Near-Critical Reflection of Internal Waves in a Stably Stratified Fluid

Thierry Dauxois[1,2], Erika E. McPhee[3] and W. R. Young[1]

[1]*Scripps Institution of Oceanography, La Jolla, CA 92093-0230, USA*

[2]*Ecole Normale Supérieure, 46, Allée d'Italie, 69007 Lyon, France*

[3]*School of Oceanography, Box 357940, Seattle, WA 98133, USA*

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The reflection of near-critical internal waves over sloping topography plays a crucial role in determining exchanges between the coastal ocean and the adjacent deep waters. Previous theoretical descriptions of this reflection process have been framed largely in terms of linear and stationary wave dynamics. These restrictions lead to the unrealistic prediction that the amplitude of the reflected wave is infinite when the angle of the slope is critical. Here we explain the role of transience and nonlinearity in healing this singularity. Our analytic solutions show good agreement with experimental visualizations of the initial stages of the reflection process. The subsequent overturning instability is responsible not only for the transition to turbulence near the slope, but presumably may also be related to the stepped temperature profile in lakes and to nepheloid layers in the oceans.

Recent direct measurements of mixing in the ocean using tracer release experiments[1] have vindicated decades of microstructure measurements and theoretical inferences. In particular, these measurements have shown[2] that most of the vertical mixing is not taking place in the ocean interior, but close to the boundaries and topographic features. These results have directed attention to the possible role of *internal waves reflection* in the boundary-mixing process.

Internal waves have different properties of reflection from a rigid boundary[3] than do sound or light waves. Instead of following the familiar Snell's law, internal waves reflect off a boundary such that the angle with respect to *gravity* is preserved upon reflection (Fig. 1). This peculiar reflection law leads not only to astonishing properties, such as the internal wave attractor[4], but also to a concentration of the reflected energy density into a narrow ray tube upon reflection. In the near-critical reflection case, the result is even more dramatic since the reflected wave grazes along the slope as in Fig. 1.

Near the bottom of a steep flank of a tall North Pacific Ocean seamount, Eriksen[5] has recently reported a signature of the internal wave reflection process: there is a clear departure from the quasi-universal Garett-Munk spectral model in a frequency band centered on the frequency at which ray and bottom slopes match. And several experimental[6–8] and numerical[9] studies have been dedicated to understanding internal waves reflection.

However, lack of theoretical understanding, even of the linear aspects of the reflection process, has hindered progress. For example, the configuration in Fig. 1 shows a 'stationary' case in which the incident beam was applied in the distant past so that the entire wave pattern (incident plus reflected) has had a great deal of time to establish itself. This is an important and unrealistic assumption, as can be seen by considering the critical case in which β, the angle between the group velocity and the horizontal, is equal to γ, the angle between the horizontal and the bottom slope. In this special case, the stationary solution predicts that the reflected rays lie along the slope, that the wavelength of the reflected wave is zero, and that the reflected energy density is infinite. But if $\beta = \gamma$ the group velocity of the reflected wave vanishes so that there can never be enough time to establish all these singularities. Similar consideration apply when β and γ are almost equal: the time required to set-up the stationary wave pattern diverges as $\beta \to \gamma$. Thus, it is crucial to understand the nearly critical (i.e., $\beta \approx \gamma$) initial value problem in which the incident wave is 'switched-on'.

To analyze the initial value problem we consider a two-dimensional, incompressible and inviscid Boussinesq fluid[10], with constant Brunt-Väisälä frequency N and a uniformly sloping bed. The incident wave is a nearly monochromatic group of internal gravity waves, which switches-on suddenly at $t = 0$. We develop a reductive approximation, based on taking a distinguished limit in which $|\beta - \gamma| \to 0$ with the maximum amplitude of the streamfunction $\psi_{\max} \propto |\beta - \gamma|^{3/2}$. The theory[10] employs also a matched asymptotic expansion in which the incident wave is in the outer region and the developing reflected disturbance is largely confined to an inner region. The inner region is essentially a boundary layer close to the slope.

Because of the quadratic terms in the equations of motion, a second harmonic is nonlinearly excited in the boundary layer where the advective terms become important. For $\beta + \gamma < \pi/3$, the group velocity of the second harmonic is nonzero; if $\beta + \gamma > \pi/3$ this second-harmonic wave is evanescent on a scale which is much greater than the boundary layer thickness. Thus, in either case, the second harmonic is not trapped in the boundary layer and consequently the inner region drives the outer region through the radiation of the second harmonic into the outer region.

One curious aspect of the weakly nonlinear theory is that the final reduced description is a linear equation (at the solvability order in the expansion all of the resonant nonlinear contributions cancel amongst themselves). However the reconstructed fields do contain nonlinearly driven second harmonics. For example, at small times the streamfunction consists of a regular array of counter-rotating vortices. However, the nonlinearity is responsible for a symmetry breaking in which alternate vortices differ in strength and size from their immediate neighbours (see Fig. 2).

Figures 3 shows the distortion of isopycnals: the experimental result are shown in panel (a) whereas panel (b) corresponds to the analytic solution of the initial value problem. There is a good visual comparison between the two. Far from the slope, the disturbance is very small and one sees essentially the initial background stratification. Closer to the slope, the disturbance folds up the isopycnals, and this produces a region of static instability. The buoyancy field is therefore unstable well before the stationary solution, corresponding to the linearly reflected wave identified by Phillips[3], can be attained for $t \to \infty$.

In the precisely critical case, $\beta = \gamma$, an explicit solution of the initial value problem shows that the buoyancy perturbation and the along-slope velocity both grow linearly with time, while the scale of the reflected disturbance is reduced as $1/t$. During the course of this scale reduction, the stratification is 'overturned' and the Miles-Howard condition for stratified shear flow stability is violated. Both of these indications of instability occur at the slope.

The transition to turbulence which eventually disrupts the laminar distortion in figure 3 is responsible for boundary mixing. In the ocean interior, the breaking of internal waves is driven by the shear instability[11]. But here, for all slope angles, the creation of statically unstable stratification by 'overturning' occurs before the Miles-Howard stability condition is violated. Thus we argue that the first instability is convective, and located close to the continental slope.

The convective instability eventually produces turbulent vortices at the boundary, and the subsequent mixing leads to intrusions of boundary layer fluid away from the mixing region, as seen in many experiments[8]. This mechanism is likely responsible for the formation of highly "stepped" temperature profiles near steep slopes in lakes[12] and, near ocean boundaries, the intrusion mechanism would lead to the seaward transport of fine particles. The formation of suspended sediment layers (nepheloid layers) at continental slopes has been linked to critical angle reflection of internal waves[13], and suggests that this process plays an important role in the transport of sediments in the fluid environment.

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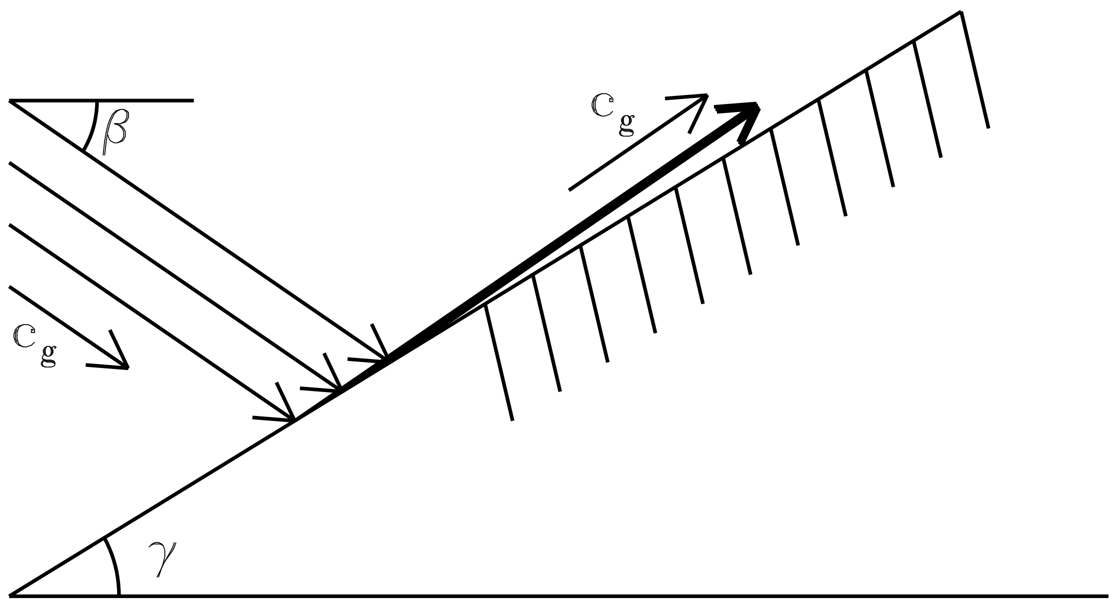

FIG. 1. Schematic view of the reflection process when the incident wave nearly satisfies the critical condition $\gamma \approx \beta$. The group velocity of the reflected wave makes a very shallow angle with the slope. $\mathbf{c}_g$ indicates the incident and reflected group velocities. The reflection law leads to a concentration of the energy density into a narrow ray tube.

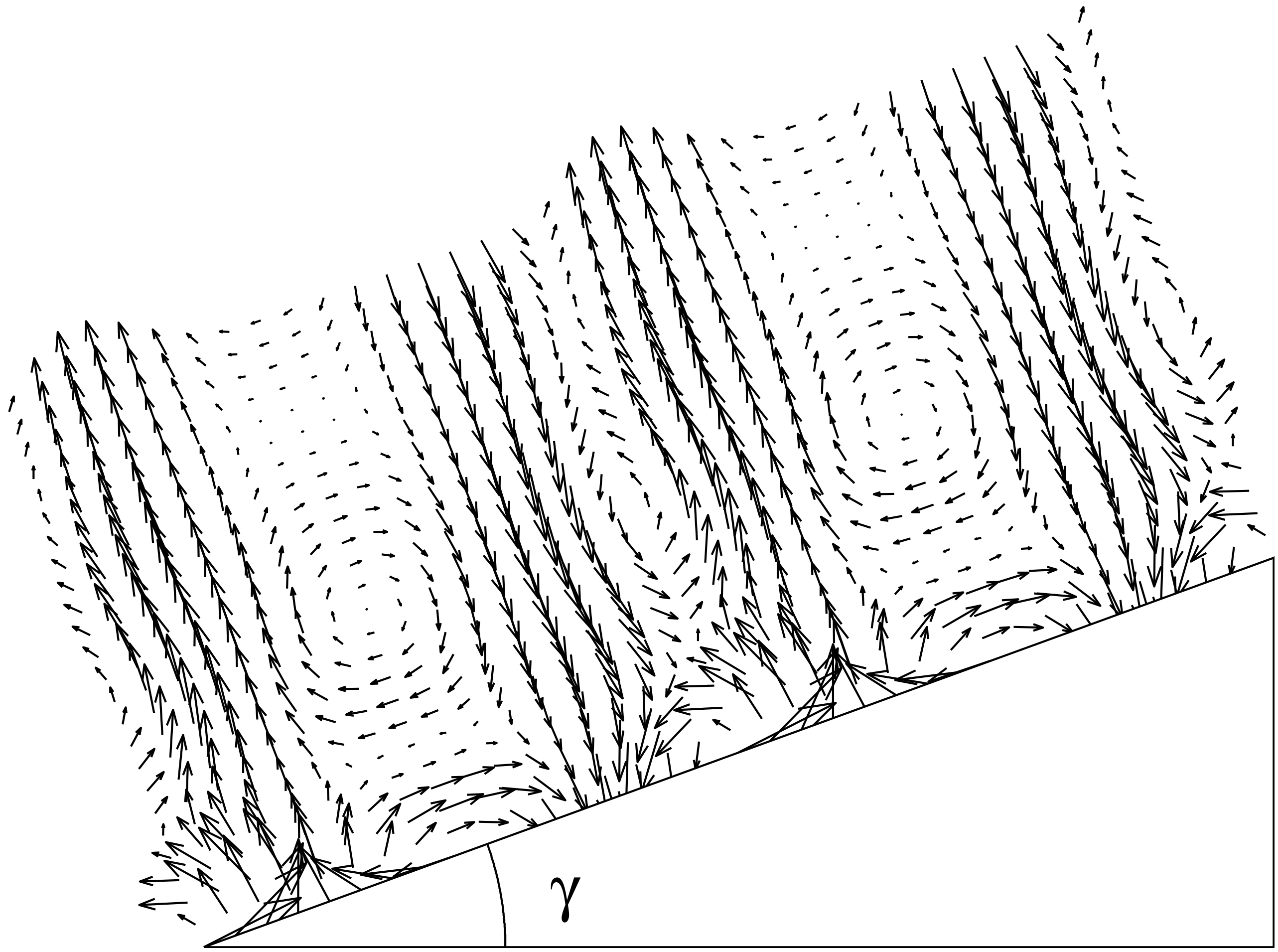

FIG. 2. "Velocity vector" visualization of the streamfunction at $N \sin\beta\ t = 3$ in the critical case $\gamma = 20°$ ($\varepsilon = 0.3$). There are two pairs of counter-rotating vortices immediately adjacent to the slope. The clockwise vortices are slower and thinner than the counter-clockwise vortices. This asymmetry is a result of the second harmonic term.

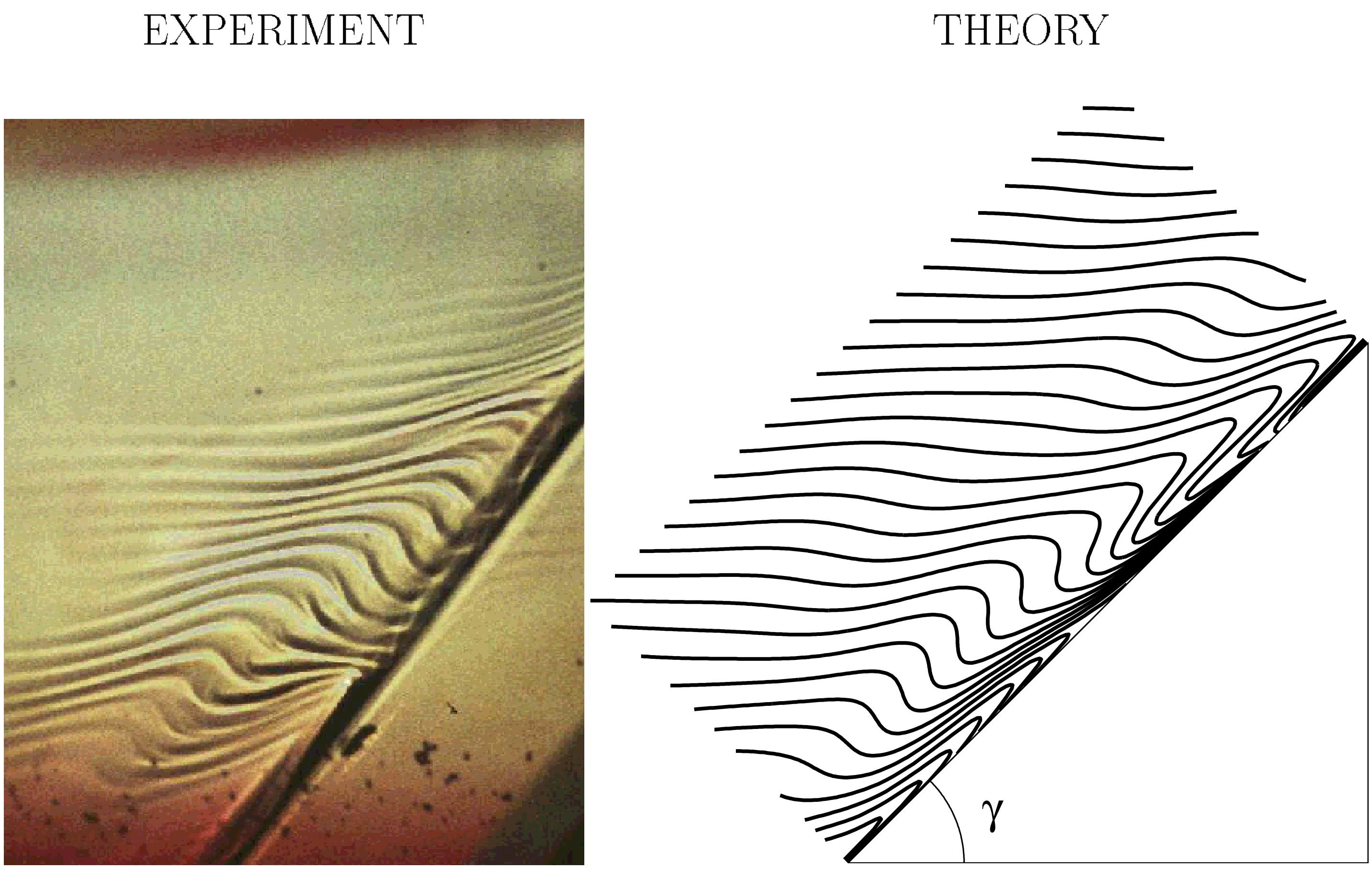

FIG. 3. *Buoyancy field.* Panel (a) shows an experimental shadowgraph picture showing the reflection of a near-critical internal wave on a slope with $\gamma = 45°$. The internal waves, created by an oscillating cylinder, are supported by a linear salt stratification with a buoyancy frequency $N \simeq 1.25$ rad/sec. The shadowgraph image is made using a slide projector as a point source of light; the light refracts off of density perturbations and produces the wavy lines, emphasizing the fold up of the isopycnals. Panel (b) shows the theoretical buoyancy field at $N \sin\beta\ t = 30$ in the near critical case $\beta = 38°$; $\gamma = 45°$; $\varepsilon = 0.45$.