Master 1ère année **2019-2020**
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Exam for "Systèmes Dynamiques et Chaos"
Thursday 19 December 2019
Duration: 2h30

Calculators and Documents are not permitted during the "Question de cours".
Permitted are printed material, hand written notes or photocopies of any kind, but not books.
Answers can be written in French or English.

1 Question de cours

Give back the copy with your answers to Part 1 no later than 20' from the start.

1(a) Donner la définition d'un système non autonome.

1(b) À partir de quel ordre, un système dynamique peut présenter des oscillations ?

1(c) Donner la forme normale d'une bifurcation nœud-col et le diagramme de bifurcation associé.

1(d) Représenter la classification des systèmes dynamiques linéaires à deux dimensions dans le plan défini par le déterminant Δ et la trace τ de la matrice $\mathbf{A}$ du système dynamique.

1(e) Donner 2 méthodes pour éliminer la possibilité d'orbites fermées.

1(f) Expliquer en 2 mots ou bien à l'aide d'un schéma à quoi correspond une bifurcation de Hopf à deux dimensions.

2 Analysis of a one-dimensional dynamical system

For $r \in R$, consider the differential equation

$$\dot{x} = rx - 2x^2 + x^3 \tag{1}$$

2(a) Show that $x^* = 0$ is a fixed point for any value of the parameter r, and determine its stability. Hence identify a bifurcation point r_1.

2(b) Show that for certain values of the parameter r there are additional fixed points.

2(c) For which values of r do these fixed points exist? Determine their stability and identify a further bifurcation point r_2.

2(d) Using a Taylor expansion of (1), determine the normal form of the bifurcation at r_1. What type of bifurcation takes place?

2(e) Similarly, determine the normal form of the bifurcation at r_2. What type of bifurcation takes place?

2(f) Sketch the bifurcation diagram for all values of r and x^*. (Use a full line to denote a curve of stable fixed points, and a dashed line for a curve of unstable fixed points!)

3 Bouncing of a particle on a springy surface

The Hamiltonian for a particle of mass m that bounces on a springy surface is approximated by

$$H(x,p) = \frac{1}{2m}p^2 + V(x) \tag{2}$$

where

$$V(x) = \begin{cases} \frac{1}{2}Cx^2 & \text{if } x \le 0 \\ mgx & \text{if } x \ge 0 \end{cases},$$

x is the position of the particle, p the momentum of the particle, $V(x)$ the potential energy, C and g are positive constants.

3(a) Write down the equations of motion for x and p in the cases where $x \ge 0$ and $x \le 0$.

3(b) Given that $m = 1$, $g = 10$, $C = 2$ and $E = 10$, sketch the contour of $H(x,p) = E$.

3(c) Solve the equations of motion for $x(t)$ and $p(t)$ for this trajectory, giving expressions for both $x \le 0$ and $x \ge 0$ separately. Hint: for $x \ge 0$, assume $x(0) = 0$ and derive $p(0)$.

3(d) Prove that the solution spends a time $T_1 = 2/\sqrt{5}$ in the region $x \ge 0$.

3(d) Similarly, compute T_2 the time spent by the solution in the region $x \le 0$.

4 Spread of an epidemic in a city

A simple model for the spread of an epidemic in a city is given by

$$\dot{S} = -\tau S I, \tag{3}$$
$$\dot{I} = \tau S I - r I, \tag{4}$$

where $S(t)$ and $I(t)$ represent the numbers of susceptible and infected individuals scaled by 1000, respectively. Assume that those who recover become immune. The time t is measured in days.

4(a) Give the physical meanings of τ and r.

4(b) Determine a value for S at which the infected population is maximum.

4(c) Given that $\tau = 0.003$ and $r = 0.5$, sketch a phase portrait showing three trajectories whose initial points are at (1000,1), (700,1) and (500,1). Give a physical interpretation in each case.

4(d) Propose a more complex model for the spread of an epidemic.

5 Water vs Alcohol in a glass

Please give your answers to this exercise on a separate page

The sloshing is the global oscillation of a liquid in a tank. We consider here the damping of the oscillations of water (viscosity $\eta_{\text{water}} = 1$ mPa.s) or ethanol (viscosity $\eta_{\text{ethanol}} = 1.4$ mPa.s) in a glass. The radius r of the glass is slightly larger than the capillary length $\sqrt{\gamma/(\rho g)}$ where γ is the surface tension, ρ the density of the liquid and g the gravity. At $t = 0$, a kick put the liquid into oscillations and the free decay of liquid oscillations in the glass is studied. We observe that the water has stopped its oscillations much earlier than ethanol.

5(a) Why is this result surprising? What is the physical phenomenon explaining such behavior?

To interpret this result, the equation of motion of the oscillation h of the free surface is given by

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}h}{\mathrm{d}t} + \mu\,\mathrm{sign}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right) + \omega_0^2 h = 0\,. \tag{5}$$

5(b) What is the physical meaning of the different term of the previous equation?

In the following, we will resolve Eq. (5) in the limit of small damping using a multi-scale analysis. We set $\alpha = \epsilon\tilde{\alpha}$ and $\mu = \epsilon\tilde{\mu}$ with $\epsilon \ll 1$.

5(c) Could you justify such analysis using an order of magnitude of the different time scales in the problem?

5(d) Using the following asymptotic form,

$$h(\tau, T) = h_0(\tau, T) + \epsilon h_1(\tau, T) + ... \tag{6}$$

involving the two time scales $\tau = t$ and $T = \epsilon t$, solve Eq. (5) at the lowest order in ϵ and show that

$$h_0(\tau, T) = A(T)\cos(\omega_0\tau + \Phi(T)). \tag{7}$$

5(e) Write Eq. (5) at the first order in ϵ. Justify why the terms proportional to $\cos(\omega_0\tau + \Phi(T))$ and $\sin(\omega_0\tau + \Phi(T))$ on the obtained equation have to disappear.

5(f) Derive a differential equation for A and another one for Φ.
Hint for any periodic function $f(\tau)$ of period τ_0

$$f(t) = \sum_{n=0}^{+\infty} a_n\cos(2\pi nt/\tau_0) + b_n\sin(2\pi nt/\tau_0) \tag{8}$$

with $a_n = (2/\tau_0)\displaystyle\int_{-\tau_0/2}^{+\tau_0/2} f(t)\cos(2\pi nt/\tau_0)\mathrm{d}t$ and $b_n = (2/\tau_0)\displaystyle\int_{-\tau_0/2}^{+\tau_0/2} f(t)\sin(2\pi nt/\tau_0)\mathrm{d}t\,.$

5(g) Determine a solution for the two equations and write the solution for $h_0(\tau, T)$ as a function of μ and α. What is the arrest time of the oscillations of the system?

5(h) Describe the shape of the enveloppe for the two limit cases for the damping. Attribute water or ethanol to each case.

Correction de l'Examen de Décembre 2019

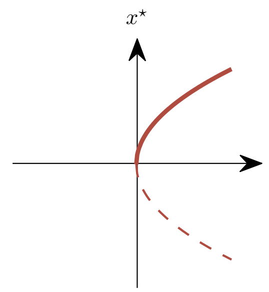
1. Question de cours

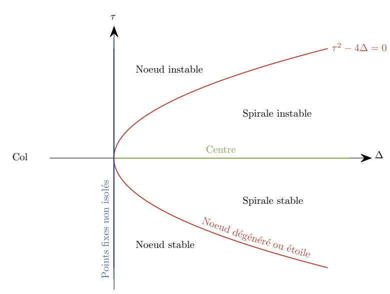
Give back the copy with your answers to Part 1 no later than 20' from the start.

1(a) Système dynamique avec une dépendance explicite du temps.

1(b) Deuxième ordre.

1(c) La forme normale de la bifurcation nœud-col est $\dot{x} = r - x^2$ et son diagramme est représenté ci-dessous.

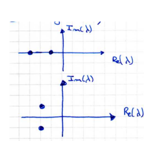




1(d)

1(e) Nous avons vu trois méthodes: Système gradient, Fonction de Lyapunov et Critère de Dulac.

1(f) À deux dimensions, la réponse se trouve dans les deux valeurs propres du Jacobien. Si le point fixe est stable, cela veut dire que la partie réelle des deux valeurs propres est négative. Les deux valeurs propres sont donc dans le demi-plan gauche. Par ailleurs, comme les valeurs propres sont solutions d'une équation du second degré à coefficients réels, il n'y a que deux possibilités. Deux valeurs propres **réelles négatives** ou bien deux valeurs propres **complexes conjuguées**. Pour que le point fixe se déstabilise, il faut donc que l'une (ou les deux !) entrent



dans la partie droite, i.e. dans la zone où la partie réelle des valeurs propres est positive. Le premier cas corresponds aux bifurcations fourches et transcritique, les bifurcations de Hopf à la seconde situation.

2. Analysis of a one-dimensional dynamical system

2(a) We have $\dot{x} = f(x)$ with $f(x) = rx - 2x^2 + x^3$. For fixed points $f(x^*) = 0$, and clearly $f(0) = 0$, so $x^* = 0$ is a fixed point for all real r.

We compute $f'(x) = r - 4x + 3x^2$. Hence $f'(0) = r$ and 0 is stable for $r < 0$ and unstable for $r > 0$.

Hence a bifurcation takes place at $r_1 = 0$.

2(b) Additional fixed points are given by $r - 2x + x^2 = 0$, which implies that there are fixed points at $1 \pm \sqrt{1-r}$, provided $r \leq 1$.

2(c) We compute $f'(1 \pm \sqrt{1-r}) = 2 - 2r \pm 2\sqrt{1-r} = 2\sqrt{1-r}(\sqrt{1-r} \pm 1)$. Hence $1 + \sqrt{1-r}$ is stable for $r < 1$, and $1 - \sqrt{1-r}$ is stable for $r < 0$ and unstable for $0 < r < 1$.

Thus $r_2 = 1$ is another bifurcation point.

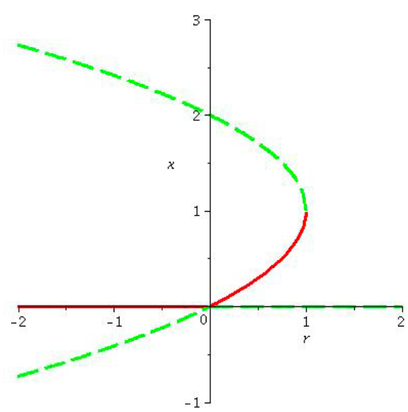
2(d) Near $r = 0$ we expand to leading order $\dot{x} \approx rx - 2x^2$. The substitution $y = 2x$ gives $\dot{y} = ry - y^2$.

This is the normal form of a transcritical bifurcation.

2(e) Near $r = 1$ we let $\tilde{r} = r - 1$ and expand to leading order in $y = x - 1$. We find $\dot{y} \approx \tilde{r} + \tilde{r}y + y^2$. Neglecting the term $\tilde{r}y$ we find $\dot{y} = r + y^2$.

This is the normal form of a saddle-node bifurcation.

2(f) A sketch of the bifurcation diagram:



3. Bouncing of a particle on a springy surface

3(a) $\dot{p} = -Cx$ if $x \leq 0$ and $\dot{p} = -mg$ if $x \geq 0$.

3(b) If $x \leq 0$, $H(x,p) = \frac{1}{2m}p^2 + \frac{1}{2}Cx^2 = E \Rightarrow p^2 + 2x^2 = 20$. A half-circle in the negative half plane.
If $x \geq 0$, $H(x,p) = \frac{1}{2m}p^2 + mgx = E \Rightarrow p^2 + 20x = 20$. A parabola in the positive half plane.

3(c) If $x(0) = 0$, as $E = 10$, $p(0) = 2\sqrt{5}$ and therefore $\dot{x}(0) = 2\sqrt{5}$.
If $x \geq 0$, $\dot{p} = mg \Rightarrow \ddot{x} = g \Rightarrow x(t) = \frac{1}{2}gt^2 + \dot{x}(0)t + x(0) \Rightarrow x(t) = 5t^2 + 2\sqrt{5}t = (\sqrt{5}t + 1)^2 - 1$.

If $x \leq 0$, $\dot{p} = -Cx \Rightarrow \ddot{x} = -2x \Rightarrow x(t) = x(0)\cos(\sqrt{2}t) + \frac{p(0)}{\sqrt{2}}\sin(\sqrt{2}t) \Rightarrow x(t) = \sqrt{\frac{5}{2}}\sin(\sqrt{2}t)$.

3(d) The expression for $x \geq 0$ vanishes for $t_1 = 0$ and $t_1' = -2/\sqrt{5}$. The particle spends therefore a time $T_1 = |t_1 - t_1'| = 2/\sqrt{5}$ in the region $x \geq 0$,

The expression for $x \leq 0$ vanishes for $t_2 = 0$ and $t_2' = \pi/\sqrt{2}$. The particle spends therefore a time $T_2 = \pi/\sqrt{2}$ in the region $x \leq 0$.

4. Spread of an epidemic in a city

4(a) τ is a constant measuring how quickly the disease is transmitted and r measures the rate of recovery.

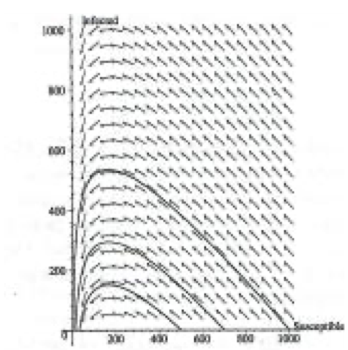
4(b) The maximum number of infected individuals occurs when $\frac{dI}{dS} = 0$. Now

$$\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} = \frac{\tau S - r}{-\tau S}. \tag{9}$$

Therefore $\frac{dI}{dS} = 0$ when $S = r/\tau$, a number called a threshold value.

The critical points for this system are found by solving the equation $\dot{S} = \dot{I} = 0$. Therefore, there is an infinite number of critical points lying along the horizontal axis.

4(c) A phase portrait showing the three trajectories is plotted in the figure below. Trajectories are only plotted in the first quadrant since populations cannot be negative.



In each case, the population of susceptibles decreases to a constant value, and the population of infected individuals increases and then decreases to zero. Note that in each case, the maximum number of infected individuals occurs at $S = r/\tau \approx 167000$.

4(d)

5. Water vs Alcohol in a glass

5(a) Although the viscosity of the water is smaller than that of ethanol, the oscillations are much more quickly damped for water than for ethanol. The qualitative difference comes from wetting conditions. Some dissipation comes from the imbalance between the advancing and the receding contact angles. This dissipation can be modeled by a static-friction force.

5(b) The different terms of the equation has the following physical meaning:

- $\frac{\mathrm{d}^2 h}{\mathrm{d}t^2}$ corresponds to the acceleration of the surface;
- $\alpha \frac{\mathrm{d}h}{\mathrm{d}t}$ is due to the bulk viscous force;
- $\mu \mathrm{sign}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)$ is due to the contact-line friction force;
- $\omega_0^2 h$ is due to gravity.

5(c) The period of oscillations of the liquid is of the order of 1 s while the decay characteristic time is of the order of 10 s. The factor 10 between these two time scales justifies such analysis.

5(d) In the limit of small damping, $\alpha = \epsilon\tilde{\alpha}$ and $\mu = \epsilon\tilde{\mu}$. At the lowest order in ϵ, the equation of motion is

$$\frac{\mathrm{d}^2 h_0}{\mathrm{d}\tau^2} + \omega_0^2 h_0 = 0\,. \tag{10}$$

The solution of the $\mathcal{O}(1)$ equation is $h_0(\tau) = A(T)\cos[\omega_0\tau + \Phi(T)]$ where $\Phi(T)$ and $A(T)$ are the phase and slowly-varying amplitude of h_0.

5(e) At the next order in ϵ, the usual two-timing substitutions give

$$\frac{\partial^2 h_1}{\partial\tau^2} + \omega_0^2 h_1 = -2\frac{\partial^2 h_0}{\partial\tau\partial T} - \tilde{\alpha}\frac{\partial h_0}{\partial\tau} - \tilde{\mu}\,\mathrm{sign}\left(\frac{\partial h_0}{\partial\tau}\right) \tag{11}$$

To avoid secular divergences, as usual, one needs that there be no terms proportional to $\cos(\omega_0\tau + \Phi)$ and $\sin(\omega_0\tau + \Phi)$ on the right-hand-side of the $\mathcal{O}(\epsilon)$ equation.

5(f) From the expression of h_0, we get

$$\frac{\partial h_0}{\partial\tau} = -A(T)\omega_0\sin[\omega_0\tau + \Phi(T)] \tag{12}$$

and

$$\frac{\partial^2 h_0}{\partial\tau\partial T} = -\frac{\mathrm{d}A}{\mathrm{d}T}\omega_0\sin[\omega_0\tau + \Phi(T)] - A\omega_0\frac{\mathrm{d}\Phi}{\mathrm{d}T}\cos[\omega_0\tau + \Phi(T)]\,. \tag{13}$$

The term due to the friction force, $\mathrm{sign}\left(\frac{\partial h_0}{\partial\tau}\right)$ has to be treated carefully. The function $\mathrm{sign}[\sin(\omega_0\tau + \Phi(T)]$ is not harmonic but is periodic of period $2\pi/\omega_0$. Indeed, $\mathrm{sign}[\sin(\omega_0\tau + \Phi(T)] = -1$ for $0 < \tau < \pi/\omega_0$ and 1 for $\pi/\omega_0 < \tau < 2\pi/\omega_0$. Therefore, it can be expressed has a Fourier series

$$\mathrm{sign}\left(\frac{\partial h_0}{\partial\tau}\right) = \sum_{n=0}^{+\infty}\left(a_n\cos(2\pi nt/\tau_0) + b_n\sin(2\pi nt/\tau_0)\right), \tag{14}$$

where

$$\begin{aligned} a_n &= \frac{\omega_0}{\pi}\left(\int_{-\pi/\omega_0}^{0}\cos(n\omega_0\tau)\mathrm{d}\tau - \int_0^{+\pi/\omega_0}\cos(n\omega_0\tau)\mathrm{d}\tau\right) & (15)\\ &= 0 & (16)\end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{\omega_0}{\pi}\left(\int_{-\pi/\omega_0}^{0}\sin(n\omega_0\tau)\mathrm{d}\tau - \int_0^{+\pi/\omega_0}\sin(n\omega_0\tau)\mathrm{d}\tau\right) & (17)\\ &= -\frac{2}{\pi n}\left[1 - (-1)^n\right]. & (18)\end{aligned}$$

The $\mathcal{O}(\epsilon)$ equation of motion can be rewritten as

$$\begin{aligned} \frac{\partial^2 h_1}{\partial \tau^2} + \omega_0^2 h_1 &= 2\frac{\mathrm{d}A}{\mathrm{d}T}\omega_0 \sin[\omega_0\tau + \Phi(T)] + 2A\omega_0\frac{\mathrm{d}\Phi}{\mathrm{d}T}\cos[\omega_0\tau + \Phi(T)] \\ &+ \tilde{\alpha}A(T)\omega_0 \sin[\omega_0\tau + \Phi(T)] - \tilde{\mu}b_1 \sin[\omega_0\tau + \Phi(T)] + N.R.T. \end{aligned} \tag{19}$$

where N.R.T. stands for non resonant Fourier terms. Since the prefactors of $\cos[\omega_0\tau + \Phi(T)]$ and $\sin[\omega_0\tau + \Phi(T)]$ on the right-hand-side have to vanish, we get

$$2A\frac{\mathrm{d}\Phi}{\mathrm{d}T} = 0 \tag{20}$$

$$\frac{\mathrm{d}A}{\mathrm{d}T} = -\frac{1}{2}\tilde{\alpha}A - \frac{2\tilde{\mu}}{\omega_0\pi} \tag{21}$$

5(g) Equation (20) infers that the phase Φ is constant and can be set to zero. Using the initial condition $A(T) = A_0$, the evolution of the amplitude is therefore

$$A(T) = \left[A_0 + \frac{4\tilde{\mu}}{\tilde{\alpha}\omega_0\pi}\right]e^{-\tilde{\alpha}T/2} - \frac{4\tilde{\mu}}{\tilde{\alpha}\omega_0\pi} \tag{22}$$

$$A(t) = \left[A_0 + \frac{4\mu}{\alpha\omega_0\pi}\right]e^{-\alpha t/2} - \frac{4\mu}{\alpha\omega_0\pi}\,. \tag{23}$$

The time of arrest of the oscillations, given by $A(t_a) = 0$, is equal to

$$t_a = \frac{2}{\alpha}\ln\left[1 + \frac{\pi\alpha A_0\omega_0}{4\mu}\right]\,. \tag{24}$$

5(h) If the damping is essentially viscous ($\alpha\omega_0 A \gg \mu$), the oscillations are exponentially damped. This is the case for ethanol. On the contrary, for $\alpha\omega_0 A \ll \mu$, the oscillations are linearly damped. This is the case for water.