$. For $p=1/2$, we obtain the equilibrium single-particle distribution as

$$\rho_{\rm eq}(\theta)=\frac{{\rm e}^{\beta(m_x\cos\theta+m_y\sin\theta-h\cos\theta)}}{\int_0^{2\theta}{\rm d}\theta\;{\rm e}^{\beta(m_x\cos\theta+m_y\sin\theta-h\cos\theta)}}. \quad (14)$$

It is worthwhile to point out that the equilibrium distribution (14) does not depend on the value of the jump length f, unlike the nonequilibrium distribution (13).

4. Numerical simulations

Choosing the jump length $f_i\ll 1$ to be the same for all particles, we show in figure 1 a comparison of the N-particle Monte Carlo simulation results for $\rho_{\rm eq}(\theta)$, obtained for $N=500$, and its theoretical form, equation (14), applicable in the Fokker–Planck approximation and in the limit $N\to\infty$. We observe an excellent agreement between simulations and theory. For the case $p\neq 1/2$ and $h\neq 0$, figures 2(a) and (b) compare simulation results and theory (equation (13)) for $\rho_{\rm st}(\theta)$ for two values of h, again illustrating an excellent agreement. Figures 2(c) and (d) compare simulation results for $f=1$ with the Fokker–Planck-limit theory valid for $f\ll 1$; in (c), we see a reasonable agreement, while in (d), the disagreement is quite large. For the latter case, we have

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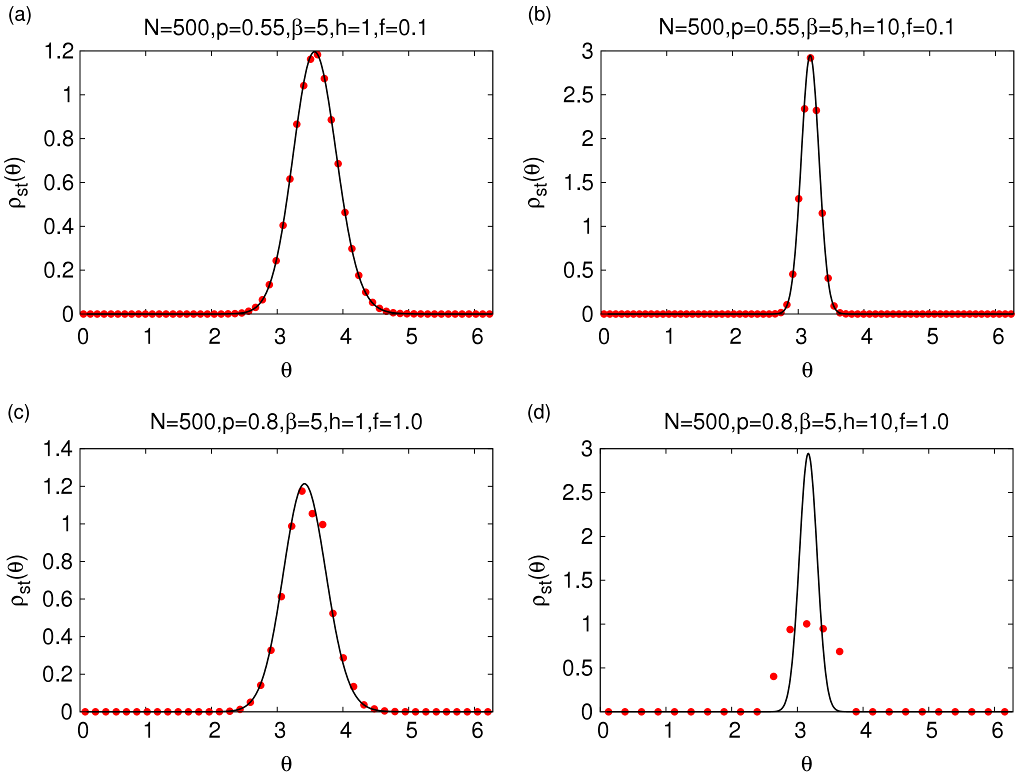

Figure 2. Stationary distribution $\rho_{\rm st}(\theta)$ for the case $p \neq 1/2$ and $h \neq 0$: panels (a) and (b) compare Monte Carlo simulation results (points) for $f = 0.1$ and theory (continuous lines) in the Fokker–Planck approximation and in the limit $N \to \infty$, given by equation (13), illustrating an excellent agreement. In panels (c) and (d) for $f = 1$, we observe the expected disagreement between simulations and the Fokker–Planck-limit theory valid for $f \ll 1$.

checked that, for the same parameter values, the mismatch between theory and simulations does not reduce with larger N, which implies that it is due to the large value of f used as compared to the Fokker–Planck limit, and not due to finiteness of N.

5. Conclusions

In this work, we introduced a model of long-range interacting systems involving N globally coupled particles moving on a circle. The system evolves under a stochastic Monte Carlo dynamics consisting of particle jumps, either symmetrically to the left and to the right, or, asymmetrically, by quenched random amounts sampled from a common distribution. The attempted new locations of the particles are accepted with transition rates chosen in such a way that, for symmetric jumps, the stationary state of the system is the Gibbs equilibrium state. For asymmetric jumps, the system at long times reaches a nonequilibrium steady state characterized by nonzero probability currents in phase space. For the associated Fokker–Planck dynamics in the thermodynamic limit $N \to \infty$, we computed exactly the steady state distribution and found that it has a nontrivial form that reduces to the Gibbs distribution in the equilibrium limit, see equations (13) and (14). We compared our theoretical predictions with N-particle Monte Carlo simulations, and found an excellent agreement between the two in the Fokker–Planck limit. The observed disagreement when the limit is not satisfied opens up the very interesting scope of analyzing and obtaining corrections to the Fokker–Planck answer. It is also of interest to treat the external fields h_is in equation (1) as quenched random variables sampled from a common distribution. It is easy to generalize our analytical framework to treat this case in the Fokker–Planck limit by considering instead of $\rho(\theta; f, t)$ the distribution $\rho(\theta; f, h, t)$ giving the density of particles with jump length f and under the action of field with strength h, which are at location θ at time t. One would then have to obtain the Fokker–Planck equation that $\rho(\theta; f, h, t)$ satisfies, by considering $f_i \ll 1$ and $h_i \ll 1\ \forall\ i$ in the master equation (1) and performing suitable Taylor series expansion of functions of f_is and h_is. With quenched h_is, one may consider the canonical dynamics introduced in this paper and its grand canonical counterpart (where, say, the number of particles N is not conserved), and investigate the issue of equivalence of the nonequilibrium steady state under the two dynamics. This issue is particularly relevant, since long-range interacting systems are known to show inequivalence in equilibrium in the presence of random fields [11].

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Appendix A. Derivation of the Fokker–Planck equation (2)

Considering the master equation (1) for $f_i \ll 1\ \forall\ i$, we expand all functions of f_is in powers of f_is. Retaining terms to second order in f_is, we get

$$\frac{\partial P}{\partial t} \approx \sum_{i=1}^{N}\Bigg[\left(P - f_i\frac{\partial P}{\partial \theta_i} + \frac{f_i^2}{2}\frac{\partial^2 P}{\partial \theta_i^2}\right) p\left\{1 - \frac{\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\left[-\frac{f_i^2}{2}\cos\Delta\theta_{ij} + f_i\sin\Delta\theta_{ij}\right]\right.\right.$$
$$\left.\left. + \ h_i\left[\frac{f_i^2}{2}\cos\theta_i - f_i\sin\theta_i\right]\right)\right\} + \left(P + f_i\frac{\partial P}{\partial \theta_i} + \frac{f_i^2}{2}\frac{\partial^2 P}{\partial \theta_i^2}\right) q$$
$$\times\ \left\{1 - \frac{\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\left[-\frac{f_i^2}{2}\cos\Delta\theta_{ij} - f_i\sin\Delta\theta_{ij}\right]\right.\right.$$
$$\left.\left. + \ h_i\left[\frac{f_i^2}{2}\cos\theta_i + f_i\sin\theta_i\right]\right)\right\}$$
$$-\ Pp\left\{1 - \frac{\beta}{2}\left(\frac{1}{2N}\sum_{j=1}^{N}\left[\frac{f_i^2}{2}\cos\Delta\theta_{ij} + f_i\sin\Delta\theta_{ij}\right]\right.\right.$$
$$\left.\left. + \ h_i\left[-\frac{f_i^2}{2}\cos\theta_i - f_i\sin\theta_i\right]\right)\right\} - \ Pq\left\{1 - \frac{\beta}{2}\right.$$

$$\times \quad \left(\frac{1}{N}\sum_{j=1}^{N}\left[\frac{f_i^2}{2}\cos\Delta\theta_{ij} - f_i\sin\Delta\theta_{ij}\right] + h_i\left[-\frac{f_i^2}{2}\cos\theta_i + f_i\sin\theta_i\right]\right)\Bigg\}\Bigg]$$

$$= \quad -\sum_{i=1}^{N}\frac{\partial J_i}{\partial\theta_i}, \tag{A.1}$$

where the probability current J_i for the ith particle is given by

$$J_i = \left[(2p-1)f_i + \frac{f_i^2\beta}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\sin\Delta\theta_{ji} + h_i\sin\theta_i\right)\right]P - \frac{f_i^2}{2}\frac{\partial P}{\partial\theta_i}. \tag{A.2}$$

Appendix B. Stationary solution of equation (10)

Here, we obtain the steady state solution of equation (10). A similar equation and the steady state solution appear in [12]. In the steady state, we have

$$\frac{\partial\rho_{\rm st}}{\partial\theta} - \left(\frac{2(2p-1)}{f} + \beta m_{\rm st}\sin(\psi_{\rm st}-\theta) + \beta h\sin\theta\right)\rho_{\rm st} = C, \tag{B.1}$$

where C is a constant independent of θ, and we have defined

$$m_{\rm st} = \sqrt{m_x^2 + m_y^2}; \qquad \psi_{\rm st} = \tan^{-1}(m_y/m_x), \tag{B.2}$$

$$(m_x, m_y) = \int {\rm d}\theta\,{\rm d}f(\cos\theta, \sin\theta)\rho_{\rm st}(\theta; f)\mathcal{P}(f). \tag{B.3}$$

Multiplying both sides of equation (B.1) by $\exp[-2(2p-1)\theta/f - \beta m_{\rm st}\cos(\psi_{\rm st}-\theta) + \beta h\cos\theta]$, and then integrating over θ, we get

$$\begin{aligned}\rho_{\rm st}(\theta; f) = {}& \rho(0; f){\rm e}^{2(2p-1)\theta/f + \beta m_{\rm st}[\cos(\psi_{\rm st}-\theta) - \cos\psi_{\rm st}] + \beta h(1-\cos\theta)} \\ & + C{\rm e}^{2(2p-1)\theta/f + \beta m_{\rm st}\cos(\psi_{\rm st}-\theta) - \beta h\cos\theta} \\ & \times \int_0^{\theta}{\rm d}\theta'\,{\rm e}^{-2(2p-1)\theta'/f - \beta m_{\rm st}\cos(\psi_{\rm st}-\theta') + \beta h\cos\theta'},\end{aligned} \tag{B.4}$$

where $\rho(0; f) = \rho(\theta; f, 0)$ is the initial condition at time $t = 0$. Requiring that $\rho_{\rm st}(\theta + 2\pi; f) = \rho_{\rm st}(\theta; f)$ fixes C to be

$$C = \frac{\rho(0; f){\rm e}^{-\beta m_{\rm st}\cos\psi_{\rm st} + \beta h}({\rm e}^{-4\pi(2p-1)/f} - 1)}{\int_0^{2\pi}{\rm d}\theta'\,{\rm e}^{-2(2p-1)\theta'/f - \beta m_{\rm st}\cos(\psi_{\rm st}-\theta') + \beta h\cos\theta'}}, \tag{B.5}$$

and hence,

$$\begin{aligned}\rho_{\rm st}(\theta; f) = {}& \rho(0; f){\rm e}^{2(2p-1)\theta/f + \beta m_{\rm st}[\cos(\psi_{\rm st}-\theta) - \cos\psi_{\rm st}] + \beta h(1-\cos\theta)} \\ & \times \left[1 + ({\rm e}^{-4\pi(2p-1)/f} - 1)\frac{\int_0^{\theta}{\rm d}\theta'{\rm e}^{-2(2p-1)\theta'/f - \beta m_{\rm st}\cos(\psi_{\rm st}-\theta') + \beta h\cos\theta'}}{\int_0^{2\pi}{\rm d}\theta'{\rm e}^{-2(2p-1)\theta'/f - \beta m_{\rm st}\cos(\psi_{\rm st}-\theta') + \beta h\cos\theta'}}\right].\end{aligned} \tag{B.6}$$

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Redefining $\rho(0;f)$, and reverting to the variables m_x and m_y, we get

$$\rho_{\rm st}(\theta;f)=\rho(0;f)\mathrm{e}^{2(2p-1)\theta/f+\beta(m_x\cos\theta+m_y\sin\theta-h\cos\theta)}$$
$$\times\left[1+(\mathrm{e}^{-4\pi(2p-1)/f}-1)\frac{\int_0^{\theta}\mathrm{d}\theta'\mathrm{e}^{-2(2p-1)\theta'/f-\beta(m_x\cos\theta'+m_y\sin\theta'-h\cos\theta')}}{\int_0^{2\pi}\mathrm{d}\theta'\mathrm{e}^{-2(2p-1)\theta'/f-\beta(m_x\cos\theta'+m_y\sin\theta'-h\cos\theta')}}\right], \tag{B.7}$$

where $\rho(0;f)$ is fixed by the normalization, $\int_0^{2\pi}\mathrm{d}\theta\ \rho_{\rm st}(\theta;f)=1$.

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