Free Electron Laser as a paradigmatic example of systems with long-range interactions

A. Antoniazzi, J. Barré, T. Dauxois, G. De Ninno, D. Fanelli*, S. Ruffo.

Abstract

We here shortly review the field of long-range interactions and presents selected issues of fundamental interest for statistical mechanics and dynamical systems theory. Applications to the case of a Single-Pass Free Electron Laser are discussed.

INTRODUCTION

Physical systems are constituted by interacting elements point particles or atoms. In presence of short-range couplings, every element is solely sensitive to the adjacent environment, being therefore uniquely subjected to the interaction with local neighbors. Conversely, when long-range forces are to be considered, the direct coupling of each element to every other element in the system must be taken into account (the system is dominated by *mean-field* effects). This crucial distinction is responsible for the enhanced degree of complexity in the treatment of long-range systems when compared to short-range ones.

Moreover, basic concepts in physics, notably in the framework of equilibrium statistical mechanics, have been developed only for short range interactions. The potential interest of such tools is however very broad since for all fundamental interactions in nature (with the exception of gravity), screening mechanisms manifest, resulting in effective short-range couplings. For this reason, it has essentially only been in the context of astrophysics and cosmology that the very specific and difficult features of long-range interactions have been tackled. Recently, however, a growing number of physical laboratory systems have emerged in which the interactions are truly long-range, e.g. unscreened Coulomb interactions, vortices in two-dimensional fluid mechanics, wave-particle systems relevant to plasma physics and Free-Electron Lasers (FELs). These developments gave new impetus [1] to attempts aiming at describing the peculiar behaviour of long-range interacting systems, in a context where, in contrast to astrophysics, laboratory experiments are possible. Moreover, a number of " toy models" have been proposed that provide the ideal ground for theoretical investigations. Among others, the Hamiltonian Mean Field (HMF) model [2] is nowadays widely analyzed for pedagogical reasons, because of its intrinsic simplicity.

In presence of long-range interactions, physics is in fact very peculiar and a wide range of striking and curious phenomena appears. Importantly, energy is non additive, hence the system under scrutiny cannot be divided into independent macroscopic parts, as it is usually the case for short-range interactions. This fact leads to unexpected consequences when performing the analysis in terms of statistical mechanics.

Single-pass FELs constitute an example of systems with long-range interactions, where the interplay between collective (wave) and individual (particles) degrees of freedom is well known to be central. This interplay being essentially non dissipative, its prototype is described by a self-consistent Hamiltonian [3], which provides a clear and intuitive picture of the basic mechanisms that drive the process of light amplification and saturation. In this respect, FELs provides a very general experimental ground to investigate the universal features that characterize systems with long range interactions.

In this paper we shall present a short review of recent progress in this field of research by focusing in particular on the relevant case of a Single Pass FEL. A wider overview on long range systems can be found in Ref. [1].

* duccio.fanelli@ki.se

DEFINITION OF LONG-RANGE SYSTEMS

Let us consider the potential energy U of a given particle positioned in the center of a sphere of radius R. Assume that the matter is homogeneously distributed with density ρ and introduce a lower cut-off $\varepsilon << R$, as depicted in Fig. 1. Focusing on the case where the interaction potential decays at large distances with a power-law with exponent α, one gets:

$$
\begin{aligned}
U &= \int_\varepsilon^R \rho \left[\frac{1}{r^\alpha}\right] 4\pi r^2 dr = \\
&= 4\pi\rho \int_\varepsilon^R r^{2-\alpha} dr \propto \left[r^{3-\alpha}\right]_\varepsilon^R \sim R^{3-\alpha} \ ,
\end{aligned}
$$

Clearly, the above energy diverges with R *if the exponent* α *is smaller than* 3, which in turn corresponds to the dimension of the physical space where the interaction is embedded.

This argument holds true in any dimension d, implying that similar divergences are found when $\alpha \leq d$. We hence *define* the interaction to be *long-range* if $\alpha \leq 3$ (resp. $\alpha \leq d$). Equivalently, it can be said that the contribution to the energy due to the surface of the sphere with respect to the bulk can be safetly neglected *only if* $\alpha > 3$. In the latter case, the interaction is of the *short-range* type.

As anticipated in the preceding discussion, long-range forces arise in many different contexts. Few physical examples of broad relevance are listed below:

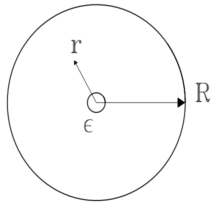

Figure 1: *The energy in a sphere of radius R diverges as $R^{3-\alpha}$ if the interaction is long-range and $\alpha \leq 3$.*

- **Gravity**: $\alpha = 1, d = 3$. In addition to the long range nature of the interaction, the system displays a singularity at the origin that needs to be carefully tackled by involving dedicated regularization schemes.
- **Coulomb interactions**: $\alpha = 1, d = 3$.
- **Dipolar interactions**: $\alpha = 3, d = 3$. The shape-dependence phenomenon is found, which makes the energy dependent on the form of the sample.
- **Onsager's 2D vortex systems** or **2D Coulomb systems**. Here $\alpha = 0$, because the interaction decays logarithmically, and $d = 2$.
- **Mean-Field**: $\alpha = 0$, due to the infinite-range interaction, and any d. This latter cathegory does not reflect a pure physical example, but rather a useful (and common) approximation to which one resorts when treating complex problems. In fact, it provides the ideal setting to gain insight into a number of different physical problems and eventually enables to perform analytical calculations. As we shall see, Single Pass FELs can be effectively described by resorting to a mean-field type of approach .

THE PROBLEM OF ADDITIVITY

The long range nature of the interaction reflects in a number of peculiar phenomena which will be shortly reviewed in the forthcoming sections. Such curious behaviours are intrinsically connected to the *non additivity* of the system, a crucial concept that one can illustrate with reference to a specific case, namely the Curie-Weiss model of magnetism. Consider the following Hamiltonian:

$$H = -\frac{J}{2N} \sum_{i,j} S_i S_j, \qquad (1)$$

where $S_i = \pm 1$ labels a spin variable, J is the ferromagnetic coupling and the sum extends over all pairs spins, N being their total number. Note that Hamiltonian (1) is *extensive*, since $H \propto N$, bu, as we shall see, *not additive*.

Refer, in fact, to the situation schematized in Fig. 2 where all the spins in the left portion of the space are equal

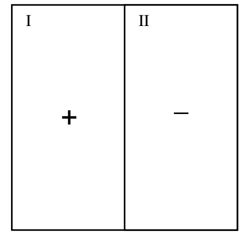

Figure 2: *A microscopic configuration with total zero magnetization, illustrating the lack of additivity in the Curie-Weiss model.*

to 1, whereas the ones in the right are assumed to be -1. Under this conditions the energy of the two regions reads $E_I = E_{II} = -(J/8)N \neq 0$. Conversely, when computing the total energy, one finds $E = -J/(2N)(N/2 - N - 2)^2 = 0$. The relation $E = E_I + E_{II}$ does not hold, and consequently the system is not additive. The underlying reason is that the above Hamiltonian is long range, since every spin interacts with all the others. This in turn implies that the energy at the interface cannot be neglected, the latter being of the same order of the energies in the two bulks. As a side remark, we shall note that the lack of additivity affects dramatically the usual construction of the canonical ensemble, and therefore peculiar behaviours are to be expected for a long-range system in contact with a thermal reservoir. Next paragraph is devoted to a short account on this topic.

THERMODYNAMICS PECULARITIES

As previosuly pointed out, the non-additivity issue is responsible for a number of important and non trivial consequences. As soon as first order phase transitions are present, "convex intruders" in the microcanonical entropy appear [4, 5]. A typical situation is displayed in Fig 3. Only if interactions are short-range and, hence, the additivity property holds, the states in the convex entropy region $\epsilon_1 < \epsilon < \epsilon_2$ of Fig 3a are unstable when compared to those obtained by combining, in appropriate portions, the two limiting states with energies ϵ_1 and ϵ_2. This is not the case if long-range interactions are to be considered and additivity is violated. The presence of stable, non-concave, entropy regions implies the appearance of *negative specific heat* in the microcanonical ensemble. As an immediate consequence *statistical ensembles are not equivalent*, since the specific heat is always positive in the canonical ensemble. Note that, *large deviation techniques* allow one to solve, both in the microcanonical and in the canonical ensemble, a large class of mean-field models and constitute therefore a powerful tool to detect possible discrepancies.

Another intriguing effect is related to temperature jumps, as depicted in Fig. 3b. In this case, two branches of the entropy exist, respectively a high and a low energy ones. When crossing each other, they generically form an angle: this sets the origin of the temperature jumps, the derivative

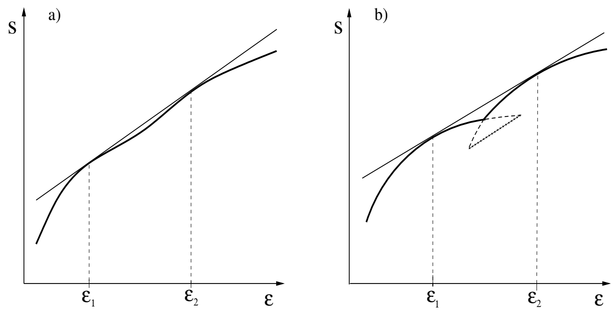

Figure 3: (a) *Entropy $s(\epsilon)$ can be non-concave in the microcanonical ensemble, giving rise to negative specific heat.* (b) *In the non-concave region, entropy can develop kinks in the slope, creating temperature jumps.*

of the entropy with respect to energy being different on the two sides of the intersection point.

NON-LINEAR DYNAMICAL ASPECTS

In this Section we shall shortly report about a selection of peculiar dynamical features, which are generically encountered when dealing with long-range interacting systems. First, the existence of *quasi-stationary states* has been often discussed in the literature with reference to various test models. Particularly important is the case of the so called Hamiltonian Mean Field (HMF) model [2], which describes the motion of N coupled rotators and is characterized by the following Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2N}\sum_{i,j=1}^{N}\left[1-\cos(\theta_j-\theta_i)\right] \qquad (2)$$

where θ_j represents the orientation of the j-th rotor and p_j is its conjugate momentum. Note that the HMF model correpsonds to the XY model with $J=1, K=0$. To monitor the evolution of the system, it is customary to introduce the magnetization, a global order parameter defined as $M = |\mathbf{M}| = |\sum \mathbf{m}_i|/N$, where $\mathbf{m}_i = (\cos\theta_i, \sin\theta_i)$ is the local magnetization vector. A second order phase transition is found at $\epsilon_c = E_c/N = 0.25$ ($T_c = 0.5$). For energy close to the transition value, the finite N system can remain trapped in *quasi-stationary* states whose life-time increases with a power of N. An example of the evolution of magnetization from an initial homogeneous (non-magnetized) water-bag state to the final maximum entropy state is displayed in Fig. 4. As N is increased the lifetime of the non-equilibrium quasi-stationary state increases: curves with growing N go from left to right. Importantly, in this intermediate regime, the magnetization is lower than predicted by the Boltzmann–Gibbs equilibrium and the system apparently displays non Gaussian velocity distributions. The above phenomena can be successfully interpreted in the framework of the statistical theory of the Vlasov equation [6, 7], a wide general approach originally introduced in the astrophysical and 2D Euler turbulence contexts.

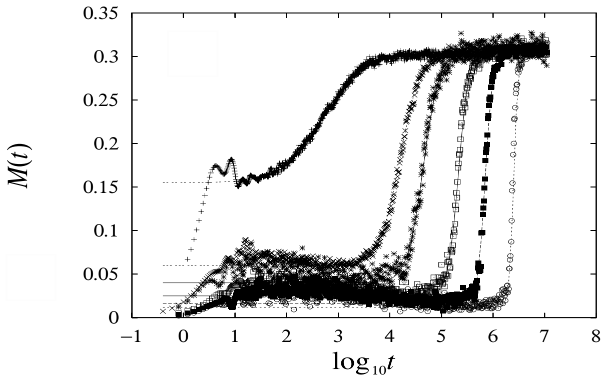

Figure 4: *Temporal evolution of the magnetization $M(t)$ for different particles numbers: $N = 10^2(10^3)$, $10^3(10^2)$, $2.10^3(8)$, $5.10^3(8)$, $10^4(8)$ and $2.10^4(4)$ from left to right (the number between brackets corresponding to the number of samples).*

Finally, let us mention that recently [8], in analyzing a Ising model with both short and long-range interactions on a ring, it has been realized that the accessible region of extensive parameters (energy, magnetization, etc.) may be non convex. This implies that *broken ergodicity* can appear, due to the fact that the accessible magnetization states at a given energy can be disconnected.

THE CASE OF A SINGLE-PASS FEL

For Single-Pass FEL amplifiers, the Colson-Bonifacio [3] model applies:

$$\frac{d\theta_j}{d\bar{z}} = p_j \qquad (3)$$

$$\frac{dp_j}{d\bar{z}} = -\mathbf{A}e^{i\theta_j} - \mathbf{A}^* e^{-i\theta_j} \qquad (4)$$

$$\frac{d\mathbf{A}}{d\bar{z}} = i\delta\mathbf{A} + \frac{1}{N}\sum_j e^{-i\theta_j} \qquad (5)$$

The complex field amplitude $\mathbf{A}$ is the degree of freedom associated to the wave, while θ_i, p_i are the conjugate variables related to the electron position and “momentum”. $\bar{z}$ is the (rescaled) longitudinal position along the undulator, N is the number of electrons and δ stands for the so-called detuning parameter. The above model can be derived from the *mean-field* Hamiltonian

$$H_N = \sum_{j=1}^{N}\frac{p_j^2}{2} - N\delta I^2 + 2I\sum_{j=1}^{N}\sin(\theta_j-\varphi)\ , \qquad (6)$$

where $\mathbf{A} = I\exp(-i\varphi)$. In this respect, FELs fall naturally in the realm of systems with long-range interactions.

The microcanonical equilibrium entropy of this model can be obtained using large deviation techniques [6]. Ensembles are equivalent, no negative specific heat or temperature jump appears. However, the evolution towards

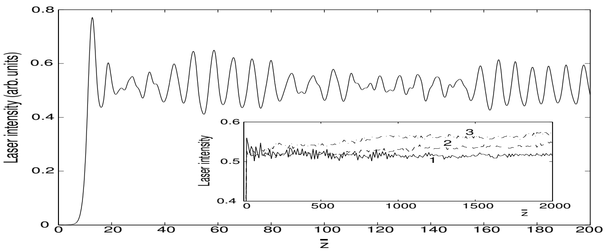

Figure 5: *Typical evolution of the radiation intensity using Hamiltonian (6). The detuning δ is set to 0, the energy per electron $H/N = 0.2$ and $N = 10^4$ electrons are simulated. The inset presents averaged simulations on longer times for different values of N: $5 \cdot 10^3$ (curve 1), 400 (curve 2) and 100 (curve 3).*

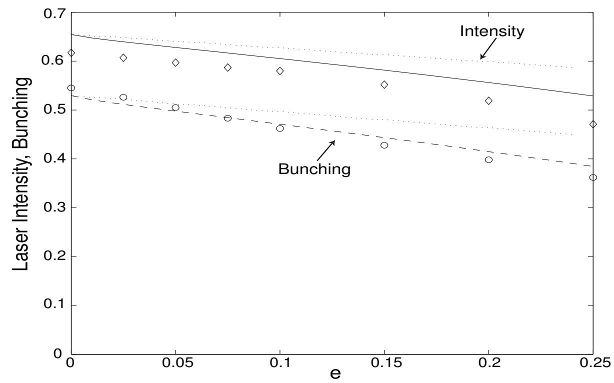

Figure 6: *Comparison between theory (solid and long-dashed lines) and simulations (symbols) for a non monoenergetic beam when the energy (e) characterizing the initial velocity dispersion of the initial electron beam, is varied. The dotted lines represent the intensity and bunching ($b = |\sum_j e^{i\theta_j}|/N$) predicted by the full (Boltzmann-Gibbs) statistical equilibrium, not very appropriate here, whereas the solid line and long-dashed lines refer to the Vlasov equilibrium defined by. The discrepancy between theory and numerical experiments is small over the whole range of explored energies.*

the maximum entropy equilibrium state is highly non trivial, as shown in Fig. 5, and shares many similarities with the case of the HMF. After the initial exponential growth, the system converges to a *quasi-stationary state* that gets more pronounced when increasing the number of simulated electrons N. Finally it relaxes to the Boltzmann-Gibbs equilibrium, driven by granularity. The intermediate quasi-stationary state is indeed the only experimentally relevant regime, due to the finite extension of the undulators. As already pointed out, the latter state, is a *Vlasov equilibrium*, sufficiently well described by Lynden-Bell's Fermi-like distribution arising from "violent relaxation" (constrained maximum entropy principle) [9, 10, 6]. Results reported in Fig. 6 provide a clear support to this conclusion.

CONCLUSIONS

In this paper, we discussed the concept of long-range interactions and shortly reviewed the fundamental properties of such systems. In particular, microcanonical and canonical ensembles disagree for long range interactions at canonical first order transitions. Negative specific heat and temperature jumps are typical signatures of this ensemble inequivalence. Dynamical non equilibrium features arise, and more specifically, *quasi-stationary* states, where the system gets trapped for long times. Their life-time increases in fact with system size N. Moreover, as an indirect signature of non-additivity, broken ergodicity manifests as a generic feature of systems with long-range interactions.

Further we focused on a simple model of the Free Electron Laser dynamics, which shares many similarities with the so called Hamiltonian Mean Field model, a paradigmatic toy-model often referred to for theoretical applications. For the case of the FEL, we have shown that collective phenomena for wave-particle interactions can be succesfully interpreted through a constrained maximum entropy principles of the associate Vlasov system. Given the above, it is here anticipated that FELs will constitute a promising experimental set up to investigate the universal pecularities that characterize systems with long range interactions.

REFERENCES

[1] T. Dauxois, S. Ruffo, E. Arimondo and M. Wilkens (eds), *Dynamics and thermodynamics of systems with long-range interactions*, Lecture Notes in Physics 602 (Springer, Berlin, 2002).

[2] M. Antoni and S. Ruffo, *Phys. Rev. E* **52** (1995) 2361.

[3] W. B. Colson, Phys. Lett. A **59**, 187 (1976); R. Bonifacio et al., Opt. Comm **50**, 373 (1984).

[4] J. Barré, D. Mukamel and S. Ruffo, *Phys. Rev. Lett.* **87** (2001) 030601.

[5] D. H. E. Gross, *Microcanonical Thermodynamics: Phase Transitions in Small Systems*, Lecture Note in Physics, **66**, World Scientific, Singapore (2001).

[6] J. Barré, T. Dauxois, G. De Ninno, D. Fanelli, S. Ruffo, *Phys. Rev. E*, Rapid Comm. **69** (2004) 045501.

[7] F. Curbis, A. Antoniazzi, G. De Ninno, D. Fanelli, submitted to *Phys. Rev. E*

[8] D. Mukamel, S. Ruffo and N. Schreiber, *Phys. Rev. Lett.*. **95**, 240604 (2005)

[9] D. Lynden-Bell, *Monthly Notices of the Royal Astronomical Society* **136**, 101 (1967).

[10] P.H. Chavanis, J. Sommeria, R. Robert, *Astrophys. J.*, **471**, 385 (1996).