

## Statistical theory of high-gain free-electron laser saturation

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We propose an approach, based on statistical mechanics, to predict the saturated state of a single-pass, high-gain free-electron laser. In analogy with the violent relaxation process in self-gravitating systems and in the Euler equation of two-dimensional turbulence, the initial relaxation of the laser can be described by the statistical mechanics of an associated Vlasov equation. The laser field intensity and the electron bunching parameter reach a quasistationary value which is well fitted by a Vlasov stationary state if the number of electrons  $N$  is sufficiently large. Finite  $N$  effects (granularity) finally drive the system to Boltzmann-Gibbs statistical equilibrium, but this occurs on times that are unphysical (i.e., excessively long undulators). All theoretical predictions are successfully tested by means of finite- $N$  numerical experiments.

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The interaction of charged particles with electromagnetic fields is a topic of paramount importance in a large variety of physical phenomena, from plasma dynamics to astrophysical systems. On the energy exchange between particles and field also relies the possibility of generating coherent and tunable radiation sources, such as free-electron lasers (FELs). In this case a relativistic electron beam propagating through a periodic magnetic field (produced by an undulator) interacts with a copropagating electromagnetic wave. Lasing occurs because the undulator field and the radiation combine to produce a beat wave that travels slower than the speed of light and can be synchronized with electrons. Among different schemes, single-pass, high-gain FELs are currently attracting growing interests [1], as they are promising sources of powerful and coherent light in the UV and X ranges [2–4]. In the high-gain regime, both the light intensity and the longitudinal bunching of the electron beam increase exponentially along the undulator, until they reach saturation due to nonlinear effects. Understanding this saturation process is important to estimate, and then *optimize*, the performance and building costs of a FEL.

Theoretical analyses usually rely on *dynamical* methods in combination with detailed, but rather complicated, numerical simulations. In this paper, we propose an approach, which is based on *statistical mechanics*, to study the saturated state of a high-gain single-pass FEL. We restrict our analysis to the steady-state regime, which amounts to neglect the variation of the electromagnetic wave within the electron pulse length (small electrons' radiation slippage). However, it is important to stress that because of its intrinsic flexibility, we believe that our statistical approach will be applicable also to alternative schemes, such as harmonic generation [5].

The starting point of our study is the Colson-Bonifacio model [6]. Under the hypotheses of one-dimensional motion

and monochromatic radiation, the steady-state equations for the  $j$ th electron of the beam coupled to radiation read

$$\frac{d\theta_j}{d\bar{z}} = p_j, \quad (1)$$

$$\frac{dp_j}{d\bar{z}} = -\mathbf{A}e^{i\theta_j} - \mathbf{A}^*e^{-i\theta_j}, \quad (2)$$

$$\frac{d\mathbf{A}}{d\bar{z}} = i\delta\mathbf{A} + \frac{1}{N}\sum_j e^{-i\theta_j}, \quad (3)$$

where  $N$  is the number of electrons in a single-radiation wavelength and  $\bar{z}=2k_u\rho z\gamma_r^2/\langle\gamma\rangle_0^2$  is the rescaled longitudinal coordinate, which plays the role of time. Here,  $\rho=(a_w\omega_p/4ck_u)^{2/3}/\gamma_r$  is the so-called Pierce parameter,  $\gamma_r$  the resonant energy,  $\langle\gamma\rangle_0$  the mean energy of the electrons at the undulator's entrance,  $k_u$  the wave vector of the undulator,  $\omega_p=(e^2n/m\epsilon_0)^{1/2}$  the plasma frequency,  $c$  the speed of light, and  $e$  and  $m$  respectively the charge and mass of one electron. Further,  $a_w=eB_w/k_umc^2$ , where  $B_w$  is the rms undulator field, for the case of a helical undulator. By introducing  $k$  as the wave number of the FEL radiation, the phase  $\theta$  is defined by  $\theta=(k+k_u)z-2\delta\rho k_u z\gamma_r^2/\langle\gamma\rangle_0^2$  and its conjugate momentum  $p=(\gamma-\langle\gamma\rangle_0)/(\rho\langle\gamma\rangle_0)$ .  $\mathbf{A}$  is the scaled field amplitude, a complex vector, transversal to  $z$ ,  $\mathbf{A}=(A_x, A_y)$  [7]. Finally, the detuning parameter is given by  $\delta=(\langle\gamma\rangle_0^2-\gamma_r^2)/(2\rho\gamma_r^2)$ , and measures the average relative deviation from the resonance condition.

Although very simple, such a model captures the main features of the dynamics of the single-pass FEL, as shown by a systematic comparison with numerical predictions based on more complete approaches [8,9]. Using this model, we are able to predict analytically the mean saturated laser intensity, the electron-beam bunching, and the electrons' velocity distribution, for a wide class of initial conditions (i.e., energy

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spread, bunching, and radiation intensity). The analytical results agree very well with numerical simulations.

The above system of equations can be derived from the Hamiltonian

$$H = \sum_{j=1}^N \frac{p_j^2}{2} - N\delta I + 2\sqrt{I} \sum_{j=1}^N \sin(\theta_j - \varphi), \quad (4)$$

where the intensity  $I$  and the phase  $\varphi$  of the wave are related to  $\mathbf{A} = A_x + iA_y = \sqrt{I}e^{-i\varphi}$ . In addition to the “energy”  $H$ , the total momentum  $P = \sum_j p_j + N\mathbf{A}\mathbf{A}^*$  is also a conserved quantity. Let us note that one can always take  $P=0$ , upon a shift in the detuning [10]; thus, we always suppose  $P=0$  in the following.

It is important to emphasize that Hamiltonian (4) models the interaction between radiation and electrons. Hence, it describes a quite universal phenomenon which is encountered in many branches of physics. As an example, in the context of plasma theory, the so-called plasma-wave Hamiltonian (4) characterizes the self-consistent interaction between a Langmuir wave and  $N$  particles, after an appropriate redefinition of the variables involved [10]. Establishing a formal bridge between these two areas allows to recast in the context of the single-pass LINAC FEL numerous results originally derived in the framework of plasma physics. In addition, Hamiltonian (4) can be viewed as a direct generalization of mean-field models [11–13], which are widely studied nowadays because of their intriguing features: statistical ensemble inequivalence, negative specific heat, and dynamical stabilization of out-of-equilibrium structures.

In plasma physics, it was numerically shown [10,14] that, in the region of instability, wave amplification occurs in two steps. One first observes an exponential growth of the wave amplitude, followed by damped oscillations around a well-defined level. However, the system does not reach a stationary state and this initial stage is followed by a slow relaxation towards the final statistical equilibrium. An example of this behavior is shown in Fig. 1.

The separation into two distinct time scales characterizes also the dynamics of self-gravitating systems and is a well-known phenomenon in astrophysics [15,16]. The intermediate quasistationary states live longer and longer as the number of particles  $N$  is increased. It is believed that galaxies ( $N \approx 10^{11}$ ) are well described by Vlasov equilibrium [15], which characterizes the quasistationary state of the  $N$ -particle system (see below). On the contrary, Boltzmann-Gibbs statistics applies to the “smaller” ( $N \approx 10^6$ ) globular clusters.

A typical evolution of the radiation intensity  $I$  as a function of the longitudinal coordinate  $\bar{z}$ , according to the free-electron laser model, Eqs. (1)–(3), is displayed in Fig. 1; starting from a very weak radiation, the intensity grows exponentially and saturates, oscillating around a well-defined value. This growth and first relaxation of the system (usually called “violent relaxation” in astrophysics) is governed by the Vlasov equation [11,17], which is rigorously derived by taking the continuum limit ( $N \rightarrow \infty$  at fixed volume and energy per particle). On longer time scales, whose duration strongly depends on the particle number  $N$  (see the inset of Fig. 1), there is a slow drift of the intensity of the beam

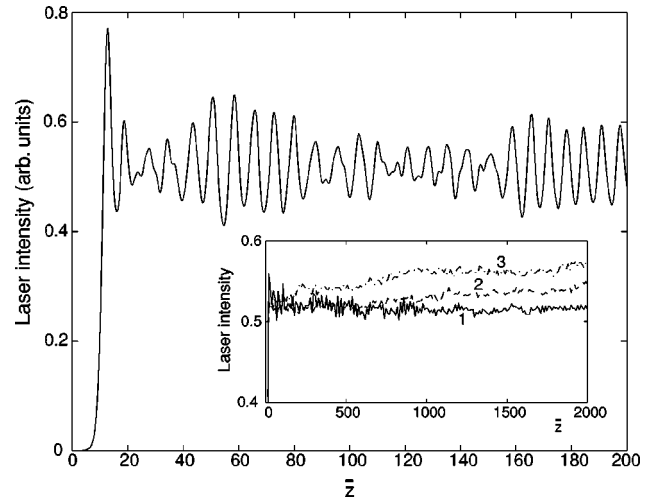


FIG. 1. Typical evolution of the radiation intensity using Eqs. (1)–(3); the detuning  $\delta$  is set to 0, the energy per electron  $H/N = 0.2$  and  $N = 10^4$  electrons are simulated. The inset presents averaged simulations on longer times for different values of  $N$ :  $5 \times 10^3$  (curve 1), 400 (curve 2), and 100 (curve 3).

towards the final asymptotic plateau determined by the Boltzmann-Gibbs statistics. Such process is driven by granularity, a finite- $N$  effect [11,15,16]. This final relaxation takes place on an extremely long time scale, well beyond the physical constraints imposed by a reasonable undulator length. We thus concentrate in the following on the Vlasov description of the dynamics.

A linear analysis [6] leads directly to the determination of the boundaries of the instability domain, which are mainly controlled by the detuning  $\delta$  and by the initial energy per electron. In the case of a monoenergetic electron beam, the instability disappears for  $\delta > \delta_c \approx 1.9$ . Linear analysis also provides estimates of the growth rate of  $I$ . However, getting insights on the saturated state requires a nonlinear study of the system; the standard approach to this problem is mainly dynamical, as for instance in Ref. [18]. In the following we discuss a procedure based on statistical mechanics.

As previously discussed, we are interested in the intermediate metastable state and, therefore, we will first consider the statistical theory of the Vlasov equation, originally introduced in the astrophysical context [15,16]. The basic idea is to coarse-grain the microscopic one-particle distribution function  $f(\theta, p, t)$ , which is stirred and filamented at smaller and smaller scales by the Vlasov time evolution. An entropy is then associated to the coarse-grained distribution  $\bar{f}$ , which essentially counts the number of microscopic configurations. Equilibrium is then computed by maximizing this entropy while imposing the dynamical constraints. A rigorous description of this procedure can be found in Ref. [19] in the context of two-dimensional Euler hydrodynamics.

In the continuum limit, Eqs. (1)–(3) lead to the following Vlasov-wave system:

$$\frac{\partial f}{\partial \bar{z}} = -p \frac{\partial f}{\partial \theta} + 2(A_x \cos \theta - A_y \sin \theta) \frac{\partial f}{\partial p}, \quad (5)$$

$$\frac{\partial A_x}{\partial \bar{z}} = -\delta A_y + \frac{1}{2\pi} \int f \cos \theta d\theta dp, \quad (6)$$

$$\frac{\partial A_y}{\partial \bar{z}} = \delta A_x - \frac{1}{2\pi} \int f \sin \theta d\theta dp. \quad (7)$$

Note that these equations have been studied numerically in a recent work by Vinokurov *et al.* [20], for the case  $\delta=0$ . The Vlasov-wave equations (5)–(7) conserve the pseudoenergy

$$\begin{aligned} \varepsilon = & \int dp d\theta \frac{p^2}{2} f(\theta, p) - \delta(A_x^2 + A_y^2) \\ & + 2 \int dp d\theta f(\theta, p)(A_x \sin \theta + A_y \cos \theta), \end{aligned} \quad (8)$$

and the total momentum

$$\sigma = \int dp d\theta p f(\theta, p) + A_x^2 + A_y^2. \quad (9)$$

For the sake of simplicity, let us suppose that the beam is initially unbunched, and that energies are distributed according to a step function, such that

$$\begin{aligned} f(\theta, p, t=0) = f_0 = & \frac{1}{4\pi\bar{p}} \quad \text{if } -\bar{p} \leq p \leq \bar{p} \\ & = 0 \quad \text{otherwise.} \end{aligned} \quad (10)$$

As far as one is dealing with small energy dispersions, the profile (10), called waterbag initial condition, represents a good approximation of a more natural Gaussian initial distribution. Numerical tests fully confirm the validity of this simple observation. According to Eq. (10),  $f$  takes only two distinct values, and coarse graining amounts to perform a local average of both. The entropy per particle associated with the coarse-grained distribution  $\bar{f}$  is then a mixing entropy [11,16] and reads

$$s(\bar{f}) = - \int dp d\theta \left[ \frac{\bar{f}}{f_0} \ln \frac{\bar{f}}{f_0} + \left(1 - \frac{\bar{f}}{f_0}\right) \ln \left(1 - \frac{\bar{f}}{f_0}\right) \right]. \quad (11)$$

As the electromagnetic radiation represents only two degrees of freedom within the  $(2N+2)$  of Hamiltonian (4), its contribution to entropy can be neglected.

The equilibrium state is computed [11] by solving the constrained variational problem:

$$\begin{aligned} S(\varepsilon, \sigma) = & \max_{\bar{f}, A_x, A_y} \left( s(\bar{f}) | H(\bar{f}, A_x, A_y) = N\varepsilon; \right. \\ & \left. \int d\theta dp \bar{f} = 1; \quad P(\bar{f}, A_x, A_y) = \sigma \right). \end{aligned} \quad (12)$$

Introducing three Lagrange multipliers  $\beta$ ,  $\lambda$ , and  $\mu$  for the energy, momentum, and normalization constraints and differentiating Eq. (12) with respect to  $\bar{f}$ , one gets the equilibrium distribution

$$\bar{f} = f_0 \frac{e^{-\beta(p^2/2 + 2A \sin \theta) - \lambda p - \mu}}{1 + e^{-\beta(p^2/2 + 2A \sin \theta) - \lambda p - \mu}}. \quad (13)$$

By differentiating Eq. (12) with respect to  $A_x$  and  $A_y$ , one obtains in addition the expression for the amplitude of the wave,

$$A = \sqrt{A_x^2 + A_y^2} = \frac{\beta}{\beta\delta - \lambda} \int dp d\theta \sin \theta \bar{f}(\theta, p). \quad (14)$$

Using the above equations for the three constraints, the statistical equilibrium calculation is now reduced to finding the values of  $\beta$ ,  $\lambda$ , and  $\mu$  as functions of energy  $\varepsilon$  and total momentum  $\sigma$ . This last step, performed numerically using for example a Newton-Raphson method, leads directly to the estimates of the main physical parameters.

Furthermore, let us stress that in the limit of a vanishing energy dispersion, the area occupied by the  $f=f_0$  level in the one-particle phase space is small, so that the coarse-grained distribution  $\bar{f}$  verifies  $\bar{f} \ll f_0$  everywhere. The second term in the entropy (11) is thus negligible, and Eq. (13) reduces to the Gibbs distribution

$$\bar{f} \propto e^{-\beta(p^2/2 + 2A \sin \theta) - \lambda p}. \quad (15)$$

Vlasov equilibrium is in that case equivalent to the full statistical equilibrium. Then, solving the constraint equations yields

$$b = A^3 - \delta A, \quad (16)$$

$$A = \left( \varepsilon - \delta A^2 + \frac{3}{2} A^4 \right) \Theta(b), \quad (17)$$

where  $b = |\sum_j e^{i\theta_j}|/N$  is the bunching parameter and  $\Theta$  is the reciprocal function of  $I_1(x)/I_0(x)$ ,  $I_n$  being the modified Bessel function of order  $n$ . Let us note that Eqs. (16) and (17) give the microcanonical solution of Hamiltonian (4). The canonical solution of the same Hamiltonian in the context of plasma physics was obtained in Ref. [21]. It turns out that the two ensembles are equivalent, which was not granted *a priori* for such a self-consistent system with infinite range interactions [13]. Let us remark that Eqs. (16) and (17) were obtained in Ref. [18] using several hypotheses, suggested only by numerical simulations. Here, a statistical mechanics approach gives a complete and self-consistent framework to justify their derivation. In particular, let us emphasize that, contrary to the previous approach, it is not necessary to choose *a priori* the distribution  $f$ , which is fully determined by the method of solution.

Figure 2 presents the comparison between the analytical predictions and the numerical simulations performed using Eqs. (1)–(3) in the case of a monoenergetic beam. Numerical data are time averaged. The agreement is remarkably good for  $\delta < 0.5$  and is accurate up to the threshold value  $\delta_c$ , although phase-space mixing is probably less effective for larger detuning. For  $\delta > \delta_c$ , there is no amplification, hence, both intensity and bunching stick to their initial vanishing values. This transition, purely dynamical, cannot be reproduced by the statistical analysis.

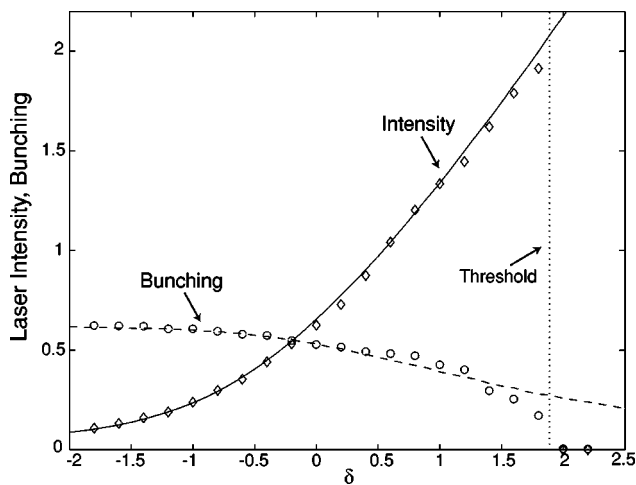


FIG. 2. Comparison between theory (solid and long-dashed lines) and simulations (symbols) for a monoenergetic beam, varying the detuning  $\delta$ . The vertical dotted line,  $\delta = \delta_c \approx 1.9$ , represents the transition from the low to the high-gain regime.

In the case of a non-monoenergetic beam, Fig. 3 presents the comparison between the estimates of the above theoretical analysis and the results of direct numerical simulations of Hamiltonian (4), after time averaging. The comparison is shown in the energy range that allows the amplification process to take place. The good agreement for intensity and bunching provides therefore an *a posteriori*, but striking, support for Vlasov statistical equilibrium.

In this paper, we have proposed an approach to study the saturated state of the Compton free-electron laser, based on a *statistical mechanics approach* in the framework of Colson-Bonifacio's model [6]. By drawing analogies with the statistical theory of violent relaxation in astrophysics and two-dimensional Euler turbulence, we have derived analytical estimates of the saturated intensity and bunching. In addition to providing a deeper insight into the physical behavior of

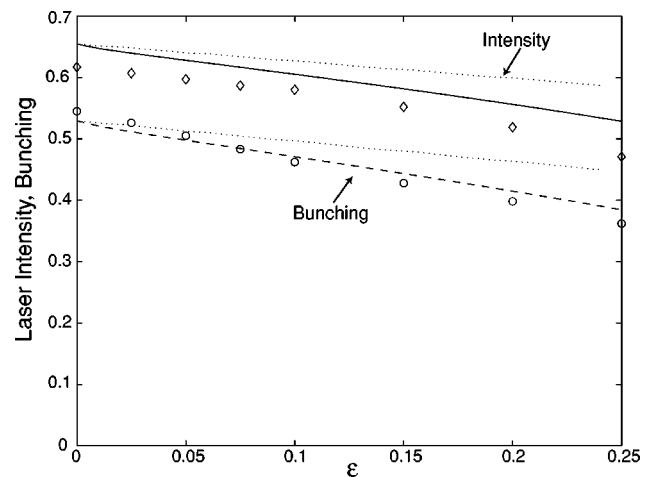


FIG. 3. Comparison between theory (solid and long-dashed lines) and simulations (symbols) for a non-monoenergetic beam when the energy  $\epsilon$ , characterizing the velocity dispersion of the initial electron beam, is varied. The dotted lines represent the intensity and bunching predicted by the full statistical equilibrium given by Eqs. (16) and (17), not very appropriate here, whereas the solid line and long-dashed lines refer to the Vlasov equilibrium defined by Eqs. (13) and (14). The discrepancy between theory and numerical experiments is small over the whole range of explored energies.

this system, the results of our theory agree very well with numerical simulations. Due to its intrinsic flexibility, it may be possible to adapt the statistical approach to more complete models and complex schemes, thus allowing a direct comparison between analytical studies and experiments on real devices. Such a statistical approach could be used as a tool to define future strategies aiming at *optimizing* FEL performance.

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