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Stability of inhomogeneous states in mean-field models with an external potential

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Abstract. The Vlasov equation is well known to provide a good description of the dynamics of mean-field systems in the $N \to \infty$ limit. This equation has an infinity of stationary states and the case of homogeneous states, for which the single-particle distribution function is independent of the spatial variable, is well characterized analytically. On the other hand, the inhomogeneous case often requires some approximations for an analytical treatment: the dynamics is then best treated in action–angle variables, and the potential generating inhomogeneity is generally very complex in these new variables. We here treat analytically the linear stability of toy models where the inhomogeneity is created by an external field. Transforming the Vlasov equation into action–angle variables, we derive a dispersion relation that we accomplish solving for both the growth rate of the instability and the stability threshold for two specific models: the Hamiltonian mean-field model with additional asymmetry and the mean-field ϕ^4 model. The results are compared with numerical simulations of the N-body dynamics. When the inhomogeneous state is a stationary stable one, we expect to observe in the N-body dynamics quasi-stationary states, whose lifetime diverges algebraically with N.

Keywords: stationary states, metastable states

Contents

1. Introduction

Long-range forces can be found in a wide variety of physical systems, including self-gravitating systems, Coulomb systems, wave–plasma interactions and two-dimensional hydrodynamics. The interest in studying long-range forces has been revived in the last decade, not only because of the broad domain of physical systems involving such forces, but also because of the presence of unusual phenomena, both at equilibrium and out of equilibrium. Let us mention negative specific heat, temperature jumps, broken ergodicity and quasi-stationary states. Reviews and books have been recently published in this field [1]–[6].

A particular, but interesting, case is the one of mean-field interactions, for which each particle is directly coupled to all the others with equal strength, whatever their distance. Although this is an idealization, it serves as a useful approximation and appears, in addition, to give at least a good trend. Moreover, there are physical situations in which particles are all in interaction via a field, whose dynamics is in turn determined by the motion of the particles themselves: this is for example the case for wave–particle interactions in plasmas [7], free electron lasers [8], collective atomic recoil lasers [9] and traveling wave tubes [10]. This self-consistent effect can also be obtained in systems composed only of particles by introducing a coupling to an order parameter, as is done for the Hamiltonian mean-field (HMF) model [11]–[13], which has been widely studied in recent years as a paradigm for systems with long-range interactions [1].

The kinetics of models with N particles and only mean-field interactions is exactly described, in the infinite-N limit, by the Vlasov equation [14, 15]. This equation exhibits an infinity of stationary solutions and its dynamical evolution starting from a generic initial state can be extremely complex. Focusing on stationary states, their stability has been studied using different methods, but mainly by restricting the analysis to homogeneous stationary states, that are characterized by a single-particle distribution function which is independent of the spatial variable. These states are of major interest in kinetic theory, because they often constitute the ‘supposed’ physical equilibrium state. For instance a globally neutral plasma has an equilibrium which is also locally neutral, giving a homogeneous charge distribution. If perturbed, this state is expected to be stable, showing a relaxation back to the homogeneous state ruled by Landau damping [16]–[18]. This phenomenology is also observed in the HMF model [19], for which the homogeneous state is stable above a given energy threshold, which depends on the initial momentum distribution.

However, below this energy, the homogeneous state is unstable and one observes a dynamical evolution towards inhomogeneous states, whose stability properties are much more difficult to determine. Inhomogeneous states appear for example in gravitational dynamics [20], because of the attractive nature of the Newton force. Their stability has been studied in the context of the Vlasov equation, yet the necessity to resort to action–angle variables [21] makes the problem analytically tricky. Apart from numerical approaches (see e.g. [22]), one can project the dynamics onto a Fourier basis, yet at a cost of performing infinite sums [25, 26]; then, only a truncation can yield tractable results. Such a technique was also used in the context of plasmas [23, 24], where the waves often generate inhomogeneous states; expanding the dynamics along modes, such as Hermite polynomials [27], requires anyway a truncation in the sums. Analytical results were also obtained for BGK modes, whose stability properties were connected, in the small inhomogeneity limit, to those of homogeneous states [28, 29]. Later on, the unstable nature of periodic BGK modes under specific perturbations was rigorously shown [30, 31], but the problem remains open for other kinds of systems and perturbations. More recently, some general criteria were proposed for deriving the stability of inhomogeneous states [32, 33].

Some toy models were also studied whose states are naturally inhomogeneous: this is typically the case for systems where an external potential is present in addition to the self-consistent one [34]–[37]. A first interesting model [34]–[36], [38] is the mean-field φ^4 model: an Ising-like spin variable is represented by a scalar field in one dimension, acted upon externally by a double-well potential which selects two states; the mean-field term of the Hamiltonian is a quadratic coupling of the scalar field at two different lattice sites. A second interesting model is a generalized version of the Hamiltonian mean-field (HMF) model to which an anisotropic external potential is added [37].

In this paper we focus on the above mentioned toy models, and show that one can treat exactly the stability of inhomogeneous states. The Vlasov equation will be rewritten in action–angle variables [39]–[41] and we will focus on those inhomogeneous stationary states whose single-particle distribution function does not depend on the angle variable, i.e. those that are homogeneous in angle. We will derive a general stability criterion which, besides giving the value of the threshold energy (action) at which these stationary states destabilize, will allow us to obtain the growth rate of the instability.

In section 2 we will introduce and discuss the Vlasov equation in action–angle variables and we will derive the stability condition for inhomogeneous states and for generic mean-field and external potentials. In sections 3 and 4 we shall apply the general method introduced in section 2 to the specific cases of the anisotropic HMF model and of the mean-field φ^4 model, deriving explicit analytical expressions for the stability thresholds and for the growth rates of the instability. These theoretical predictions will be then compared with numerical simulations performed with N-body Hamiltonians. Finally, in section 5, we will draw some conclusions and we will discuss some perspectives of this work.

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2. The Vlasov equation in action–angle variables and the stability relations

Let us consider N particles in one dimension whose positions and momenta are (q_j, p_j), $j = 1, \ldots, N$. They interact through the two-body (symmetric) potential $v(q_j, q_k)$ and, in addition, each particle is trapped into the external potential $W(q_j)$. Hamilton's equations for such a system are

$$\dot{q}_j = p_j, \tag{1}$$

$$\dot{p}_j = -W'(q_j) - \partial_{q_j} V[\{q_k\}](q_j), \tag{2}$$

where $V[\{q_k\}](q_j) = (1/N)\sum_k v(q_j, q_k)$ stands for the mean-field potential acting on particle j. The $1/N$ term is a rescaling factor [42] which allows one to take the mean-field limit discussed in [14, 15]. The prime will denote, from now on, the derivative with respect to the position variable q. Equations (1) and (2) can be derived from the following Hamiltonian:

$$H = \sum_j \left(\frac{p_j^2}{2} + W(q_j) + \frac{1}{2} V[\{q_k\}](q_j) \right), \tag{3}$$

where the (q_j, p_j) are couples of canonically conjugate variables. Let us introduce the so-called empirical measure

$$f(q, p, t) = \frac{1}{N} \sum_{j=1}^{N} \delta(q - q_j(t))\delta(p - p_j(t)). \tag{4}$$

It can be shown [15] that, in the $N \to \infty$ limit, the single-particle distribution function $f(q, p, t)$ obeys the following Vlasov equation:

$$\partial_t f + p\partial_q f - (W'(q) + V'[f](q))\partial_p f = 0, \tag{5}$$

where

$$V[f](q, t) = \int\!\!\int \mathrm{d}q'\, \mathrm{d}p'\, f(q', p', t) v(q, q'), \tag{6}$$

is the averaged mean-field potential. One can also show that the N-body dynamics is well described by the Vlasov equation over times that are at least of order $\ln N$ [15]. This makes the Vlasov framework a natural one for studying such systems when a large number of particles is involved.

The Vlasov equation can also be written in Hamiltonian form using the following functional:

$$H[f] = \int\!\!\int \mathrm{d}q\, \mathrm{d}p\, f(q, p, t) \left(\frac{p^2}{2} + W(q) + \frac{1}{2} V[f](q) \right). \tag{7}$$

After having introduced the appropriate Poisson brackets for the functionals $A[f]$ and $B[f]$:

$$\{A, B\} = \int\!\!\int \mathrm{d}q\, \mathrm{d}p\, f(q, p, t) \left(\frac{\partial}{\partial p}\frac{\delta A}{\delta f}\frac{\partial}{\partial q}\frac{\delta B}{\delta f} - \frac{\partial}{\partial q}\frac{\delta A}{\delta f}\frac{\partial}{\partial p}\frac{\delta B}{\delta f} \right), \tag{8}$$

the dynamics of $A[f]$ is given by

$$\partial_t A = \{H, A\}. \tag{9}$$

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If one rewrites the single-particle distribution function in the functional form $f(q,p,t) = \int\int \mathrm{d}q'\,\mathrm{d}p'\, f(q',p',t)\delta(q-q')\delta(p-p')$, one obtains the evolution equation

$$\partial_t f(q,p,t) + \frac{\partial h}{\partial p}\frac{\partial f(q,p,t)}{\partial q} - \frac{\partial h}{\partial q}\frac{\partial f(q,p,t)}{\partial p} = \partial_t f(q,p,t) + \{h[f](q,p), f(q,p,t)\} = 0 \quad (10)$$

where $h[f](q,p) = p^2/2 + W(q) + V[f](q)$ and the brackets are now the standard Poisson brackets. This equation is nothing but the Vlasov equation (5).

It is straightforward to check that the Boltzmann–Gibbs equilibrium distribution $f_{\mathrm{BG}}(q,p) = Z^{-1}\exp(-\beta h(q,p))$, with β an arbitrary constant and Z a normalization constant, is a stationary solution of this equation (i.e. $\partial_t f_{\mathrm{BG}} = 0$). In fact, all distributions that depend on (q,p) only through h are stationary. The existence of an infinity of stationary distributions is actually responsible for the peculiar out-of-equilibrium regimes in which N-body long-range systems get trapped over very long times [1]. More specifically, starting from a generic unstable distribution, a long-range system typically relaxes towards a 'quasi-stationary' state, which can be significantly different from the Boltzmann–Gibbs equilibrium. Quasi-stationary states (QSS) can be interpreted as stable stationary states of the Vlasov equation in the $N \to \infty$ limit. The relaxation to statistical equilibrium occurs on much longer time scales, that were observed to diverge either algebraically [19] or logarithmically [37] with N, depending on whether the 'quasi-stationary' state corresponds to a stable or an unstable stationary state of the Vlasov equation. Relaxation to equilibrium is not due to collisions but due to finite-N effects (also called 'granularity'), which can be modeled by convenient kinetic equations, like Landau or Lenard–Balescu equations [1, 17, 18]. Stable stationary states of the Vlasov equation are therefore of paramount importance for understanding the dynamics of long-range systems. It is therefore crucial to determine the general conditions for stationarity and stability, for both homogeneous and inhomogeneous states.

Let us consider the stationary state $f_0(q,p)$. If one focuses on the Lagrangian trajectory of a single particle, one immediately realizes that it is a constant energy trajectory of the energy functional

$$h[f_0](q,p) = \frac{p^2}{2} + W(q) + V[f_0](q), \quad (11)$$

which is a straightforward consequence of equations (1) and (2). Hence, it is convenient to cast the dynamics into the appropriate variables associated with this trajectory, namely the 'action–angle' variables

$$J(h) = \frac{1}{2\pi}\oint p(h,q')\,\mathrm{d}q' = \frac{1}{2\pi}\oint \sqrt{2\left(h - W(q') - V[f_0](q')\right)}\,\mathrm{d}q' \quad (12)$$

$$\phi = \omega\int_0^q \frac{\mathrm{d}q'}{\sqrt{2(h - W(q') - V[f_0](q'))}}, \quad (13)$$

where the frequency ω is given by

$$\omega = \left(\frac{1}{2\pi}\oint \frac{\mathrm{d}q'}{\sqrt{2(h - W(q') - V[f_0](q'))}}\right)^{-1} = \frac{\partial h}{\partial J}. \quad (14)$$

It is important to note that the conjugate variables (J,ϕ) are not action–angle *stricto sensu*: since Vlasov dynamics is infinite dimensional and only a specific set of conserved quantities can be typically identified (e.g. the Hamiltonian, total momentum, the Casimirs $\int\int \mathrm{d}q\mathrm{d}p\, C(f(q,p))$, with C an analytic function), its integrability is not generic [43]. The term action–angle variables comes from the fact that the dynamics of a Lagrangian test particle is integrable if the single-particle distribution function is stationary. Indeed, for a stationary distribution f_0, the potential $V[f_0]$ is constant in time. Therefore, the dynamics of the test particle is that of a one-degree-of-freedom system with the associated conserved quantity $h[f_0]$, and hence integrable. A dependence of the potential on time caused by a non-stationary distribution $f(q,p,t)$ would introduce an extra 1/2 degree of freedom, thus breaking integrability *a priori*.

In this single-particle framework and for stationary distributions, the energy h depends only on the action J, so a particle evolves on a trajectory of constant 'action' J at the constant action-dependent angular speed $\dot{\phi} = \partial_J h(J) = \omega(J)$. The change of variables $(q,p) \rightarrow (\phi, J)$ being canonical, the corresponding Poisson brackets, which apply to functions of the phase space, are equivalent:

$$\{a,b\}_{q,p} = \partial_p a \partial_q b - \partial_q a \partial_p b = \{a,b\}_{\phi,J} = \partial_J a \partial_\phi b - \partial_\phi a \partial_J b. \tag{15}$$

Using this equivalence and the condition $\partial_\phi h = 0$, the Vlasov equation (5) for f_0 can be recast in the following form:

$$\partial_J h(J) \partial_\phi f_0 = \omega(J) \partial_\phi f_0 = 0. \tag{16}$$

Hence, the stationarity condition, $\partial_t f_0 = 0$, leads to $f_0 = f_0(J)$. This means in particular that the stationary distributions are those that are homogeneous in angle, with any distribution in action J. Such a result highlights the relevance of action–angle variables for the analysis of Vlasov stationary dynamics, but also for the study of QSS.

We shall now consider a perturbation δf around f_0, that is $f(\phi,J) = f_0(J) + \delta f(\phi,J)$. The linearity of the potential V with respect to the distribution, as emphasized by its definition in equation (6), implies that $V[f] = V[f_0] + V[\delta f]$. Using property (15) for the Vlasov equations (5) and (10) and neglecting second-order terms in δf leads to the linearized Vlasov equation

$$\partial_t \delta f + \omega(J) \partial_\phi \delta f - (\partial_p f_0) V'[\delta f](\phi, J) = 0, \tag{17}$$

where the factor $\partial_p f_0$ should be expressed in terms of (ϕ, J) and the derivative of V is with respect to q, and then it is also expressed in terms of (ϕ, J). The study of this equation in full generality would imply the solution of an initial value problem using a Laplace–Fourier transform and then a transformation back to action–angle variables using a Bromwich contour [17, 18]. We will be less ambitious here and we will focus on the study of an eigenmode $\delta f(\phi, J; t) = \mathrm{e}^{\lambda t} \bar{f}(\phi, J)$ with the eigenvalue λ determining the stability properties. Inserting this ansatz solution in equation (17), one gets

$$(\lambda + \omega(J)\partial_\phi)\bar{f} - (\partial_p f_0) V'[\bar{f}](\phi, J) = 0. \tag{18}$$

Assuming a non-zero ω (the frequency ω typically only vanishes on the separatrices of the single-particle phase space), the above equation turns into

$$\partial_\phi \left(\mathrm{e}^{\lambda\phi/\omega(J)} \bar{f}\right) - \frac{\mathrm{e}^{\lambda\phi/\omega(J)}}{\omega(J)} (\partial_p f_0) V'[\bar{f}](\phi, J) = 0. \tag{19}$$

After integration over the angle ϕ, and assuming that the integration constant vanishes, one gets

$$\bar{f} - \frac{\mathrm{e}^{-\lambda\phi/\omega(J)}}{\omega(J)} \int_0^\phi \mathrm{d}\phi'\, \mathrm{e}^{\lambda\phi'/\omega(J)} (\partial_p f_0)(\phi', J) V'[\bar{f}](\phi', J) = 0. \tag{20}$$

This equation can be fully cast into action–angle variables using the following relation:

$$\frac{\partial f_0}{\partial p}(q, p) = \frac{\partial J}{\partial p}\frac{\partial f_0(J)}{\partial J} = \frac{\partial h}{\partial p}\frac{\partial J}{\partial h} f_0'(J) = \frac{p}{\omega} f_0'(J), \tag{21}$$

which, inserted into equation (20), results in the following dispersion relation:

$$\bar{f} - f_0'(J)\frac{\mathrm{e}^{-\lambda\phi/\omega(J)}}{\omega^2(J)} \int_0^\phi \mathrm{d}\phi'\, p(\phi', J)\, \mathrm{e}^{\lambda\phi'/\omega(J)}\, V'[\bar{f}](\phi', J) = 0. \tag{22}$$

It is convenient to express the integral in this latter equation in terms of the position variable q'. Indeed, using equation (13), the differential $\mathrm{d}\phi'$ can be calculated as a function of q' at constant action J, which means along a single-particle trajectory. One gets

$$\mathrm{d}\phi' = \frac{\omega\, \mathrm{d}q'}{\sqrt{2\,(h - W(q') - V[f_0](q'))}} = \frac{\omega}{p}\, \mathrm{d}q'\ , \tag{23}$$

which allows one to put equation (22) into the following form:

$$\bar{f} - f_0'(J)\frac{\mathrm{e}^{-\lambda\phi/\omega(J)}}{\omega(J)} \int_0^q \mathrm{e}^{\lambda\phi'/\omega(J)}\, V'\left[\bar{f}\right](q')\, \mathrm{d}q' = 0, \tag{24}$$

in which the integration is performed at constant action J. The interest of this alternative formula is that it may be easier to solve in some cases. In particular, if one focuses on the stability threshold, given by taking $\lambda = 0$, the integral over q' can be performed straightforwardly and equation (24) can be rewritten as

$$\bar{f} = \frac{f_0'(J)}{\omega(J)} V[\bar{f}](q), \tag{25}$$

where q is, in general, a function of both action and angle.

Since all functions in angle are 2π-periodic, it is common to project the dispersion relation in a Fourier basis [39, 41]. However, since in equation (22) both the p term and the potential $V[\bar{f}]$ have generically a non-trivial dependence on the angles, one ends up with expressions where all Fourier modes are coupled. The modes are decoupled only when momentum does not depend on angle, which is the case for homogeneous states, for which the momentum coincides with the action (modulo a sign).

In what follows we will discuss a method which allows us to compute the stability threshold and the growth rate λ without resorting to a Fourier expansion. The method is, however, not generic and its application depends on the specific form of the interaction potential. We will therefore discuss two examples separately.

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3. The HMF model with additional asymmetry

Introduced in [37], the HMF model with additional cosine on-site potential is a generalization of the paradigmatic HMF model [11]–[13]. Besides the mean-field term $v(q_j, q_k) = -\cos(q_j - q_k)$, an external potential W of amplitude κ is present:

$$W(q_j) = \kappa \cos^2 q_j. \tag{26}$$

The Hamiltonian (7) reads

$$H[f] = \int\int \mathrm{d}q\,\mathrm{d}p\, f(q,p)\left[\frac{p^2}{2} + \kappa\cos^2 q - \frac{1}{2}\int\int \mathrm{d}q'\,\mathrm{d}p'\, f(q',p')\cos(q-q')\right]. \tag{27}$$

At variance with the HMF model case, the spatially homogeneous state is no longer a stationary state of the Vlasov equation, due to the presence of the on-site potential.

Using formula (22), one easily gets the dispersion relation for this model:

$$\bar{f} - f_0'(J)\frac{\mathrm{e}^{\lambda\phi/\omega(J)}}{\omega^2(J)}\int_0^{\phi} \mathrm{d}\phi'\, p(\phi',J)\mathrm{e}^{\lambda\phi'/\omega(J)}[M_x[\bar{f}]\sin(q(\phi',J)) - M_y[\bar{f}]\cos(q(\phi',J))] = 0, \tag{28}$$

where

$$\mathbf{M}[f] = M_x[f] + \mathrm{i}M_y[f] = \int\int \mathrm{d}q\,\mathrm{d}p f(q,p)\cos q + \mathrm{i}\int\int \mathrm{d}q\,\mathrm{d}p f(q,p)\sin q \tag{29}$$

stands for the magnetization. For the sake of simplicity, q, q' and p' will respectively refer to $q(\phi,J)$, $q(\phi',J)$ and $p(\phi',J)$ in the remainder of this section. Equation (28) can be solved by multiplying each term by either $\cos q$ or $\sin q$, and then integrating over phase space. One gets the following equations:

$$M_x[\bar{f}](1 - I^{\lambda}_{X,Y}[f_0]) + M_y[\bar{f}] I^{\lambda}_{X,X}[f_0] = 0, \tag{30}$$

$$-M_x[\bar{f}] I^{\lambda}_{Y,Y}[f_0] + M_y[\bar{f}]\left(1 + I^{\lambda}_{Y,X}[f_0]\right) = 0, \tag{31}$$

where

$$I^{\lambda}_{X,Y}[f_0] = \int \mathrm{d}J\,\frac{f_0'(J)}{\omega(J)}\oint \mathrm{d}\phi\,\mathrm{e}^{-\lambda\phi/\omega(J)}\, X(q)\int_0^{q} \mathrm{d}q'\,\mathrm{e}^{\lambda\phi'/\omega(J)}\, Y(q'), \tag{32}$$

and the label X (resp. Y) stands for the cos (resp. sin) function. The integration $\oint$ is performed over a single-particle trajectory.

Inhomogeneous stationary states of the Vlasov equation correspond to solutions of the linear system of equations (30)–(31) with non-vanishing (M_x, M_y). They can be found only when the determinant vanishes. This condition allows us to rewrite the dispersion relation in the form

$$(1 - I^{\lambda}_{X,Y}[f_0])(1 + I^{\lambda}_{Y,X}[f_0]) + I^{\lambda}_{Y,Y}[f_0] I^{\lambda}_{X,X}[f_0] = 0. \tag{33}$$

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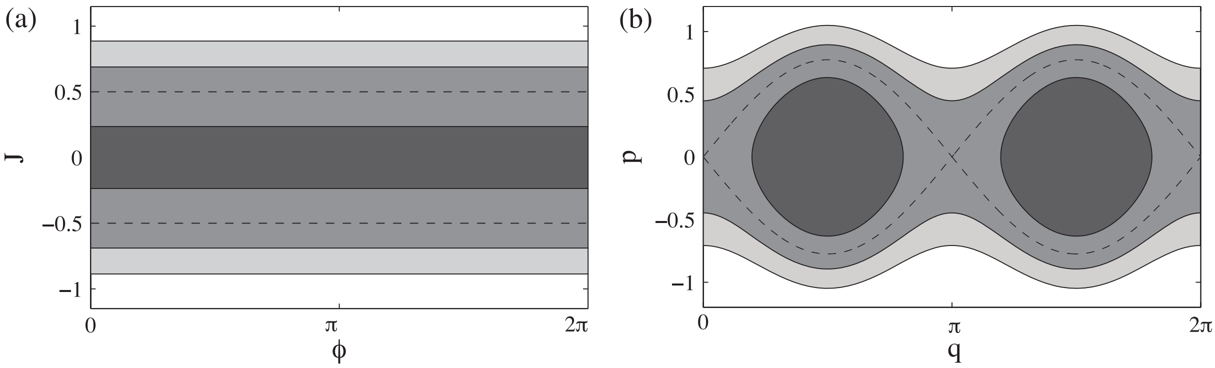

Figure 1. Waterbags in action–angle (panel (a)) and in (q, p) space (panel (b)). The waterbags have increasing boundary energies $U = 0.2$, 0.4 and 0.55 and they are represented by filled contours of lighter and lighter gray as the energy is increased. The dashed line corresponds to the separatrix, which has energy $U_s = 0.3$ and action $J_s = 0.5$.

The numerical resolution of this equation can be performed by using the explicit expressions for the action–angle coordinates [45], for a particle of energy h and position q:

$$J_{\text{in}}(h) = \frac{2\sqrt{2\kappa}}{\pi}\left[\mathscr{E}\left(\frac{h}{\kappa}\right) - \left(1 - \frac{h}{\kappa}\right)\mathscr{K}\left(\frac{h}{\kappa}\right)\right] \qquad \phi_{\text{in}}(q, h) = \frac{\pi}{2}\sqrt{\frac{\kappa}{h}}\frac{\mathscr{F}(q, h/\kappa)}{\mathscr{K}(h/\kappa)} \tag{34}$$

$$J_{\text{out}}(h) = \frac{2\sqrt{2h}}{\pi}\mathscr{E}\left(\frac{\kappa}{h}\right) \qquad \phi_{\text{out}}(q, h) = \frac{\pi}{2}\frac{\mathscr{F}(q, \kappa/h)}{\mathscr{K}(\kappa/h)}, \tag{35}$$

where the label in/out stands for inside/outside of the separatrix of the potential $\kappa\cos^2 q$, while $\mathscr{E}$, $\mathscr{K}$ and $\mathscr{F}$ are elliptic integrals of the first kind.

In order to compute the growth rate $\text{Re}(\lambda)$ from equation (33) it is necessary to choose a specific unperturbed stationary distribution $f_0(J)$. We here consider 'waterbag' distributions in action–angle space that are homogeneous in angle: these are two-level distributions, which are non-zero and homogeneous between two lines of constant action $J = J_1$ and $J = J_2$:

$$f_0(J) = \frac{1}{2\pi(J_2 - J_1)}(\Theta(J - J_1) - \Theta(J - J_2)), \tag{36}$$

where the first factor guarantees the normalization of the density f_0, while Θ is the Heaviside step function. Moreover, we here focus on waterbags delimited by a given energy U, i.e. we consider all trajectories with energies $h \leq U$ (so $J_2 = J_2(U)$ and $J_1 = 0$), such as those represented in figures 1(a) and (b). It is interesting to note that, since the change of variables $(q, p) \leftrightarrow (\phi, J)$ is canonical, $f_0(q, p)$ is also a two-step distribution with the boundary given by the curve $h(q, p) = U$. It should be pointed out that, although the action fixes the energy unequivocally, a trajectory of given energy is always split in two: those with positive and negative momentum p for $U > U_s = 0.3$, the separatrix energy, and the ones with $0 < q < \pi$ and $\pi < q < 2\pi$ for $U < U_s$. This has the consequence that, when performing integrations over the action–angle space, the two trajectories give separate

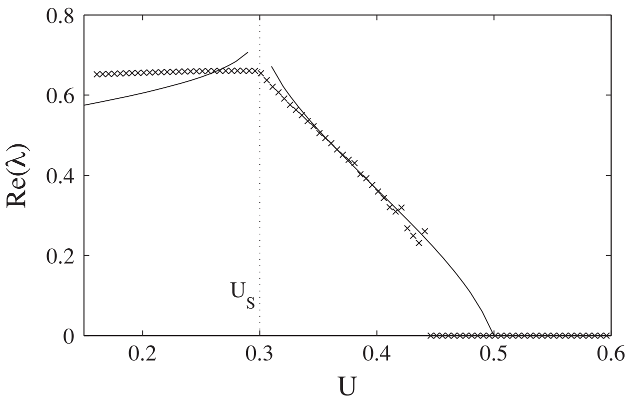

Figure 2. Growth rate $\mathrm{Re}(\lambda)$ (full line) of the instability of the inhomogeneous waterbag states obtained by solving equation (33) for waterbags with boundary energy U. The crosses are the results of numerical simulations of the N-body Hamiltonian. The agreement between theory (which describes the $N \to \infty$ limit) and numerics (performed at $N = 3 \times 10^5$) is reasonably good apart from the region near the separatrix energy $U_\mathrm{s} = 0.3$ and the one near the critical energy $U_\mathrm{c} = 0.498$, which is theoretically determined by solving equation (42).

contributions. Related to this remark is for example the evaluation of the normalization of f_0: the total area of the waterbag is indeed $2 \times 2\pi(J(U) - J(0)) = 4\pi J(U)$.

For the waterbag initial conditions, the integral in equation (32) reads

$$I^\lambda_{X,Y}[f_0] = \frac{1}{2\pi(J_2 - J_1)}\left[\frac{1}{\omega(J_1)} \oint \mathrm{d}\phi\, \mathrm{e}^{-\lambda\phi/\omega(J_1)} X(q) \int_0^q \mathrm{d}q'\, \mathrm{e}^{\lambda\phi'/\omega(J_1)} Y(q') \right.$$
$$\left. - \frac{1}{\omega(J_2)} \oint \mathrm{d}\phi\, \mathrm{e}^{-\lambda\phi/\omega(J_2)} X(q) \int_0^q \mathrm{d}q'\, \mathrm{e}^{\lambda\phi'/\omega(J_2)} Y(q') \right]. \tag{37}$$

The numerical solutions of equation (33), using equation (37), are then compared with the results of simulations performed with the N-body Hamiltonian using a sixth-order integration scheme [46] with time step 0.1. Figure 2 shows the growth rate $\mathrm{Re}(\lambda)$ obtained theoretically (full line) as a function of the boundary energy U. The growth rate is determined numerically by fitting an exponential to the short-time increase of the magnetization. One notices the existence of a threshold energy $U_\mathrm{c} = 0.498$ (determined more precisely in the following), which separates a region where the waterbag is stable ($U > U_\mathrm{c}$) from one where the waterbag is unstable ($U < U_\mathrm{c}$, $\mathrm{Re}(\lambda) > 0$). When the waterbag is stable, the N-body dynamics shows a QSS regime with zero magnetization but with an inhomogeneous distribution of particles in the q spatial coordinate. Let us remark that the theoretical results show a divergence of $\mathrm{Re}(\lambda)$ at the separatrix energy $U = U_\mathrm{s} = 0.3$ where the frequency $\omega(J_\mathrm{s}) = 0$: this divergence is not reproduced by the N-body dynamics. Moreover, in the N-body dynamics, the threshold energy is found to be around $U \approx 0.44$, well below the theoretical value. Indeed, in the energy region $0.44 < U < U_\mathrm{c}$ the growth of the magnetization is spoiled by finite-N effects, and its exponential character is no longer clear. However, the energy U_c is really the one where we numerically observe a destabilization of the zero-magnetization state.

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The critical energy U_c beyond which the waterbags become stable can be explicitly derived using equation (33) and by imposing $\lambda = 0$. Let us first note that, in this equation, the last term vanishes, since both $I^0_{X,X}[f_0]$ and $I^0_{Y,Y}[f_0]$ yield an integral of $\sin q \cos q$ over a trajectory. Consequently, the product $(1 - I^0_{X,Y}[f_0])(1 + I^0_{Y,X}[f_0])$ should be zero. Then, considering that $|p| = \sqrt{2U_\mathrm{c}}\sqrt{1-(\kappa/U_\mathrm{c})\cos^2 q}$, integrating over q', and using equation (23) and then equation (12), we finally get

$$I^0_{X,Y}[f_0] = \frac{2}{4\pi\omega_\mathrm{c} J_\mathrm{c}} \oint \mathrm{d}\phi \, \cos^2 q \tag{38}$$

$$= \frac{1}{2\pi J_\mathrm{c}} \oint \mathrm{d}q \, \frac{\cos^2 q}{p} \tag{39}$$

$$= \frac{1}{2} \frac{\oint |\mathrm{d}q| \cos^2 q / \sqrt{U_\mathrm{c} - \kappa \cos^2 q}}{\oint |\mathrm{d}q| \sqrt{U_\mathrm{c} - \kappa \cos^2 q}}, \tag{40}$$

$$I^0_{Y,X}[f_0] = -\frac{1}{2} \frac{\oint |\mathrm{d}q| \sin^2 q / \sqrt{U_\mathrm{c} - \kappa \cos^2 q}}{\oint |\mathrm{d}q| \sqrt{U_\mathrm{c} - \kappa \cos^2 q}}. \tag{41}$$

Let us explain the meaning of the uncommon notation $|\mathrm{d}q|$. When integrating over segments of the single-particle trajectory where p is negative, q decreases. Thus, both $\mathrm{d}q$ and p are negative, so their ratio or product is positive. The use of the differential $|\mathrm{d}q|$ allows us to unify notation for both the cases in which p and $\mathrm{d}q$ are positive or negative. The coefficient 2 in front of the first integral originates from the double boundary of the waterbag, be it inside or outside the separatrix. It can be shown that both expressions (40) and (41) are strictly decreasing functions of U_c. Moreover, integral (40) tends to 1 in the $U_\mathrm{c} \to \kappa$ limit, so $1 - I^0_{X,Y}[f_0]$ is always positive. The threshold of stability is thus given by solving the implicit equation

$$\oint |\mathrm{d}q| \frac{\sin^2 q}{\sqrt{U_\mathrm{c} - \kappa \cos^2 q}} = 2 \oint |\mathrm{d}q| \sqrt{U_\mathrm{c} - \kappa \cos^2 q}. \tag{42}$$

The numerical resolution of the above equation for $\kappa = 0.3$ yields the value $U_\mathrm{c} \approx 0.498$, in excellent agreement with the energy value at which $\mathrm{Re}(\lambda)$ vanishes (see figure 2).

We note that the above derivation of the threshold energy U_c corroborates the result derived in [37], where the same result was obtained by developing the single-particle distribution as a sum of derivatives of Dirac distributions. The truncation of the expansion to the very first term allowed the authors of [37] to obtain the same implicit equation (42). The approach presented here is more general, since it provides a dispersion relation for any stationary distribution, and allows us to derive the stability condition without any additional hypothesis.

We devote the final part of this section to the derivation of the growth rate of the instability and of the threshold energy for the HMF model, in the limit where the on-site potential is turned off ($\kappa = 0$). Although this result was already obtained [11, 44], its derivation in this new context allows us to point out the connection between action–angle variables (ϕ, J) and the canonical ones (q, p). In fact, when only the mean-field potential couples the particles, the non-magnetized inhomogeneous stationary states become homogeneous in q and, correspondingly, the action–angle variables reduce, modulo a sign, to the canonical coordinates

$$J = \frac{1}{2\pi} \oint p(h,q)\,\mathrm{d}q = |p|, \tag{43}$$

$$\omega = \frac{\partial h}{\partial J} = |p|, \tag{44}$$

$$\phi = \omega \int^q \frac{\mathrm{d}q'}{p} = \mathrm{sgn}(p)\,q. \tag{45}$$

The presence of absolute values is due to the fact that the action–angle variables take into account the direction of the motion along the trajectories, which are now ballistic. Then, inserting the following relations:

$$\int^q \mathrm{e}^{\lambda q'/p} \sin q'\,\mathrm{d}q' = \frac{\mathrm{e}^{\lambda q/p}}{1+\lambda^2/p^2}\left(\frac{\lambda}{p}\sin q - \cos q\right), \tag{46}$$

$$\int^q \mathrm{e}^{\lambda q'/p} \cos q'\,\mathrm{d}q' = \frac{\mathrm{e}^{\lambda q/p}}{1+\lambda^2/p^2}\left(\sin q + \frac{\lambda}{p}\cos q\right), \tag{47}$$

into equation (32), one can explicitly write the dispersion relation (33) as

$$\left(1 + \pi \int \mathrm{d}p\, \frac{f_0'(p)}{p\left(1+\frac{\lambda^2}{p^2}\right)}\right)^2 + \left(\lambda\pi \int \mathrm{d}p\, \frac{f_0'(p)}{p^2\left(1+\frac{\lambda^2}{p^2}\right)}\right)^2 = 0. \tag{48}$$

The waterbag distribution is now homogeneous in q and symmetric in p:

$$f_0(p) = \frac{1}{2\pi}\frac{1}{2\Delta p}\left(\Theta(p+\Delta p) - \Theta(p-\Delta p)\right), \tag{49}$$

and its derivative is given by

$$f_0'(p) = \frac{1}{2\pi}\frac{1}{2\Delta p}\left(\delta(p+\Delta p) - \delta(p-\Delta p)\right). \tag{50}$$

The second quadratic term in equation (48) vanishes, and one obtains

$$0 = 1 + \pi \int \mathrm{d}p\, \frac{f_0'(p)}{p\left(1+\frac{\lambda^2}{p^2}\right)} = 1 - \frac{1}{2\Delta p^2\left(1+\frac{\lambda^2}{\Delta p^2}\right)}. \tag{51}$$

We finally obtain the complex growth rate

$$\lambda = \pm\sqrt{\tfrac{1}{2} - \Delta p^2}, \tag{52}$$

which shows that the waterbag is stable beyond the threshold energy $U_{\mathrm{c}} = 1/12$, since the energy of the system is given by $U = \Delta p^2/6$.

Figure 3 shows the comparison of this analytical prediction with the numerical results obtained for the N-body simulations of the HMF model; the agreement is excellent.

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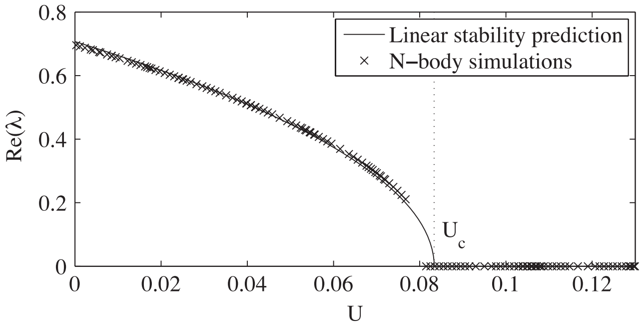

Figure 3. Growth rate $\mathrm{Re}(\lambda)$ of the instability (full line) as a function of the energy U for the HMF model (model (27) with $\kappa = 0$), as obtained analytically using formula (52). The crosses are the results of exponential fits of the short-time evolution of the magnetization for the N-body HMF Hamiltonian.

4. The mean-field φ^4 model

The second example that we consider is the mean-field φ^4 model introduced by Desai and Zwanzig [38]. It is a system where the particles are trapped in an external double-well potential, and are in addition coupled via an infinite-range force. It is described by the following Hamiltonian:

$$H[f] = \int\int \mathrm{d}q\,\mathrm{d}p\, f(q,p) \left[\frac{p^2}{2} + \left(\frac{q^4}{4} - (1-\theta)\frac{q^2}{2} \right) - \frac{\theta}{2} q \int\int \mathrm{d}q'\,\mathrm{d}p'\, f(q',p') q' \right]. \tag{53}$$

Notice that positive (resp. negative) values of the parameter θ correspond to attractive (resp. repulsive) mean-field forces. We have used the same parameterization as was introduced in [38], which can be shown to be minimal by conveniently rescaling the variables and time. The magnetization M is now defined as $M[f] = \int\int \mathrm{d}q\mathrm{d}p\, f(q,p)\, q$, so the mean-field potential is given by $V[f](q) = -(\theta/2)qM[f]$, whereas the external potential is $W(q) = q^4/4 - (1-\theta)q^2/2$. It displays a double well for $\theta < 1$ and a single well otherwise. The solution in the canonical ensemble has been recently derived in [34], emphasizing that the system exhibits a second-order phase transition. When $\theta = 1/2$, the critical temperature has been found to be $T_c \simeq 0.264$, corresponding to a critical energy $U_c^* = T_c/2 \simeq 0.132$. The model has also been solved in the microcanonical ensemble and the entropy as a function of energy and magnetization has been derived using large deviations [1, 35, 36], giving equivalent results. However, it has been shown that, in the microcanonical ensemble, the magnetic susceptibility can be negative [36, 1].

For this system, the dispersion relation (24) takes the following form:

$$\bar{f} + \theta M[\bar{f}] f_0'(J) \frac{\mathrm{e}^{-\lambda\phi/\omega(J)}}{\omega(J)} q \int_0^q \mathrm{e}^{\lambda\phi'/\omega(J)} \mathrm{d}q' = 0. \tag{54}$$

The magnetization $M[\bar{f}]$ can be factored out by multiplying this latter expression by q and by integrating it over the phase space. One gets

$$1 + \theta \int \mathrm{d}J\, f_0'(J) \oint \mathrm{d}\phi\, \frac{\mathrm{e}^{-\lambda\phi/\omega(J)}}{\omega(J)} q^2 \int_0^q \mathrm{e}^{\lambda\phi'/\omega(J)}\, \mathrm{d}q' = 0. \tag{55}$$

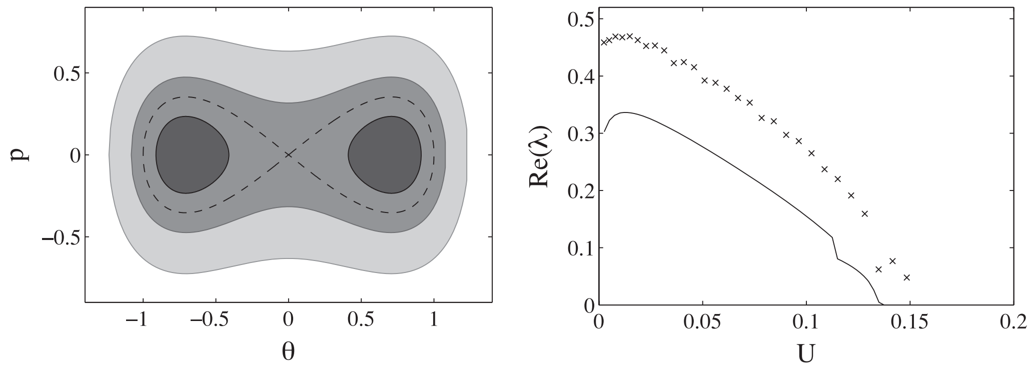

Figure 4. Left panel: representation in the (q, p) plane of the waterbags in action–angle form. The boundary energies are $U = -0.035$, 0.05 and 0.2 for lighter and lighter gray levels. The dashed line corresponds to the separatrix, which has energy $h = 0$. Right panel: growth rate $\mathrm{Re}(\lambda)$ (full line) computed by numerically solving equation (55). The crosses represent the short-time exponential rate of growth of the magnetization obtained in numerical simulations of the N-body Hamiltonian with $N = 10^6$.

Before proceeding to the numerical solution of the above dispersion relation, let us explicitly derive the expression that allows us to obtain the stability threshold by setting $\lambda = 0$ in the previous formula. The last integral in equation (55) gives trivially q, while $\mathrm{d}\phi/\omega$ can be rewritten as $\mathrm{d}q/p$ thanks to equation (23). One finally gets

$$1 + \theta \int \mathrm{d}J\, f_0'(J) \oint \frac{q^2}{p}\, \mathrm{d}q = 0. \tag{56}$$

Let us now restrict to those stationary distributions for which the mean field vanishes, i.e. $M[f_0] = 0$. This case includes those distributions that are symmetric with respect to $q = 0$. For purposes of clarity, we shall also restrict to waterbag distributions that have a boundary energy $U > 0$, i.e. $f_0(J)$ is constant for all actions $0 < J < J(U)$ and zero for $J > J(U)$. Waterbags with both positive and negative boundary energy U are shown in figure 4.

On introducing the following set of variables:

$$q = x\bar{q}, \tag{57}$$

$$\bar{q} = \sqrt{\sqrt{4h + (1-\theta)^2} - (1-\theta)}, \tag{58}$$

$$\rho = \sqrt{\frac{\sqrt{4h + (1-\theta)^2} + (1-\theta)}{\sqrt{4h + (1-\theta)^2} - (1-\theta)}}, \tag{59}$$

the momentum of a particle with positive energy h can be written as

$$p = \pm\sqrt{2(h - W(q))} = \pm\frac{\bar{q}^2}{\sqrt{2}}\sqrt{(\rho^2 - x^2)(1 + x^2)}. \tag{60}$$

Note that x varies in the range $[-\rho;\rho]$, so the maximum position along a trajectory is $\rho\bar{q}$. Now, the action–angles variables (12), (13) assume the following form:

$$J = \frac{\bar{q}^3}{2\sqrt{2}\pi}\oint\sqrt{(\rho^2-x^2)(1+x^2)}\,\mathrm{d}x \tag{61}$$

$$= \frac{\bar{q}^3\sqrt{2}}{3\pi}\left[(\rho^2-1)\mathscr{E}(-\rho^2)+(\rho^2+1)\mathscr{K}(-\rho^2)\right], \tag{62}$$

$$\omega^{-1} = \frac{\sqrt{2}}{2\pi\bar{q}}\oint\frac{\mathrm{d}x}{\sqrt{(\rho^2-x^2)(1+x^2)}} = \frac{2\sqrt{2}}{\pi\bar{q}}\mathscr{K}(-\rho^2), \tag{63}$$

$$\phi = \omega\int_0^q\frac{\mathrm{d}x}{\sqrt{(\rho^2-x^2)(1+x^2)}} = \omega\mathscr{F}\left(\frac{x}{\rho},-\rho^2\right). \tag{64}$$

Using the following relation:

$$\oint\frac{q^2}{\sqrt{2(h-W(q))}}|\mathrm{d}q| = \sqrt{2}\bar{q}\oint\frac{x^2\mathrm{d}x}{\sqrt{(\rho^2-x^2)(1+x^2)}} \tag{65}$$

$$= 4\sqrt{2}\bar{q}\left[\mathscr{E}(-\rho^2)-\mathscr{K}(-\rho^2)\right], \tag{66}$$

one can show, taking also equations (62) and (63) into account, that

$$\oint\frac{q^2}{\sqrt{2(h-W(q))}}|\mathrm{d}q| = \frac{12\pi J}{\bar{q}^2(\rho^2-1)} - \frac{4\pi\bar{q}^2\rho^2}{(\rho^2-1)\omega}. \tag{67}$$

Considering that $\bar{q}^2(\rho^2-1)=2(1-\theta)$ and $\bar{q}^2\rho^2/(\rho^2-1)=2h/(1-\theta)$, we eventually get the following expression for the stability threshold:

$$1+\frac{2\pi\theta}{1-\theta}\int_{J_0}^{\infty}f_0'(J)\left(3J-4\frac{h(J)}{\omega(J)}\right)\mathrm{d}J = 0. \tag{68}$$

Let us now consider the case of the waterbag defined by equation (36) with $J_1=0$ and $J_2=J(U)$. The dispersion relation for this waterbag reads

$$1-\frac{\theta}{1-\theta}\left(3-4\frac{U}{\omega(U)J(U)}\right) = 0. \tag{69}$$

Solving this latter equation numerically for $\theta=1/2$ gives the threshold energy $U_\mathrm{c}\simeq 0.144$, which turns out to be pretty close to the value of the statistical transition energy U_c^* found in [34]. We can also solve the dispersion relation (55) numerically and obtain the growth rate $\mathrm{Re}(\lambda)$. In figure 4 this growth rate is compared to a fit of the short-time exponential growth of the magnetization obtained by integrating numerically the N-body Hamiltonian. Unfortunately the agreement is only qualitative, although the stability threshold is correctly reproduced.

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5. Concluding remarks

Systems with mean-field interactions are well described by the Vlasov equation in the $N \to \infty$ limit. An infinity of stationary states exists for such an equation and the study of their stability is a subject of paramount importance. Many exact results concerning homogeneous stationarity have appeared in the literature and several stability criteria have been applied. Also inhomogeneous states have been treated, but the study of their stability is more complex [32, 33, 37], [39]–[41]. Characterizing analytically the stability of stationary solutions of the Vlasov equation will have an impact also on the characterization of the slow convergence to equilibrium observed in systems with long-range interactions [1]–[6], in particular on the study of quasi-stationary states (QSS), which are ubiquitous long-lived states in the N-body dynamics of long-range systems. It has been shown that the lifetime of QSS diverges algebraically with N in some simple models and it has been conjectured that this can happen only when the QSS corresponds to a stable stationary state of the Vlasov equation [1] (see also [47] for an interesting mathematical result along these lines). Again, most of the studies on QSS are for the homogeneous case.

In this paper, we have discussed a class of models where, besides the mean-field interaction, particles are subjected to an external potential. The effect of the external potential is that of creating an inhomogeneity in the spatial distribution. Hence, these models are naturally endowed with inhomogeneous stationary states. Upon rewriting the Vlasov equation in action–angle variables, we have shown that some of these inhomogeneous stationary states in conjugate coordinates transform into homogeneous stationary states that are homogeneous in angle. We have therefore applied the standard tools of linear stability of the Vlasov equation to derive a dispersion relation, given in formula (24), which is the key result of this paper. We have specialized this formula for two models: the HMF model with additional asymmetry [37] and the mean-field φ^4 model [34]–[36], [38]. For these two models it is possible to further simplify the dispersion relation and to obtain implicit equations that, solved numerically, give both the growth rate of the instability and the stability threshold. When the real part of the growth rate vanishes, the state is a stable stationary state of the Vlasov equation. We have checked these results against the numerical simulation of the Hamiltonian dynamics of the corresponding N-body system. The stability thresholds are in general in good agreement with the theoretical predictions, but for the growth rate the agreement is only qualitative for the φ^4 model. A case in which the growth rate turns out to be in perfect agreement with the simulations is that of the HMF model [11]–[13].

Those inhomogeneous stationary states that are also stable are good candidates for becoming QSS at finite N. We have therefore pointed out the existence of a new class of inhomogeneous QSS, for which it will be possible in the future to study the law of divergence of the lifetime with system size.

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