Master 1ère année **2017-2018**
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Examen du cours "Systèmes Dynamiques et Chaos"
Jeudi 11 Janvier 2018
Durée: 2h30

Documents et calculatrices interdits lors de la question de cours.
Notes de cours autorisées pour le reste de l'examen.
Les différentes parties du problème sont indépendantes.

1 Question from the lectures

Give back the copy with your answers to Part 1 no later than 20' from the start.

1(a) What is the name of the theorem that allows one to show that a limit cycle exists in 2D?

1(b) Draw schematically the phase portrait of the pendulum in absence of damping.

1(c) Let's consider the dynamical system $\dot{x} = r - x - e^{-x}$. Determine graphically the number of fixed points. Show that a bifurcation exists when one modifies the parameter r. At which value? What is the name of this bifurcation?

1(d) Draw the successive iterated values through the application $x_{n+1} = \cos x_n$.

1(e) Define what is a Lyapunov exponent with a simple sentence.

2 Bead on a horizontal wire

A bead of mass m is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support at a distance h from the wire (Figure 1). Finally … d by a viscous damping force $b\dot{x}$.

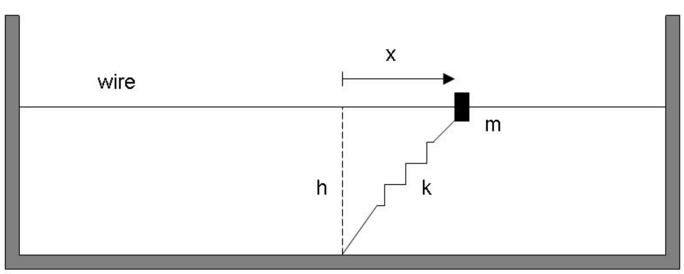


Figure 1: Schematic représentation of the mechanical system.

2(a) Write Newton's law for the motion of the bead.

2(b) Find all possible equilibria, i.e. fixed points, as functions of k, h, m, b, and L_0.

2(c) Justify that the overdamped regime corresponds to suppose $m = 0$. In this case, draw the vector field, classify the stability of all the fixed points, and draw a bifurcation diagram.

2(d) If $m \neq 0$, how small does m have to be to be considered negligible? In what sense is it negligible?

2(e) Solve the problem in the general case.

2(f) Draw the phase portrait.

3 Faint young Sun paradox

The faint young Sun paradox describes the apparent contradiction between observations of liquid water early in Earth's history and the astrophysical expectation that the Sun's output would be only 70 percent as intense during that epoch as it is during the modern epoch. The issue was raised by astronomers Carl Sagan and George Mullen.

The energetic flux R_S received by the Earth from the Sun is schematically distributed as follows

- 30% reflected (albédo) by the atmosphere. It is characterized by the parameter α that is a decreasing function of the temperature T as schematically presented Fig. 2,
- 20% absorbed by the atmosphere,
- 50% reach the ground.

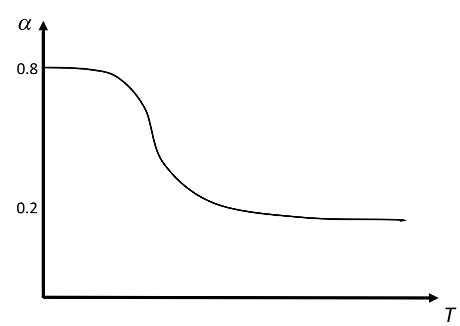


Figure 2: Schematic evolution of the albédo parameter α.

3(a) To describe the evolution of the temperature on Earth, justify why one uses the following thermodynamic equation

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = R_S(1-\alpha) - \sigma T^4 \tag{1}$$

3(b) What are the conditions at equilibrium?

3(c) How many states are possible? Discuss schematically their stability?

3(d) Justify that one switches from the two possible generic conditions through a bifurcation described during the lecture.

3(e) Could you explain why this simple analysis leads to a paradox, called the faint young Sun paradox?

4 Simple model for a child playing on a swing

A simple model for a child playing on a swing is

$$\ddot{x} + (1 + \varepsilon\gamma + \varepsilon \cos(2t)) \sin x = 0 \tag{2}$$

where ε and γ are parameters and $0 < \varepsilon \ll 1$. The variable x measures the angle between the swing and the downward vertical.

The problem is about the following very important question: Starting near the fixed point $x = 0$ and $\dot{x} = 0$, can the child get the swing going by pumping her/his legs this way, or does she/he need a push?

4(a) Comment the different terms in the equation (2).

4(b) For small values of x, the equation may be replaced by

$$\ddot{x} + (1 + \varepsilon\gamma + \varepsilon \cos(2t))x = 0. \tag{3}$$

Using the following asymptotic form,

$$x = x_0(t, T) + \varepsilon x_1(t, T) + ... \tag{4}$$

involving the two time scales t and $T = \varepsilon t$, solve (3) at the lowest order in ε, and show that

$$x_0(t, T) = r(T) \cos(t + \phi(T)). \tag{5}$$

4(c) To find the differential equations governing r and ϕ, insert (5) in the $O(\varepsilon)$ equation. Use prime to denote differentiation with respect to slow time $T = \varepsilon t$, i.e. $r' = dr/dT$.

4(d) Justify why the terms proportional to $\cos(t + \phi)$ and $\sin(t + \phi)$ on the right hand side of the $O(\varepsilon)$ equation have to disappear.

Show that the averaged equations become

$$r' = \frac{1}{4} r \sin(2\phi) \qquad \text{and} \qquad \phi' = \frac{1}{2}\left(\gamma + \frac{1}{2}\cos(2\phi)\right). \tag{6}$$

4(e) Show that the fixed point $r^\star = 0$ is unstable to exponentially growing oscillations, i.e. $r(T) = r_0\, e^{kT}$ with $k > 0$, if $|\gamma| < \gamma_c$ where γ_c is to be determined.

4(f) For $|\gamma| < \gamma_c$, write a formula for the growth rate k in terms of γ.

4(g) In this regime, what is (are) the physical reason(s) why r is not diverging in practice?

4(h) What could your prove when $|\gamma| > \gamma_c$?

4(i) Interpret the results physically.

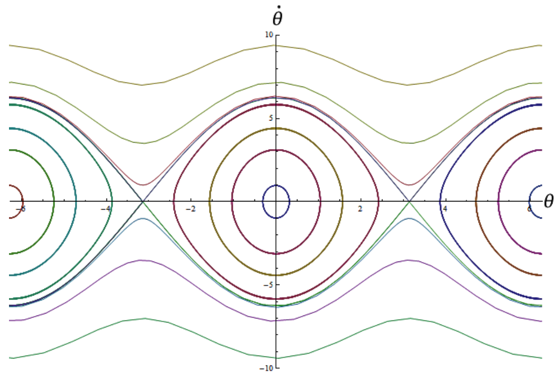
4(j) Can the child get the swing going by pumping her/his legs this way, or does she/he need a push?

Correction de l'Examen de Janvier 2018

1. Question from the lectures

1(a) Théorème de Poincaré-Bendixson.

1(b) Portrait de phase du pendule sans dissipation. La périodicité selon θ se retrouve dans le plan de phase, que l'on peut représenter sous forme cylindrique (périodicité selon θ mais pas selon $\dot{\theta}$).



1(c) On trace les deux courbes "$r - x$" et e^{-x} pour trouver le nombre de points fixes. Pour r grand, on a deux points fixes, pour r petit, il n'y a plus de points fixes.

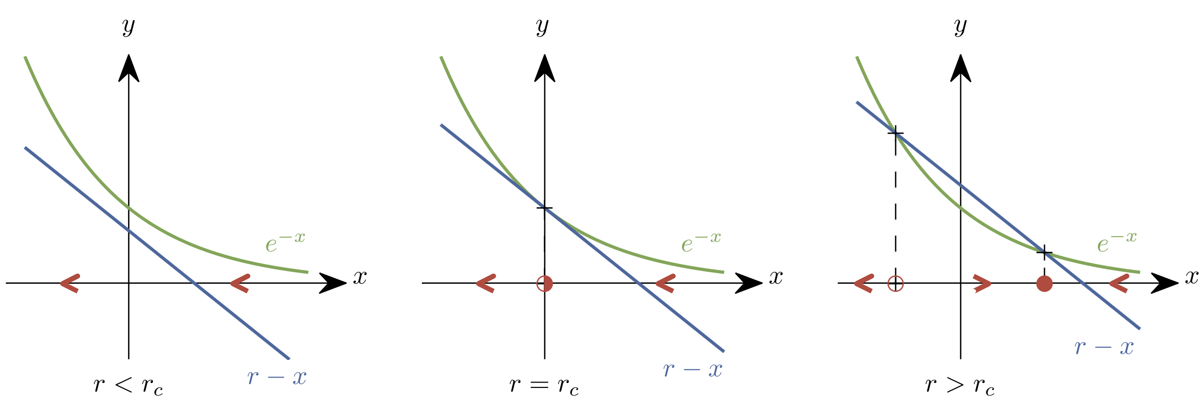


Figure 3: Résolution graphique de l'équation $\dot{x} = r - x - e^{-x}$.

Pour déterminer le point de bifurcation, c'est-à-dire la valeur critique r_c, on impose que les deux courbes se croisent tangentiellement. On doit donc avoir les fonctions et leurs dérivées égales en ce point. $r_c - x = e^{-x}$ et -1=$-e^{-x}$ qui impliquent $x = 0$ et r_c =1. C'est une bifurcation nœud-col

1(d) Itérés de $x_{n+1} = \cos x_n$.

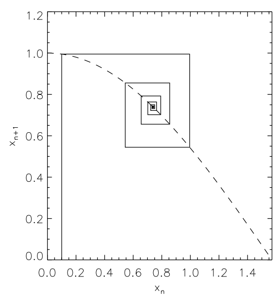


Figure 4: Représentation de la courbe $x_{n+1} = \cos x_n$ en traitillés. La courbe en continue correspond à la toile d'araignée qui converge vers le point fixe $x^\star = 0739...$

1(e) Un exposant de Lyapunov mesure la divergence exponentielle entre deux orbites infiniment proches.

2. Bead on a horizontal wire

2(a) There are two forces opposing the motion of the bead. The first is F_{spring} the force due to the spring and the second is the viscous damping. It is important to note that we are only concerned with forces in the horizontal direction so the force from the spring is

$$F_{spring} = k(\ell - L_0)\cos\theta \tag{7}$$

where ℓ is the distance from the connection $\ell = \sqrt{x^2 + h^2}$ and $\cos\theta = x/\ell$. Using these substitutions

$$F_{spring} = k(\sqrt{x^2 + h^2} - L_0)\frac{x}{\sqrt{x^2 + h^2}} = kx\left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right). \tag{8}$$

Now putting together the acceleration and the damping force, we get

$$m\ddot{x} = -F_{spring} - F_{damping} \tag{9}$$

that leads to

$$m\ddot{x} + b\dot{x} + kx\left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) = 0. \tag{10}$$

2(b) To find the fixed points, one set the derivatives equal to 0 and solve the equation for x. One gets three solutions

$$x^\star = \pm\sqrt{L_0^2 - h^2} \qquad \text{and} \qquad 0. \tag{11}$$

2(c) In the overdamped regime, the dissipative term is dominating the inertia term. It is strictly identical to set $m = 0$.

In this part, we will make the change of variable $z = x/h$ and the substitution $\alpha = L_0/h$. After setting the mass to 0, the equation becomes

$$\dot{z} + \frac{k}{b} z \left(1 - \frac{\alpha}{\sqrt{z^2+1}}\right) = 0. \tag{12}$$

The fixed points for this system occur at $z^\star = \pm\sqrt{\alpha^2 - 1}$ and 0.

To get the stability, one just has to proceed as shown during the lectures. The vector fields are shown in Fig. 5 and the bifurcation plot is presented in Fig. 6.

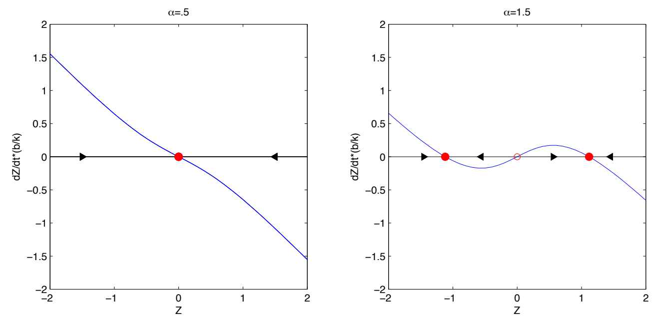


Figure 5: Vector fields for two different values of α. One smaller than 1 on the left, and one greater on the right.

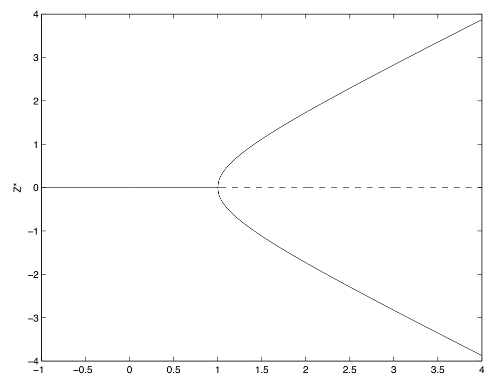


Figure 6: Bifurcation diagram showing $z^\star$ as a function of α.

2(d) For the final portion of the question, we will use the substitution $T = kt/b$. This changes the equation (10) to the following

$$\frac{mk^2}{b^2}\frac{d^2x}{dT^2} + b\frac{k}{b}\frac{dx}{dT} + kx\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right) = 0 \tag{13}$$

$$\varepsilon\frac{d^2x}{dT^2} + \frac{dx}{dT} + x\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right) = 0 \tag{14}$$

where $\varepsilon = mk/b^2$. For the mass to be negligible, the first term must be much smaller than the other two meaning $\varepsilon \ll 1 \Rightarrow m \ll b^2/k$.

2(e) In the general case, it is necessary to compute the Jacobian of the dynamical system. Let's rewrite Eq. (10) as a 2D dynamical system as follows

$$\begin{cases} \dot{x} = y\,, \\ \dot{y} = \ddot{x} = -\dfrac{b}{m}\dot{x} - \dfrac{k}{m}x\left(1 - \dfrac{L_0}{\sqrt{x^2+h^2}}\right) \end{cases} \tag{15}$$

that leads to the Jacobian matrix associated to the fixed point $(x^\star, y^\star)$,

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m}\left(1 - \frac{L_0}{\sqrt{x^{\star 2}+h^2}}\right) + \frac{kx^\star}{m}\frac{L_0(-1/2)2x^\star}{(x^{\star 2}+h^2)^{3/2}} & -\frac{b}{m} \end{pmatrix}. \tag{16}$$

- Fixed Point $(0, 0)$:

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m}\left(1 - \frac{L_0}{h}\right) & -\frac{b}{m} \end{pmatrix}. \tag{17}$$

  With a discriminant $\Delta = (k/m)\,(1 - L_0/h)$ and a trace $\tau = -b/m$ that is negative. So, if $L_0 > h$, $\Delta < 0$, i.e. a saddle point. While, if $L_0 < h$, $\Delta > 0$, one has either a stable spirale (when $\tau^2 - 4\Delta > 0$) or a stable node.

- Fixed Point $(\pm\sqrt{L_0^2 - h^2}, 0)$:

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m}\left[1 - \left(\frac{h}{L_0}\right)^2\right] & -\frac{b}{m} \end{pmatrix}. \tag{18}$$

  The discriminant $\Delta = (k/m)\left[1 - (h/L_0)^2\right]$ is positive since $L_0 > h$ for these fixed points to exist. The trace $\tau = -b/m$ is always negative, so the fixed points are always stable, either node or spirale depending on the sign of $\tau^2 - 4\Delta$.

2(f) If $L_0 > h$, the origin is a saddle, while the tow other fixed points are stable. On the contrary, if $L_0 < h$, the origin is a stable fixed point (spirale or node).

3. Faint young Sun paradox

3(a) If C is the heat capacity of the Earth and T its temperature, one gets

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = R_S(1-\alpha) - R_E \tag{19}$$

in which $R_S(1-\alpha)$ is the energy of the Sun that reaches the ground.

For what concerns R_E the energy radiated by the Earth, it is a rather usual approximation to consider that the power radiated from a black body in terms of its temperature is given by the Stefan–Boltzmann law. Specifically, the Stefan–Boltzmann law states that the total energy radiated per unit surface area of a black body across all wavelengths per unit time is directly proportional to the fourth power of the black body's thermodynamic temperature T. The constant of proportionality σ is called the Stefan–Boltzmann constant and derives from other known constants of nature. One has therefore $R_E = \sigma T^4$.

3(b) At equilibrium, on has $\mathrm{d}T/\mathrm{d}t = 0$, that corresponds to $R_E = R_S(1-\alpha)$.

3(c) As shown by Fig. 7, there are two generic possibilities: either 3 intersections corresponding to 3 fixed points (2 stable and 1 instable), or only one possible fixed point for smaller (or very large), although not too small values of α.

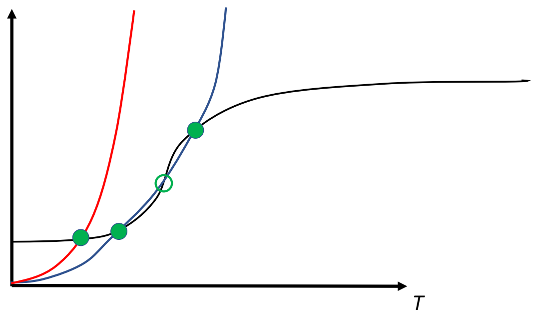


Figure 7: Schematic evolution of the radiated fluxes: the flux received from the Sun is plotted in black, while the flux emitted from the Earth for a large (in red) or an intermediate value of the Stefan–Boltzmann constant α.

3(d) One gets from the first case to the second through a saddle-node bifurcation, in which two fixed points merge (one stable and one unstable) before disappearing.

3(e) As explained in the introduction of the exercice, early in Earth's history, the Sun's output would have been only 70 percent as intense as it is during the modern epoch. A lower Sun's output is analog to a larger Stefan-Boltzmann constant, and, as shown in Fig. 7, leads to a single fixed point at low temperature.

In the environmental conditions existing at that time, this solar output would have been insufficient to maintain a liquid ocean, in apparent contradiction with observations of liquid water early in Earth's history. Astronomers Carl Sagan and George Mullen pointed out in 1972 that this is contrary to the geological and paleontological evidence.

Indeed, according to the Standard Solar Model, stars similar to the Sun should gradually brighten over their main sequence lifetime due to contraction of the stellar core caused by fusion. However, with the predicted solar luminosity 4 billion years ago and with greenhouse gas concentrations the same as are current for the modern Earth, any liquid water exposed to the surface would freeze. However, the geological record shows a continually relatively warm surface in the full early temperature record of Earth, with the exception of a cold phase, the Huronian glaciation, about 2.4 to 2.1 billion years ago. Water-related sediments have been found dating to as early as 3.8 billion years ago. Hints of early life forms have been dated from as early as 3.5 billion years, and the basic carbon isotopy is very much in line with what is found today (cf. Wikipedia).

The presence of water-related sediments and hints of early life despite a weak Sun radiations is the Faint young Sun paradox. Explanations of this paradox have taken into account greenhouse effects, astrophysical influences, or a combination of the two.

4. Simple model for a child playing on a swing

4(a) The term $1+\varepsilon\gamma+\varepsilon\cos(2t)$ models the effects of gravity. The "1" corresponds to the frequency of the empty swing, while $\varepsilon\gamma$ models the modification of the frequency introduced by the presence of the child that changes the center of gravity. Finally, $\varepsilon\cos(2t)$ corresponds to the periodic pumping of the child's legs at approximately twice the natural frequency of the swing.

4(b) The usual two-timing substitutions give

$$
\begin{aligned}
O(1): \quad & \partial_{\tau\tau}x_0 + x_0 = 0 && (20)\\
O(\varepsilon): \quad & \partial_{\tau\tau}x_1 + x_1 = -2\partial_{\tau T}x_0 - (\gamma+\cos(2t))\,x_0. && (21)
\end{aligned}
$$

The solution of the $O(1)$ equation is $x_0(t,T) = r(T)\cos(t+\phi(T))$ where $r(T)$ and $\phi(T)$ are the slowly-varying amplitude and phase of x_0.

4(c) This yields

$$
\partial_{\tau\tau}x_1 + x_1 \quad = \quad 2\left[r'\sin(t+\phi) + r\phi'\cos(t+\phi)\right] - (\gamma+\cos(2t))r(T)\cos(t+\phi(T)) \qquad (22)
$$

4(d) As usual one needs that there be no terms proportional to $\cos(t+\phi(T))$ and $\sin(t+\phi(T))$ on the right-hand-side of the $O(\varepsilon)$ equation,

With the substitution $\theta = t+\phi$ and using the averaged equation $<(22)\times\sin\theta>$, one gets

$$
\begin{aligned}
r' &= <r\cos\theta(\gamma+\cos(2t))\sin\theta> && (23)\\
&= <r\gamma\cos\theta\sin\theta> + <r\cos\theta\sin\theta\cos(2t)> && (24)\\
&= r\left(\gamma<\cos\theta\sin\theta> + <\cos\theta\sin\theta\cos(2t)>\right) && (25)\\
&= r(\gamma\times(0) + \frac{1}{4}\sin(2\phi)) && (26)\\
r' &= \frac{1}{4}r\sin(2\phi) && (27)
\end{aligned}
$$

where we used that

$$
\begin{aligned}
<\cos(2t)\cos\theta\sin\theta> &= \frac{1}{2}<\cos(2\theta-2\phi)\sin(2\theta)> && (28)\\
&= \frac{1}{2}<(\cos(2\theta)\cos(2\phi)+\sin(2\theta)\sin(2\phi))\sin(2\theta)> && (29)\\
&= \frac{1}{4}<\sin(2\phi)>. && (30)
\end{aligned}
$$

For ϕ', using the averaged equation $<(22)\times\cos\theta>$, one gets

$$
\begin{aligned}
r\phi' &= <r\cos^2\theta(\gamma+\cos(2t))> && (31)\\
&= r<\gamma\cos^2\theta + \cos^2\theta\cos(2t)> && (32)
\end{aligned}
$$

qui se simplifie en

$$\begin{aligned}
\phi' &= <\gamma\cos^2\theta> + <\cos^2\theta\cos(2t)> && (33)\\
&= \frac{1}{2}\gamma + <\cos^2\theta\cos(2\theta-2\phi)> && (34)\\
&= \frac{1}{2}\gamma + <\cos^2\theta(\cos(2\theta)\cos(2\phi)+\sin(2\theta)\sin(2\phi))> && (35)\\
&= \frac{1}{2}\gamma + <\cos^2\theta(\cos^2\theta-\sin^2\theta)\cos(2\phi)+2\cos\theta\sin\theta\sin(2\phi)> && (36)\\
&= \frac{1}{2}\gamma + <\cos^4\theta\cos 2\phi - \cos^2\theta\sin^2\theta\cos 2\phi + 2\cos^3\theta\sin\theta\sin(2\phi)> && (37)\\
&= \frac{1}{2}\gamma + \frac{3}{8}\cos(2\phi) - \frac{1}{8}\cos(2\phi) + 0 && (38)\\
&= \frac{1}{2}\left(\gamma + \frac{1}{2}\cos(2\phi)\right) && (39)
\end{aligned}$$

4(e) The fixed points satisfy ($r' = 0 \Rightarrow r^\star = 0$ or $\phi^\star = k\pi/2$) and ($\phi' = 0 \Rightarrow \cos(2\phi^\star) = -2\gamma$ if $|\gamma| < 1/2$).
When $|\gamma| < 1/2$, there are two fixed points, defined by $r^\star = 0$ and $\phi^\star_\pm = \pm\mathrm{Arccos}(-2\gamma)/2$. Equations (6) lead to the following Jacobian matrix

$$\begin{aligned}
\mathbf{J} &= \begin{pmatrix} \sin(2\phi^\star)/4 & r^\star\cos(2\phi^\star)/2 \\ 0 & -\sin(2\phi^\star)/2 \end{pmatrix} && (40)\\
&= \begin{pmatrix} \sin(2\phi^\star)/4 & 0 \\ 0 & -\sin(2\phi^\star)/2 \end{pmatrix} \quad \text{if} \quad r^\star = 0. && (41)
\end{aligned}$$

As the Jacobian matrix is already diagonal, we see that one has two different eigenvalues of opposite signs. The eigenvalue $(-\sin(2\phi^\star)/2)$ associated to the stability of the field ϕ could be either positive or negative. Let's define $\phi^\star_+$ the solution corresponding to $\sin(2\phi^\star_+) > 0$ and $\phi^\star_-$ to $\sin(2\phi^\star_-) < 0$. So $\phi^\star_-$ is unstable with respect to the field ϕ, while stable to the field r. On the contrary $\phi^\star_+$ is stable with respect to the field ϕ, while unstable to the field r. So, in the regime $|\gamma| < 1/2$, the phase will be locked to the value $\phi^\star_+$, while the field r will be unstable. So the threshold is $\gamma_c = 1/2$.
For the evolution of the radius r, when $|\gamma| < 1/2$, the phase rapidly reaches the fixed point $\phi^\star_+$, leading to the exponential growth result for the radius r, since one has $r' = \frac{1}{4}r\sin(2\phi^\star_+)$. Outside this region $|\gamma| < 1/2$, the phase is continually changing and the fixed point $r^* = 0$ is stable.

4(f) For the region $|\gamma| < \gamma_c$, there is a fixed growth rate of the amplitude since one has $r' = kr$ with

$$k = \frac{1}{4}\sin(2\phi^\star_+) = \frac{1}{4}\sqrt{1-\cos^2(2\phi^\star_+)} = \frac{1}{4}\sqrt{1-(2\gamma)^2} = \frac{1}{4}\sqrt{1-4\gamma^2}. \quad (42)$$

4(g) Nonlinearity and damping.

4(h) When $|\gamma| > 1/2$, there are no real fixed point. However, $r^\star = 0$ corresponds to a fixed point in the physical space since the phase is continuously evolving but physically unimportant, since the radius is zero. In that case, there are two ranges:

- For values of $\gamma < -1/2$, ϕ' will always be negative and therefore ϕ will decrease to $-\infty$.

- For values of $\gamma > +1/2$, ϕ' will always be positive and therefore ϕ will increase to $+\infty$.

If one uses the Jacobian matrix (41), one sees that the eigenvalue associated to the field r is $\sin(2\phi^\star)/4$ that is not always negative; nor always positive. One cannot claim that the fixed point is unstable.

Actually, it is possible to exhibit a conserved quantity. From Eqs. (6), one can indeed derive

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\frac{1}{4}\sin(2\phi)}{\gamma + \frac{1}{2}\cos(2\phi)} = -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\phi}\ln(\gamma + \frac{1}{2}\cos(2\phi)) \tag{43}$$

that can be simplified in

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\ln r^4 + \ln\left[\,\gamma + \frac{1}{2}\cos(2\phi)\,\right]\right) = 0 \tag{44}$$

and leads to

$$r^4\left(\gamma + \frac{1}{2}\cos(2\phi)\right) = \text{Cste.} \tag{45}$$

Because of this conserved quantity, this system behaves like a Hamiltonian system, and the trajectory will be a closed orbit, prohibiting any escape toward infinity.

4(i) In the case where γ is outside the range $-1/2$ and $1/2$, the system has a fixed point at $r = 0$, neither stable nor unstable, meaning there are small oscillations around this point but the value never goes off to infinity. The child can pump as hard as he likes but will not be able to get swinging unless (maybe !) he is pushed outside the region where the small angle assumption is valid.

In the case where γ is inside the range $-1/2$ and $1/2$, the amplitude goes off to infinity meaning that the swing goes back and forth with increasing amplitude for a system slightly perturbed from the fixed point. This means that for values inside this range, it is possible to start swinging without any external forcing.

4(j) A child will correspond to a smaller distance between the rotation point and the center of mass than for an adult. The square of the oscillation frequency of the pendulum is inversely proportional to this length. It means that the corresponding value γ for the child will be larger than the one for the adult. One thus realizes that a too light child will have a γ above the critical value γ_c. Pushing is therefore crucial if γ is too large. Only if your child is heavy enough, you can be lazy...!