Master 1ère année **2018-2019**
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Exam for "Systèmes Dynamiques et Chaos"
Vendredi 21 Décembre 2018
Duration: 3h

Calculators and Documents are not permitted during the "Question de cours".
Permitted are printed material, hand written notes or photocopies of any kind, but not books.
Answers can be written in French or English.

1 Question from the lectures

Give back the copy with your answers to Part 1 no later than 20' from the start.

1(a) What defines a conservative dynamical system?

1(b) What is the difference between a conservative dynamical system and a Hamiltonian dynamical system?

1(c) Explain the main differences between a supercritical and a subcritical bifurcation.

1(d) Explain what a Hopf bifurcation is.

1(e) Explain what a fractal (strange) attractor is.

1(f) How would you explain a global warming skeptic that weather cannot be predicted next week does not imply that the climate cannot be predicted 20 years from now?

2 Predator-Prey system

Consider the predator-prey system in which y (resp. x) stands for the number of predators (resp. prey).

$$\dot{x} = x(x(1-x)-y) \quad (1)$$
$$\dot{y} = y(x-a), \qquad \text{with} \quad 0<a<1 \quad (2)$$

2(a) Find and draw the null-clines ($\dot{x}=0$ ou $\dot{y}=0$) of the system (1)-(2).

2(b) Find the fixed points.

2(c) Argue from the fixed points alone that a state exists where the predator goes extinct and the prey is still alive.

2(d) Discuss the stability of the fixed points where one or two of the species are extinct.

2(e) A Hopf bifurcation for the fixed point $x>0$, $y>0$ occurs at a critical values $a=a_c$. Determine a_c.

2(f) Show that the fixed point $x>0$, $y>0$ is unstable for $0<a<a_c$.

3 Study of an iterated application

Consider a population dynamics model that is written with r a real parameter, as the 1-d map

$$x_{n+1} = f(x_n) = x_n \exp(r(1 - x_n)). \tag{3}$$

3(a) Show that the map always has an extremal point x_m. Determine for which values of r it is a maximum and for which values of r it is a minimum.

3(b) Find the fixed point(s) $x^\star$ of (3).

3(c) Determine the stability interval of the fixed point(s) as a function of r.

3(d) Using a simple cobweb diagram reveals how flip bifurcations can give rise to a period doubling when $f'(x^\star) = -1$, if the graph of f is concave near $x^\star$.

3(e) Determine the parameter value $r = r_p$ where (one of) the fixed point(s) undergoes period doubling.

3(f) Let $f(x, r)$ denote a unimodal map that undergoes a period-doubling route to chaos as r increases , and x_m is the maximum of f. The superstability condition (of a fixed point or a cycle) is $\partial f/\partial x|_{x=x^\star} = 0$ where $x^\star$ is the fixed point, or a point of a cycle.

Write the superstable two-cycle as a function of r and show that it leads to the condition $2r - 1 - \exp(r - 1) = 0$.

3(g) Find a trivial solution $r = r_s$ to this equation. Which two-cycle does that correspond to ?

4 Weakly nonlinear oscillators

Consider the second-order differential equation

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0 \tag{4}$$

where $h(x, \dot{x}) = (x^2 - a)\dot{x} + (\dot{x}^2 - b)x$, $a > 0$, $b > 0$ and $0 < \varepsilon \ll 1$.

4(a) Using the following asymptotic form,

$$x = x_0(\tau, T) + \varepsilon x_1(\tau, T) + ... \tag{5}$$

involving the two time scales τ and $T = \varepsilon\tau$, solve (4) at the lowest order in ε, and show that

$$x_0(\tau, T) = r(T)\cos(\tau + \phi(T)) = r(T)\cos\theta(\tau, T). \tag{6}$$

4(b) To find the differential equations governing r and ϕ, insert (6) in the $O(\varepsilon)$ equation. Use prime to denote differentiation with respect to slow time T, i.e. $r' = dr/dT$.

4(c) Explain why the terms $\cos\theta$ and $\sin\theta$ lead to difficulties in the perturbative expansion. How can this difficulty be overcome?

4(d) Derive an equation for r' and for ϕ'. Hint: Use $\cos^3\theta = (\cos(3\theta) + 3\cos\theta)/4$ and $\sin^3\theta = -(\sin(3\theta) - 3\sin\theta)/4$, or the averaging method.

4(e) Find the fixed points for $r(T)$ in terms of a and discuss their stability.

4(f) For $a = 2$ and with the initial condition $r(0) = 1$, show after integrating by partial fraction the solution is $r(T) = \sqrt{8/(1 + 7\exp(-(2T)))}$. Sketch the graph of $r(T)$ and determine the value of $r(T)$ as $T \to \infty$.

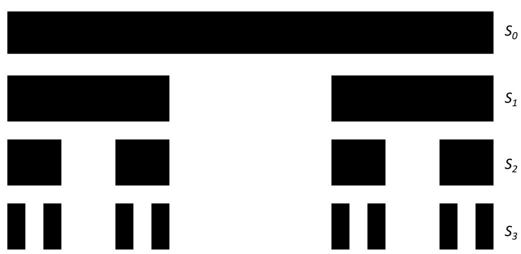
4(g) Insert the stable fixed point for r and determine an equation for ϕ' in terms of a and b.

4(h) What is the resulting frequency ω for the limit cycle?

5 Box-counting dimension

The figures below show the first few generations S_0, S_1, S_2 and S_3 in the construction of modified versions of the middlethirds Cantor set. For each figure the fractal set is obtained by iterating to generation S_n with $n \to \infty$.

5(a) Start by a two-dimensional strip of finite width Analytically find the box-counting dimension D_0 of the fractal set, obtained by at each generation removing the middle third horizontal interval out of three equally sized horizontal intervals:



5(b) Start by a two-dimensional strip of finite width and height. Analytically find the box-counting dimension D_0 of the fractal set, obtained by at each generation removing both the middle third horizontal and vertical intervals out of three equally sized horizontal and vertical intervals:

5(c) Discuss how the results in a) and b) are related to the box-counting dimension of the (middle-third) Cantor set discussed in class.

Correction de l'Examen de Décembre 2018

1. Question from the lectures

Give back the copy with your answers to Part 1 no later than 20' from the start.

1(a) div $\mathbf{f}$=0.

1(b) A Hamiltonian system is a dynamical system governed by Hamilton's equations.

1(c) The important difference is that the first nonlinear term is destabilizing (as the linear one) for a subcritical bifurcation, while it is stabilizing for a supercritical bifurcation. The subcritical case is always much more dramatic, and potentially dangerous in engineering applications. After the bifurcation, the trajectories must jump to a distant attractor, which may be a fixed point, a limit cycle, infinity or a chaotic attractor.

1(d) A Hopf bifurcation corresponds to the case in which two complex conjugates eigenvalues cross the imaginary axis into the right half-plane where their real part becomes positive.

1(e) An attractor is called strange if it has a fractal structure. This is often the case when the dynamics on it is chaotic.

2. Predator-Prey system

2(a) Null clines: $\dot{x} = 0$ for $x = 0$, $y = x(1 - x)$ and $\dot{y} = 0$ for $y = 0$, $x = a$

2(b) $(x^\star, y^\star) = (0, 0), (1, 0), (a, a - a^2)$.

2(c) For the fixed point $(x^\star, y^\star) = (1, 0)$ the predator is extinct while prey exists.

2(d)

$$J = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - a \end{pmatrix}$$

$(x^\star, y^\star) = (0, 0)$, $\lambda = -a$ or 0, one direction stable, one marginal.
$(x^\star, y^\star) = (1, 0)$, $\lambda = -1$ or $1 - a$, one direction stable, one unstable. This is a saddle point.

2(e) $(x^\star, y^\star) = (a, a - a^2) \Rightarrow \tau = -2a^2 + a, \Delta = -a^3 + a^2$. The Hopf occurs when $\tau = a - 2a^2 = 0 \Rightarrow a_c = 1/2$ while $\Delta > 0$, with a change from complex stable to complex unstable.

2(f) For $0 < a < 1/2 \Rightarrow \tau > 0$ so unstable.

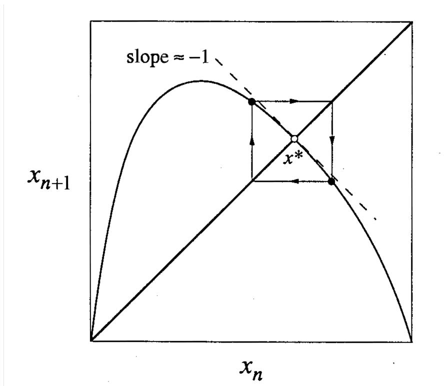
3. Study of an iterated application

3(a) $f'(x_n) = (1 - rx_n)\exp(r(1-x_n)) \Rightarrow x_m = 1/r$, minimum for $r < 0$, maximum for $r > 0$

3(b) $x^\star = 0$ or 1.

3(c) $f'(0) = \exp(r) \Rightarrow$ unstable for $r > 0$, stable for $r < 0$.
$f'(1) = 1 - r \Rightarrow$ unstable for $r < 0$ and $r > 2$, stable for $0 < r < 2$.

3(d) A cobweb diagram reveals how flip bifurcations can give rise to a period doubling. Near a fixed point where $f'(x^\star) = -1$, if the graph of f is concave near $x^\star$ (see figure), the cobweb tends to produce a small stable 2-cycle close to the fixed point.



3(e) The period doubling occurs therefore when $f'(x^\star) = -1$, which can occur only for $x^\star = 1$. One gets $f'(1) = 1 - r = -1 \Rightarrow r_p = 2$

3(f) $x_m = 1/r \to 1/r\exp(r-1) \to 1/r\exp(r-1)\exp(r - \exp(r-1)) = 1/r \Rightarrow \exp(r-1)\exp(r - \exp(r-1)) = 1 \Rightarrow 2r - 1 - \exp(r-1) = 0$

3(g) The value $r_s = 1$ is a solution. The superstable fixed point is also a two-cycle.

4 Weakly nonlinear oscillator

4(a) The usual two-timing substitutions give

$$O(1): \quad \partial_{\tau\tau}x_0 + x_0 = 0. \tag{7}$$

The solution of the $O(1)$ equation is $x_0(t,T) = r(T)\cos(t + \phi(T)) = r(T)\cos\theta(t,T)$ where $\theta(T)$ and $r(T)$ are the phase and slowly-varying amplitude of x_0.

4(b) At the next order, one gets

$$O(\varepsilon): \quad \partial_{\tau\tau}x_1 + x_1 = -(x_0^2 - a)\frac{\partial x_0}{\partial \tau} - \left[\left(\frac{\partial x_0}{\partial \tau}\right)^2 - b\right] x_0 - 2\partial_\tau\partial_T x_0. \tag{8}$$

Using $x_0 = r(T)\cos\theta$ and $\dot{x}_0 = -r(T)\sin\theta$, the $O(\varepsilon)$ equation yields

$$\begin{aligned} \partial_{\tau\tau}x_1 + x_1 &= -(r^2\cos^2\theta - a)(-r\sin\theta) - \left(r^2\sin^2\theta - b\right) r\cos\theta + 2\left[r'\sin\theta + r\cos\theta\phi'\right] \quad (9) \\ &= r^3\cos^2\theta\sin\theta - ar\sin\theta - r^3\sin^2\theta\cos\theta + br\cos\theta + 2\left[r'\sin\theta + r\cos\theta\phi'\right] \quad (10) \end{aligned}$$

4(c) To avoid secular divergences, as usual, one needs that there be no terms proportional to $\cos\theta$ and $\sin\theta$ on the right-hand-side of the $O(\varepsilon)$ equation.

4(d)

$$\begin{aligned}
\partial_{\tau\tau}x_1 + x_1 &= r^3\left[\sin\theta - \cos\theta - \sin^3\theta + \cos^3\theta\right] - ar\sin\theta + br\cos\theta + 2\left[r'\sin\theta + r\cos\theta\phi'\right] \\
&= r^3\left[\sin\theta - \cos\theta + \frac{1}{4}\left(\sin(3\theta) - 3\sin\theta\right) + \frac{1}{4}\left(\cos(3\theta) + 3\cos\theta\right)\right] - ar\sin\theta + br\cos\theta + 2\,[r'\,s \\
&= \left[\frac{1}{4}r^3 - ar + 2r'\right]\sin\theta + \left[-\frac{1}{4}r^3 + br + 2r\phi'\right]\cos\theta + \frac{r^3}{4}\left[\sin(3\theta) + \cos(3\theta)\right]
\end{aligned}$$

To avoid secular divergences, one has to cancel the prefactors of $\cos\theta$ and $\sin\theta$ on the right-hand-side. That leads to

$$r'(T) = -\frac{1}{8}r^3 + \frac{1}{2}ar \qquad \text{and} \qquad r\phi'(T) = \frac{1}{8}r^3 - \frac{1}{2}br. \tag{14}$$

The averaging method is an alternative method to get to the very same result. With the substitution $\theta = t + \phi$ and using the averaged equation $\langle(13)\times\sin\theta\rangle$, one gets

$$\begin{aligned}
r'(T)2\langle\sin^2\theta\rangle &= -r^3\langle\cos^2\theta\sin^2\theta\rangle + ar\langle\sin^2\theta\rangle + r^3\langle\cos\theta\sin^3\theta\rangle - br\langle\cos\theta\sin\theta\rangle && (15)\\
&= -r^3\frac{\langle\sin^2(2\theta)\rangle}{4} + ar\langle\sin^2\theta\rangle + r^3\langle\cos\theta\sin^3\theta\rangle - br\frac{\langle\sin(2\theta)\rangle}{2} && (16)\\
r'(T)2\frac{1}{2} &= -r^3\frac{1/2}{4} + ar\frac{1}{2} + 0 + 0 && (17)\\
r'(T) &= -\frac{1}{8}r^3 + \frac{1}{2}ar && (18)
\end{aligned}$$

Using the averaged equation $\langle(13)\times\cos\theta\rangle$, one gets

$$\begin{aligned}
r\phi'(T)2\langle\cos^2\theta\rangle &= -r^3\langle\cos^3\theta\sin\theta\rangle + ar\langle\cos\theta\sin\theta\rangle + r^3\langle\cos^2\theta\sin^2\theta\rangle - br\langle\cos^2\theta\rangle && (19)\\
r\phi'(T) &= \frac{1}{8}r^3 - \frac{1}{2}br && (20)
\end{aligned}$$

that leads to $\phi'(T) = \frac{1}{8}r^2 - \frac{1}{2}b$.

4(e) $r'(T) = 0 \Rightarrow r^\star = 0$ or $r^\star = 2\sqrt{a}$. $r^\star = 0$ is unstable, while $r^\star = 2\sqrt{a}$ is stable.

4(f) $a = 2 : r'(T) = \frac{1}{8}r(8 - r^2) \Rightarrow \frac{8dr}{r(8-r^2)} = dT \Rightarrow \left(\frac{1}{r} + \frac{r}{8-r^2}\right)dr = dT \Rightarrow \ln r - \frac{1}{2}\ln(8 - r^2) = T + C$. The initial condition $r(0) = 1$ results in $C = -\frac{1}{2}\ln 7$. Inserting this gives after some algebra the solution $r(T) = \sqrt{\frac{8}{1+7\exp(-2T)}}$. From this equation, one gets that $r(T) \to \sqrt{8}$.

4(g) $r^\star = 2\sqrt{a} \Rightarrow \phi'(T) = (a - b)/2$.

4(h) $\omega = 1 + \varepsilon\dot{\phi} = 1 + \varepsilon\,(a - b)/2$.

5. Box-counting dimension

5(a) Covering the fractal with boxes of side length $\varepsilon_n = 3^{-n}$, the total number of boxes is proportional to $N(\varepsilon) = 3^n 2^n$. The box-counting dimension becomes

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \lim_{n \to \infty} \frac{\ln\left[(2*3)^n\right]}{\ln\left(3^n\right)} = 1 + \frac{\ln 2}{\ln 3}. \tag{21}$$

5(b) Similarly, covering the fractal with boxes of side length $\varepsilon_n = 3^{-n}$, the total number of boxes is proportional to $N(\varepsilon) = 2^n * 2^n$. The box-counting dimension becomes

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} = \frac{\ln(2*2)^n}{\ln 3^n} = 2 * \frac{\ln 2}{\ln 3}. \tag{22}$$

5(c) The case a) corresponds to the (middle-third) Cantor set in the horizontal direction and to the dimension 1, in the vertical direction. It is just normal that one gets $1 + \ln 2/\ln 3$.
The case b) corresponds to the (middle-third) Cantor set in the horizontal and vertical direction One thus gets $\frac{\ln 2}{\ln 3} + \frac{\ln 2}{\ln 3} = 2(\ln 2/\ln 3)$.