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Nonlinear stability of counter-rotating vortices

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Recently, Mallier and Maslowe [Phys. Fluids A **5**, 1074 (1993)] found an exact nonlinear solution of the inviscid, incompressible, two-dimensional Navier–Stokes equations, representing an infinite row of counter-rotating vortices, which extended the previous Kelvin–Stuart vortices. The aim of this work is to establish explicit sufficient conditions for the nonlinear stability of this solution. The result is derived with the energy-Casimir stability method as a function of the parameters of the solution and the domain size. The size of the domain over which the street of vortices is unstable is exhibited.

In the case of inviscid and incompressible fluid, the vorticity equation for two-dimensional motion of the fluid is of the form:

$$\frac{\partial \omega}{\partial t}=\{\psi,\omega\} \tag{1}$$

where ψ is the streamfunction, $\omega=-\Delta\psi$ the vorticity, t the time and $\{,\}$ the usual Poisson bracket. The solution of Eq. (1), introduced by Mallier and Maslowe,[1] that we would like to discuss is

$$\psi_e=\psi(\omega_e)=\log\left(\frac{\cosh \varepsilon y-\varepsilon \cos x}{\cosh \varepsilon y+\varepsilon \cos x}\right)=-2\ \mathrm{Arcth}\left(\frac{\varepsilon \cos x}{\cosh \varepsilon y}\right) \tag{2}$$

which describes a stationary pattern in the form of a street of counter-rotating vortices, arranged periodically along the x-axis at a distance equal to π. A typical aspect of the solution is shown in the Fig. 1 for $\varepsilon=0.9$. The parameter ε characterizes the density of vorticity: when $\varepsilon=\pm 1$, we recover the point vortices solution and when $\varepsilon=0$ we have $\psi=0$. Thus, as ε ranges from 0 to 1, the flow represented by Eq. (2) ranges from the fluid at rest, to the flow due to a set of point vortices on the x-axis.

This solution has to be connected to the row of identical vortices, which was introduced by Stuart[2] and which are the streamlines of the celebrated "Kelvin–Stuart's cat's eyes." The solution (2), because it is an exact consequence of certain equations, is of theoretical and illustrative value, and especially, it would be of great interest to know from a stability analysis whether this solution is stable in a finite domain. Indeed, the analytical expressions of the identical or counter-rotating street vortices are specially relevant for the studies of the stability or instability of experimental fluid flows.[3] In the two already mentioned limiting cases, the answer is simple. On the one hand, the fluid at rest ($\varepsilon=0$) is clearly stable; and on the other hand, the flow due to a set of point counter-rotating vortices ($|\varepsilon|=1$) is unstable.[4,5] The aim of this paper is to establish explicit sufficient stability conditions for all values of ε.

In order to study the nonlinear stability of the counter-rotating vortices in a domain D of the plane R^2, we will use the total energy on this domain, which takes the form:[6]

$$\begin{aligned} H(\omega)&=\int_D \frac{1}{2}|v|^2\,dx\,dy\\ &=\frac{1}{2}\int_D \omega(-\nabla^2)^{-1}\omega\,dx\,dy\\ &\quad+\frac{1}{2}\sum_i \psi_{|(\partial D)_i}\int_{(\partial D)_i}\frac{\partial\psi}{\partial n}\,ds \end{aligned} \tag{3}$$

where n is the outward unit normal of the boundary ∂D and ds the scalar infinitesimal arc element. As the fluid is inviscid, this quantity is conserved, and more generally, one can show also,[7] that the functionals $C_\Phi(\omega) = \int_D\Phi(\omega)dx\,dy$, called Casimir, are also conserved for any function Φ in R. To establish sufficient conditions for the nonlinear stability, we will employ convexity properties of $H_\Phi=H+C_\Phi$ to find an explicit norm. We get

$$\begin{aligned} H_\Phi(\omega)&=\int_D\left(\frac{1}{2}\omega(-\nabla^2)^{-1}\omega+\Phi(\omega)\right)dx\,dy\\ &\quad+\frac{1}{2}\sum_{i=1}^{g}\psi_{|(\partial D)_i}\int_{(\partial D)_i}\frac{\partial\psi}{\partial n}\,ds \end{aligned} \tag{4}$$

and integrating twice by parts and using the boundary conditions, it reads

$$DH_\Phi(\omega_e)\delta\omega=\int_D(\psi(\omega_e)+\Phi'(\omega_e))\delta\omega\,dx\,dy. \tag{5}$$

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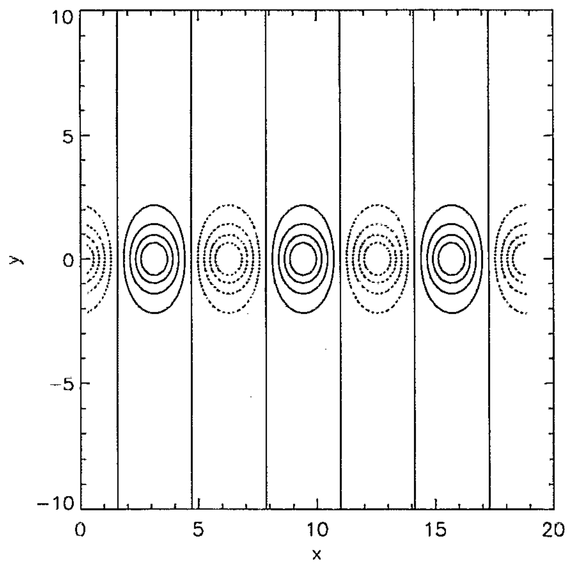

FIG. 1. Streamlines of the solution (2) for $\varepsilon=0.9$. The dashed lines correspond to the negative contour lines and the full lines to the positive ones.

Let us choose the function Φ so that $DH_\Phi(\omega_e)=0$. As the solution (2) satisfies[1] the sinh-Gordon equation $\Delta\psi=-((1-\varepsilon^2)\sinh 2\psi)/2$, we have $\psi(\omega_e)=\frac{1}{2}\operatorname{Arcsinh}(2\omega_e/(1-\varepsilon^2))$. Therefore, we obtain

$$\Phi(\lambda)=\int_0^\lambda -\psi(s)ds$$
$$=-\frac{1}{2}\left[\lambda\log\left(\frac{2\lambda}{(1-\varepsilon^2)}+\sqrt{\frac{4\lambda^2}{(1-\varepsilon^2)^2}+1}\right)\right.$$
$$\left.-\sqrt{\lambda^2+\frac{(1-\varepsilon^2)^2}{4}}+\frac{(1-\varepsilon^2)}{2}\right]. \quad (6)$$

Differentiating this twice, we get $\Phi''(\lambda)=-2\{\lambda^2+[(1-\varepsilon^2)^2/4]\}^{-1}\leqslant 0$; it means that the function Φ is concave in R, but Φ'' is unbounded from below. To study the stability of the solution in a finite domain, consider a finite perturbation $\delta\omega$. The quantity

$$\hat{H}_\Phi(\delta\omega)=H_\Phi(\omega_e+\delta\omega)-H_\Phi(\omega_e)-DH_\Phi(\omega_e)\delta\omega$$
$$=\int_D\left(\frac{1}{2}\delta\omega(\nabla^2)^{-1}\delta\omega+\Phi(\omega_e+\delta\omega)\right.$$
$$\left.-\Phi(\omega_e)-D\Phi(\omega_e)\delta\omega\right)dx\,dy \quad (7)$$

is a nonlinear constant of motion since we have chosen Φ so that $DH_\Phi(\omega_e)=0$. To establish the Lyapunov stability estimates with $\tilde{H}$, we will modify the function Φ to a function $\tilde{\Phi}$, which gives $DH_{\tilde{\Phi}}(\omega_e)=0$ and, which has its second derivative bounded above and below on the L^2 norm.

From Eq. (2), we obtain

$$\omega_{\min}=-\frac{(1-\varepsilon^2)}{2}\sinh(4\operatorname{Arcth}\varepsilon)$$
$$\leqslant\omega_e=-\Delta\psi_e\leqslant\frac{(1-\varepsilon^2)}{2}\sinh(4\operatorname{Arcth}\varepsilon)=\omega_{\max} \quad (8)$$

since the extremal values are obtained for $(x,y)=(0,0)$ and $(\pi,0)$ respectively. Thus, from the expression of Φ'', we see that on the interval $[\omega_{\min},\omega_{\max}]$,

$$\Phi''(0)\leqslant\Phi''(\omega_e)=-\frac{1}{2\sqrt{\omega_e^2+[(1-\varepsilon^2)^2/4]}}$$
$$\leqslant\Phi''(\omega_{\min})=\Phi''(\omega_{\max}), \quad (9)$$

i.e.,

$$-\frac{1}{(1-\varepsilon^2)}\leqslant\Phi''(\omega_e)\leqslant-\frac{1-\varepsilon^2}{1+6\varepsilon^2+\varepsilon^4}<0. \quad (10)$$

Note that this equation is *not* valid for the extreme cases $|\varepsilon|=1$.

Let us construct the function $\tilde{\Phi}$ in such a way that it coincides with Φ on the interval $[\omega_{\min},\omega_{\max}]$, and with

$$\tilde{\Phi}(\lambda)=-\left(\frac{1-\varepsilon^2}{1+6\varepsilon^2+\varepsilon^4}\right)\frac{\lambda^2}{2}+\alpha_\pm\lambda+\beta_\pm \quad (11)$$

on the two intervals $]-\infty,\omega_{\min}]$ and $[\omega_{\max},+\infty[$. The constants $\alpha_\pm$ and $\beta_\pm$ are determined from continuity criteria, so that $\tilde{\Phi}$ is a C^2-function.

By construction, the function $(-\tilde{\Phi})$ is convex, i.e.,

$$\frac{1-\varepsilon^2}{1+6\varepsilon^2+\varepsilon^4}\frac{(\delta\omega)^2}{2}$$
$$\leqslant-\tilde{\Phi}(\omega_e+\delta\omega)+\tilde{\Phi}(\omega_e)+\tilde{\Phi}'(\omega_e)\delta\omega$$
$$\leqslant\frac{1}{(1-\varepsilon^2)}\frac{(\delta\omega)^2}{2}. \quad (12)$$

With the use of (7), we get

$$\int_D\left(\frac{1-\varepsilon^2}{1+6\varepsilon^2+\varepsilon^4}(\delta\omega)^2+\delta\omega(\nabla^2)^{-1}\delta\omega\right)dx\,dy$$
$$\leqslant-2\hat{H}_{\tilde{\Phi}}(\delta\omega) \quad (13)$$

and, keeping in mind that $(-\hat{H}_{\tilde{\Phi}}(\omega))$ is a conserved quantity and that $(\nabla^2)^{-1}$ is negative, we obtain

$$-2\hat{H}_{\tilde{\Phi}}(\delta\omega)=-2\hat{H}_{\tilde{\Phi}}(\delta\omega_0)$$
$$\leqslant\int_D\left(\frac{(\delta\omega_0)^2}{(1-\varepsilon^2)}+\delta\omega_0(\nabla^2)^{-1}\delta\omega_0\right)dx\,dy$$
$$\leqslant\int_D\frac{(\delta\omega_0)^2}{(1-\varepsilon^2)}dx\,dy \quad (14)$$

where $\delta\omega_0$ is the initial value of perturbation.

Finally

$$\frac{(1-\varepsilon^2)}{(1+6\varepsilon^2+\varepsilon^4)}||\delta\omega||_{L^2}+\int_D \delta\omega(\nabla^2)^{-1}\delta\omega\, dx\, dy$$

$$\leqslant -2\hat{H}_{\bar{\Phi}}(\delta\omega) \leqslant \frac{1}{(1-\varepsilon^2)}||\delta\omega_0||_{L^2}; \tag{15}$$

or, on the domain D, if one notes $k^2_{\min}$ the minimal eigenvalue of the operator $(-\nabla^2)$, we obtain[6]

$$\int_D \delta\omega(\nabla^2)^{-1}\delta\omega\, dx\, dy \geqslant -k^{-2}_{\min}||\delta\omega||^2_{L^2} \tag{16}$$

just by setting

$$\delta\omega=\sum_{i=0}^{+\infty} c_i\Phi_i \tag{17}$$

where the functions Φ_i are an L^2 orthonormal basis of eigenfunctions.

Since $\hat{H}$ is time invariant, the *a priori* estimate provide suitable norms bounding the growth of disturbances, provided that Eq. (13) is satisfied. We obtain therefore the nonlinear stability condition

$$\left[\frac{(1-\varepsilon^2)}{(1+6\varepsilon^2+\varepsilon^4)}-k^{-2}_{\min}\right]||\delta\omega||^2_{L^2} \leqslant \frac{1}{(1-\varepsilon^2)}||\delta\omega_0||^2_{L^2}. \tag{18}$$

Consider for the domain D a rectangular box, with length $2\pi N$ in the x-axis and 2ℓ in the y-axis; the minimal eigenvalue of the operator $(-\nabla^2)$ is $k^2_{\min}=(1/N^2)+(\pi^2/\ell^2)$, since the eigenfunctions, vanishing on the boundary, are $f(x,y)=\cos(x/N)\sin(\pi y/\ell)$. We can then write the following theorem.

The Mallier–Maslowe solution of the bidimensional Euler equation is nonlinearly stable in the L^2 norm on vorticities for perturbations of the initial vorticity which preserve the flow rate ($\psi=$ constant on the boundaries) and the circulations, in the domain $D=\{(x,y)/x\in[0,2N\pi],y\in[-\ell,+\ell]\}$ provided that ε and ℓ satisfy the following condition:

$$\frac{\pi}{\ell} \geqslant \sqrt{\frac{1+6\varepsilon^2+\varepsilon^4}{1-\varepsilon^2}-\frac{1}{N^2}}. \tag{19}$$

Therefore we exhibit a transverse size of the domain D over which the street of vortices is unstable.

The implication of this conditional stability is that the violation of the convexity conditions for the the given flow is a necessary condition for its instability. Figure 2 presents the stability region of the counter-rotating vortices in the plane (ℓ,ε) for $N=1$. One can check that the particular case, where the fluid is at rest $(\varepsilon=0)$ is always stable, since the condition goes to infinity. On the other hand, one notes that the point vortices $(\varepsilon=\infty)$ are always unstable. The figure shows also that the solution is unstable for all values of ε if ℓ is infinite. We emphasize that, contrary to the case of identical Stuart vortices,[6] the analytical calculations are here tractable in a rectangular box, and not in a hypothetical one, which follows the streamlines.

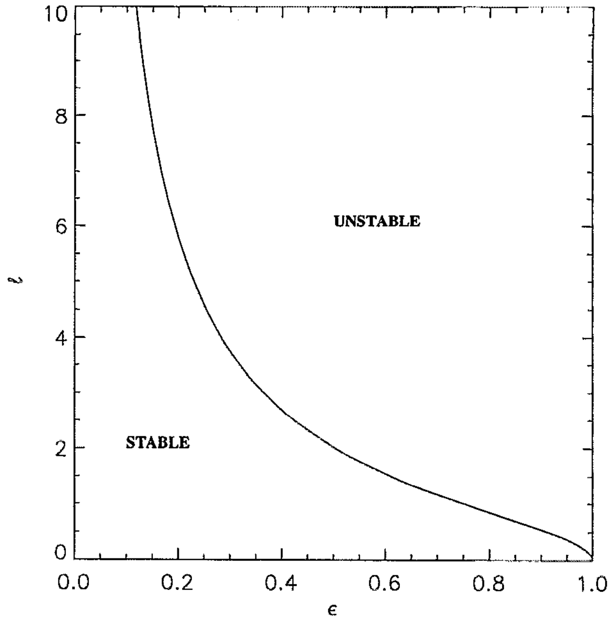

FIG. 2. Domain of stability of a pair of counter-rotating vortices ($N=1$) in the plane (ℓ,ε). ℓ is the transverse size of the box and ε characterizes the density of vorticity. The solid line is defined by Eq. (19).

In conclusion, in this work, we derive explicitly the nonlinear stability condition for the counter-rotating vortices solutions in a rectangular box. This work should lead to a better understanding of the role of the nonlinearity in the instability of electromagnetically forced counter-rotating vortices[3] when the viscosity is present. Work along this line is in progress.

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