

Exam of the lecture “Physics of Long-Range Interacting Systems”

Lundi 21 Mars 2016

13h30-16h30 : Handwritten lecture notes allowed

## A. Hot Plasma

### 1 Temperature of a plasma

- 1(a) Give the definition of a plasma
- 1(b) Give two reasons to explain why plasma are long-range interacting systems?
- 1(c) Considering a gas in thermal equilibrium, give the distribution of particles velocity  $v$  that one considers usually. Comment.
- 1(d) Why is it customary in plasma physics to express temperatures in units of energy? What is the temperature of 1 eV plasma?
- 1(e) Why does one often characterize a plasma with several temperatures?
- 1(f) Discuss the relaxation between these temperatures?
- 1(g) Why in presence of a magnetic field  $\mathbf{B}$  even a single species, say ions, can have two distinct temperatures?

### 2 Debye shielding and quasi-neutrality

- 2(a) Consider now that a charge is placed in the plasma at  $\mathbf{r} = 0$  and assume for simplicity that the ions of mass  $m_i$  do not move and constitute a uniform background of positive charges. How would you justify this assumption?
- 2(i) Give the Poisson equation in terms of the ions density  $n_i$  and the electron density  $n_e$ . Give the electron velocity distribution  $f_e(\mathbf{r}, \mathbf{v})$  in presence of the electric potential field  $\phi$  and the link between  $f_e$  and  $n_e$ .
- 2(c) Solve this equation in the regime  $|e\phi/(k_B T)| \ll 1$ . Hint: it can be useful to introduce  $V = r\phi$ . Discuss the physical interpretation of the final expression for the potential  $\phi(r)$ . We remind that  $\Delta f = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right)$  for a function  $f$  with spherical symmetry.
- 2(d) Given this result, define in rigorous terms the “quasi-neutrality” concept that is usually advocated in the definition of a plasma.

### 3 Klimontovich and Vlasov-Maxwell equations

Let us consider a plasma composed of a mixture of  $s$  charged species, with  $s = i, e$  for ions ( $i$ ) and electrons ( $e$ ). The total number of particles with mass  $m_s$  for each species is denoted  $N_s$ . This particle is located in the position  $\mathbf{r}_i$  in the 3D configuration space and its velocity is  $\mathbf{v}_i$ .

- 3(a) Define the associate discrete time-dependent density function  $f_d^s$ .
- 3(b) Give the equations of motion of the point particle  $i$ .
- 3(c) By differentiating with respect to time the density function, derive the Klimontovich equation.
- 3(d) What are the advantages and the difficulties of the Klimontovich equation?
- 3(e) Explain the strategy to go further.
- 3(f) Derive the Vlasov equation for the Coulombic interaction.

### 4 Bernstein-Greene-Kruskal modes

We will here discuss a special class of non linear waves: the BGK modes, named after their inventors Bernstein, Greene and Kruskal. To this end we shall need to consider a Vlasov equilibrium distribution in presence of a spatially varying electrostatic potential. We will restrict to the  $x$ -direction.

- 4(a) Imagine that the electrons are subject to an electric field  $E(x) = -d\phi(x)/dx$  oriented along the  $x$ -direction. Explain why any function of the type  $f_0^s(v_x^2 + 2q_s\phi(x)/m_s)$  is a stationary Vlasov solution.
- 4(b) A BGK mode is characterized by such a distribution with the electric potential self-consistently produced by the very same distribution, via the Poisson equation. For the sake of simplicity, we consider the time independent setting. Give the Poisson equation.
- 4(c) We would like to determine the potential  $\phi$  solution of this self-consistent integro-differential equation. To progress with the analytical study, we assume that both ions and electrons are initially distributed in cold beams. In practice, we require that each particle of species  $s$  has the same speed in a given spatial position. Mathematically, we impose that the particles obey  $f_0^s(x, v) = 2n_0v_s \delta[v^2 + 2q_s\phi(x)/m_s - v_s^2]$ , where  $q_s$  is the charge of the species  $s$ .  
Simplify the Poisson equation using the general property  $\delta[f(y)] = \delta(y - y_0) / \left| \frac{df}{dy} \right|_{y=y_0}$ , with  $y_0$  a solution of  $f(y_0) = 0$ .
- 4(d) Show that the Poisson equation can be cast in the compact form  $\partial^2\phi/\partial x^2 = -\partial V(\phi)/\partial\phi$ , and give the expression of the pseudo-potential  $V(\phi)$ .
- 4(e) For the specific example  $m_e v_e^2 = m_i v_i^2 \equiv k_B T$ , plot the pseudo-potential as a function of  $\phi$ . What is the dynamics of the pseudo-particle?
- 4(f) Propose a linear approximation in the limit of very small  $\phi_0$ . What means small?
- 4(g) Discuss this result.

## B. Ising model combining long- with short-range interactions

We will consider the Hamiltonian

$$H_N = -\frac{1}{2N} \left( \sum_{i=1}^N S_i \right)^2 - \frac{K}{2} \sum_{i=1}^N (S_i S_{i+1} - 1), \quad (1)$$

where  $S_i = \pm 1$ , originally introduced by Nagle (1970), and then elaborated later by Khardar (1983).

### 5 Introduction

- 5(a) Discuss the different terms of this Hamiltonian: their properties, their difficulties. Why no prefactor has been kept in front of the first one?
- 5(b) At  $T = 0$ , determine if the model has a phase transition between ferro and antiferro states.
- 5(c) Determine if the model has a phase transition for  $K = 0$ .
- 5(d) Draw the basic lines of the phase diagram that one can derive from these two limiting cases?

### 6 The solution in the canonical ensemble

- 6(a) Write the canonical partition function of this model.
- 6(b) Show that it can be simplified by introducing  $f_0(\beta, x)$ , the free energy of the nearest-neighbour Ising model with an external field  $x$ .
- 6(c) Which method would you use to compute  $f_0(\beta, x)$ ?
- 6(d) Using the result  $f_0(\beta, x) = -\frac{1}{\beta N} \ln(\lambda_+^N + \lambda_-^N)$  with  $\lambda_{\pm} = e^{\beta K/2} \cosh(\beta x) \pm \sqrt{e^{\beta K} \sinh^2(\beta x) + e^{-\beta K}}$ , give the final expression for the rescaled free energy  $\tilde{\phi}(\beta, x) = \beta \tilde{f}(\beta, x)$ . Plot the schematical behavior of  $\tilde{f}(\beta, x)$  as a function of  $x$  for several values of  $\beta$ .
- 6(e) Compute the canonical tricritical point.
- 6(f) How does one compute the the first order line ?
- 6(g) Draw the phase diagram in the canonical ensemble.

### 7 The solution in the microcanonical ensemble

- 7(a) Give the expression of the total energy in terms of the magnetization  $M = \sum_{i=1}^N S_i$  and and  $\mathcal{U}$  the number of “kinks” in the chain, i.e. links between two neighbouring spins of opposite sign.
- 7(b) Introducing the number of spins up,  $N_+$ , and of spins down,  $N_-$ , compute the number of microstates in a chain of  $N$  spins corresponding to an energy  $E$  and a total magnetization  $M$ .
- 7(c) Derive the entropy  $\tilde{s}(\varepsilon, m)$  as a function of  $m = M/N$ ,  $u = \mathcal{U}/N$  and  $\varepsilon = E/N$ .
- 7(d) From this result, how does one get the entropy  $s(\varepsilon)$ ?
- 7(e) Compute the critical lines and the microcanonical critical points.

# Correction

## A. Hot Plasma

### 1. Temperature of a plasma

- 1(a) A plasma is a *quasineutral* gas of charged particles and neutral particles which exhibits *collective behaviour*: the atoms are dissociated into positive ions (*i*) and negative electrons (*e*).
- 1(b) Plasmas are first made of charged particles: the coulombic interaction is a long-range one. Second, as charges move, they generate local concentrations of positive and negative charges, which in turn give rise to *electric and magnetic fields*. These latter induce long-range forces that solicit the motion of other particles resulting in a rich and complex *self-consistent* scenario.
- 1(c) Considering a gas in thermal equilibrium, the particles velocity  $v$  obeys the Maxwell distribution which, in one dimension, we can write as

$$f(v) = n \left( \frac{m}{2\pi k_B T} \right)^{1/2} \exp \left( -\frac{\frac{1}{2}mv^2}{k_B T} \right) \quad (2)$$

where  $k_B$  is the Boltzmann constant and  $n = \int dv f(v)$  stands for the particle density.  $m$  label the particles' masses, supposed to be identical. It is straightforward to show that the average kinetic energy is equal to  $k_B T/2$ , the factor 1/2 being replaced by 3/2 if one extends the reasoning to the relevant three-dimensional case.

As it is a long-range interacting system, one should take a priori with a lot of care this expression in the canonical ensemble.

- 1(d) Since temperature and energy are closely related, it is customary in plasma physics to express temperatures in units of energy. To fix ideas, for a 1 eV plasma, we intend a temperature of the charged gas of particles of  $T = 1.610^{-19}[\text{J}]/1.3810^{-23}[\text{J/K}] = 11600$  K.
- 1(e) A plasma can have several temperatures at the same time, since electrons and ions have separate Maxwell distributions, each characterized by a specific temperature, resp.  $T_e$  and  $T_i$ . In Nicolas Plihon experiment in the Laboratoire de Physique, for example, electrons are at 3 to 6 eV, while ions are around 0.1 eV.
- 1(f) Collisions drive the equilibration process. The thermalization time among ions *or* electrons themselves is definitely faster than the thermalization time between ions *and* electrons. Then each species can be in its own (thermal) equilibrium while the plasma may not last enough to allow for the two temperatures to equalise.
- 1(g) The force acting along the direction parallel to  $\mathbf{B}$  is different from that which is experienced along the perpendicular direction. Velocities can consequently attain different distribution profiles (resp. along the parallel and perpendicular directions as defined by the vector  $\mathbf{B}$ ), being characterized by different temperature values.

## 2. Debye shielding and quasi-neutrality

**2(a)** This ansatz is motivated by recalling that  $m_i \gg m_e$ , the mass of the electrons: the inertia of the ions is so large that it prevents them from moving on the timescale of the considered experiment.

**2(b)** The electric potential field  $\phi$  is given by the Poisson equation

$$\nabla^2 \phi = -\frac{e(n_i - n_e)}{\varepsilon_0}. \quad (3)$$

The ions density  $n_i$  is constant by virtue of the uniformity condition. In presence of a potential energy  $-e\phi$ , the electron velocity distribution gets modified, so to incorporate the apposite Boltzmann factor, to become

$$f_e(\mathbf{r}, \mathbf{v}) = A' \exp\left(-\frac{\frac{1}{2}mv^2 - e\phi(\mathbf{r})}{k_B T}\right). \quad (4)$$

The physical meaning of the above formula is transparent: there are more electrons where  $\phi$  is positive, while there are fewer electrons where  $\phi$  is negative: in the latter case this is due to the fact that not all electrons have enough energy to get there.

Integration of  $f_e(\mathbf{r}, \mathbf{v})$ , given by Eq. (4), over  $v$  yields

$$n_e = n_i \exp\left(\frac{e\phi}{k_B T}\right). \quad (5)$$

Then, inserting (5) into Eq. (3), one gets

$$\nabla^2 \phi = \frac{en_i}{\varepsilon_0} [\exp(e\phi/k_B T) - 1]. \quad (6)$$

**2(c)** In the region of interest to us,  $|e\phi/k_B T| \ll 1$ , we can then expand the exponential in Taylor series. Keeping only the linear terms, we get

$$\nabla^2 \phi = \frac{e^2 n_i}{\varepsilon_0 k_B T} \phi \quad (7)$$

Defining  $\lambda_D = \sqrt{\varepsilon_0 k_B T / (e^2 n_i)}$ , we get

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = \frac{\phi}{\lambda_D^2}, \quad (8)$$

which can be rewritten introducing  $V = r\phi$  as

$$\frac{d^2 V}{dr^2} = \frac{V}{\lambda_D^2}, \quad (9)$$

which leads to the solution  $V = Ce^{-r/\lambda_D} + C'e^{r/\lambda_D}$ .  $C'$  has to vanish because of the impossibility of divergence in  $+\infty$ . We finally get

$$\phi(r) = \phi_0(r) \exp(-r/\lambda_D), \quad (10)$$

where  $\phi_0(r) = (-e)/(4\pi\epsilon_0 r)$  is the usual potential that one would obtain for the charge in vacuum, and  $\lambda_D$  is the so called Debye length, a quantitative measure of the shielding distance. Notice that as  $n_i$  is increased, the Debye length  $\lambda_D$  decreases, as it should since each spherical layer of plasma contains more electrons. Furthermore,  $\lambda_D$  increases with increasing the temperature.

- 2(d)** If the dimensions of a system  $L$  are much larger than  $\lambda_D$ , then local concentrations of charges get shielded out in a distance that is short when compared to  $L$ , so leaving the bulk of the plasma free from large electric fields. The plasma is hence quasi-neutral, or neutral enough so that it is legitimate to assume  $n_i \simeq n_e = n$ , this latter quantity being termed the plasma density.

### 3. Klimontovich and Vlasov-Maxwell equations

- 3(a)** The discrete density  $f_d^s(\mathbf{r}, \mathbf{v}, t)$  of  $N_s$  such particles, belonging to species  $s$ , in the six-dimensional phase space  $(\mathbf{r}, \mathbf{v})$  reads

$$f_d^s(\mathbf{r}, \mathbf{v}, t) = \frac{1}{N_s} \sum_{i=1}^{N_s} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)), \quad (11)$$

where  $\delta(\cdot)$  stands for the Dirac delta function.

- 3(b)** The position  $\mathbf{r}_i$  satisfies the trivial equation  $\dot{\mathbf{r}}_i = \mathbf{v}_i$  and the velocity of particles  $i$  obeys the Lorentz force equation

$$m_s \dot{\mathbf{v}}_i = q_s \left[ \mathbf{E}(\mathbf{r}_i, t) + \mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i, t) \right], \quad (12)$$

where  $q_s$  is the charge of the particle, and  $\mathbf{E}$  and  $\mathbf{B}$  are the microscopic electric and magnetic field, respectively. These latter are in turn governed by the Maxwell equations

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\epsilon_0} \quad (13)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (14)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \quad (15)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{J}(\mathbf{r}, t) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t}. \quad (16)$$

The description is completed by the microscopic charge density relation

$$\rho(\mathbf{r}, t) = \sum_s N_s q_s \int d\mathbf{v} f_d^s(\mathbf{r}, \mathbf{v}, t), \quad (17)$$

while the microscopic current is given by

$$\mathbf{J}(\mathbf{r}, t) = \sum_s N_s q_s \int d\mathbf{v} \mathbf{v} f_d^s(\mathbf{r}, \mathbf{v}, t). \quad (18)$$

**3(c)** An exact equation for the evolution of the system of charged particles is obtained by taking the time derivative of the density  $f_d^s(\cdot)$ . This immediately yields

$$\frac{\partial f_d^s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = -\frac{1}{N_s} \sum_{i=1}^{N_s} \dot{\mathbf{r}}_i \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)) - \frac{1}{N_s} \sum_{i=1}^{N_s} \dot{\mathbf{v}}_i \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)),$$

where  $\nabla_{\mathbf{r}} = (\partial_x, \partial_y, \partial_z)$  and  $\nabla_{\mathbf{v}} = (\partial_{v_x}, \partial_{v_y}, \partial_{v_z})$ .

By inserting the equations of motion into Eq. (19), one eventually obtains

$$\begin{aligned} \frac{\partial f_d^s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = & - \frac{1}{N_s} \sum_{i=1}^{N_s} \mathbf{v}_i \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)) \\ & - \frac{1}{N_s} \sum_{i=1}^{N_s} \frac{q_s}{m_s} (\mathbf{E}(\mathbf{r}_i, t) + \mathbf{v}_i \times \mathbf{B}(\mathbf{r}_i, t)) \cdot \nabla_{\mathbf{v}} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)). \end{aligned} \quad (20)$$

Using the properties of the Dirac function  $a \delta(a - b) = b \delta(a - b)$  yields

$$\begin{aligned} \frac{\partial f_d^s(\mathbf{r}, \mathbf{v}, t)}{\partial t} = & - \frac{1}{N_s} (\mathbf{v} \cdot \nabla_{\mathbf{r}} \sum_{i=1}^{N_s} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t)) \\ & - \frac{q_s}{m_s} (\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)) \cdot \nabla_{\mathbf{v}} \sum_{i=1}^{N_s} \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{v} - \mathbf{v}_i(t))), \end{aligned} \quad (21)$$

and recalling the definition of the density function  $f_d^s(\cdot)$  the previous equation can be cast in the form

$$\frac{\partial f_d^s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_d^s(\mathbf{r}, \mathbf{v}, t) + \frac{q_s}{m_s} (\mathbf{E}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{r}, t)) \cdot \nabla_{\mathbf{v}} f_d^s(\mathbf{r}, \mathbf{v}, t) = 0. \quad (22)$$

This is the *Klimontovich equation* which, together with the Maxwell equations, provides an exact and general description of any electromagnetic wave-particles system in mutual, self-consistent evolution.

**3(d)** The above equation needs to be complemented by proper initial conditions. By assigning initial particles position and velocity, one can clearly reconstruct the associated densities  $f_d^s(\mathbf{r}, \mathbf{v}, t = 0)$ . The initial fields are instead chosen to be consistent with the aforementioned Maxwell equations. The system is hence completely deterministic and both density and fields can be in principle traced as a function of time and in any position of the generalized phase space.

However the amount of information embedded in the Klimontovich description is enormous. Equation (22) explicitly contains in fact the orbits of each individual microscopic entity belonging to the system under inspection.

**3(e)** When looking at average quantities, one is primarily interested in knowing how many particles of type  $s$  can be found in a small volume  $(\Delta \mathbf{r}, \Delta \mathbf{v})$  of phase space, positioned in  $(\mathbf{r}, \mathbf{v})$ . In other terms, it is tempting to invoke an appropriate ensemble average  $\langle \cdot \rangle$ . As explained in the lecture, this is an average over realizations of the system prepared according to assigned and reproducible prescriptions. Therefore, we focus the attention on the smooth function

$$f_0^s(\mathbf{r}, \mathbf{v}, t) = \langle f_d^s(\mathbf{r}, \mathbf{v}, t) \rangle. \quad (23)$$

An equation for the time evolution of the distribution function  $f_0^s(\mathbf{r}, \mathbf{v}, t)$  can be recovered from the Klimontovich Eq. (22) by this ensemble averaging. To this end we define the quantities  $\delta f^s$ ,  $\delta E$  and  $\delta B$  as obeying the following relations

$$f_d^s(\mathbf{r}, \mathbf{v}, t) = f_0^s(\mathbf{r}, \mathbf{v}, t) + \frac{1}{\sqrt{N}} \delta f^s(\mathbf{r}, \mathbf{v}, t), \quad (24)$$

$$\mathbf{E}(\mathbf{r}, \mathbf{v}, t) = \mathbf{E}_0(\mathbf{r}, \mathbf{v}, t) + \frac{1}{\sqrt{N}} \delta \mathbf{E}(\mathbf{r}, \mathbf{v}, t), \quad (25)$$

$$\mathbf{B}(\mathbf{r}, \mathbf{v}, t) = \mathbf{B}_0(\mathbf{r}, \mathbf{v}, t) + \frac{1}{\sqrt{N}} \delta \mathbf{B}(\mathbf{r}, \mathbf{v}, t), \quad (26)$$

with  $N = \sum_s N_s$ . The index 0 labels the averaged quantities, namely  $\mathbf{E}_0 = \langle \mathbf{E} \rangle$  and  $\mathbf{B}_0 = \langle \mathbf{B} \rangle$ , and the factor  $1/\sqrt{N}$  takes into account the typical size of relative fluctuations.

**3(f)** Inserting this definition in Eq. (22) and performing the ensemble averaging, we get

$$\begin{aligned} \frac{\partial f_0^s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_0^s(\mathbf{r}, \mathbf{v}, t) + \frac{q_s}{m_s} (\mathbf{E}_0(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}_0(\mathbf{r}, t)) \cdot \nabla_{\mathbf{v}} f_0^s(\mathbf{r}, \mathbf{v}, t) \\ = -\frac{1}{N} \frac{q_s}{m_s} \langle (\delta \mathbf{E} + \mathbf{v} \times \delta \mathbf{B}) \cdot \nabla_{\mathbf{v}} \delta f^s(\mathbf{r}, \mathbf{v}, t) \rangle. \end{aligned} \quad (27)$$

The right-hand side of Eq. (27) is sensitive to the discrete nature of the fluid, while the left-hand side deals with collective variables. In the limit of large systems ( $N \rightarrow \infty$ ), one can formally neglect the right-hand side and consequently obtain the Vlasov equation

$$\frac{\partial f_0^s(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_0^s(\mathbf{r}, \mathbf{v}, t) + \frac{q_s}{m_s} (\mathbf{E}_0(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}_0(\mathbf{r}, t)) \cdot \nabla_{\mathbf{v}} f_0^s(\mathbf{r}, \mathbf{v}, t) = 0. \quad (28)$$

The ensemble averaged fields  $\mathbf{E}_0$  and  $\mathbf{B}_0$  must satisfy the ensemble averaged Maxwell equations, which we give here

$$\nabla \cdot \mathbf{E}_0(\mathbf{r}, t) = \frac{\rho_0}{\varepsilon_0}, \quad (29)$$

$$\nabla \cdot \mathbf{B}_0(\mathbf{r}, t) = 0, \quad (30)$$

$$\nabla \times \mathbf{E}_0(\mathbf{r}, t) = -\frac{\partial \mathbf{B}_0(\mathbf{r}, t)}{\partial t}, \quad (31)$$

$$\nabla \times \mathbf{B}_0(\mathbf{r}, t) = \mu_0 \mathbf{J}_0(\mathbf{r}, t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}_0(\mathbf{r}, t)}{\partial t}, \quad (32)$$

with

$$\rho_0 = \sum_s N_s q_s \int d\mathbf{v} f_0^s(\mathbf{r}, \mathbf{v}, t), \quad (33)$$

$$\mathbf{J}(\mathbf{r}, t) = \sum_s N_s q_s \int d\mathbf{v} \mathbf{v} f_0^s(\mathbf{r}, \mathbf{v}, t). \quad (34)$$

## 4. Bernstein-Greene-Kruskal modes

**4(a)** Introducing the convective derivative  $D/Dt$ , the Vlasov equation can be rewritten as

$$\left( \frac{\partial}{\partial t} + v \cdot \nabla_x + \dot{v} \cdot \nabla_v \right) f_0^s(x, v, t) = \frac{D f_0^s(x, v, t)}{\partial t} = 0. \quad (35)$$



It is then straightforward to note that a particle distribution function  $f_0^s(\cdot)$  for the species  $s$  which can be expressed as a function of selected constants of motions, here termed  $C_j(\mathbf{x}, \mathbf{v}, t)$ , is a stationary solution of the Vlasov equation since

$$\frac{D}{Dt} f_0^s(\dots, C_j(\mathbf{x}, \mathbf{v}, t), \dots) = \sum_j \frac{\partial f_0^s}{\partial C_j} \frac{D C_j}{Dt} = 0. \quad (36)$$

The energy  $m_s v_x^2/2 + q_s \phi(x)$  being a constant of motion, any function  $f_0^s(v_x^2 + 2q_s \phi(x)/m_s)$  is a stationary Vlasov solution.

**4(b)** The Poisson equation reads

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e(n_e - n_i)}{\varepsilon_0} = \frac{e}{\varepsilon_0} \int_{-\infty}^{+\infty} dv [f_0^e(v^2 - 2e\phi(x)/m_e) - f_0^i(v^2 + 2e\phi(x)/m_i)]. \quad (37)$$

Solving this integro-differential equation for  $\phi(x)$  subject to proper boundary conditions return a closed solution of the inspected problem.

**4(c)** Selecting only the positive root inside the delta function, one gets  $f_0^e(x, v) = n_0 \frac{v_e}{v} \delta(v - \tilde{v}_e)$  where  $\tilde{v}_e = \sqrt{v_e^2 + 2e\phi(x)/m_e}$ , and analogously  $f_0^i(x, v) = n_0 \frac{v_i}{v} \delta(v - \tilde{v}_i)$  with  $\tilde{v}_i = \sqrt{v_i^2 - 2e\phi(x)/m_i}$ . Here  $v_e$  and  $v_i$  are positive constants that we choose large enough so that  $\tilde{v}_e$  and  $\tilde{v}_i$  are real positive quantities. Notice that we have chosen the same normalization constant  $n_0$  both in the definition of  $f_0^e(x, v)$  and  $f_0^i(x, v)$ , an assumption that implies dealing with an overall neutral plasma.

The Poisson equation can therefore be written as

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} \left[ n_0 \frac{v_e}{\tilde{v}_e} - n_0 \frac{v_i}{\tilde{v}_i} \right]. \quad (38)$$

**4(d)** The pseudo-potential is defined by

$$V(\phi) = -\frac{n_0}{\varepsilon_0} \left[ m_e v_e^2 \left( 1 + \frac{2e\phi(x)}{m_e v_e^2} \right)^{1/2} + m_i v_i^2 \left( 1 - \frac{2e\phi(x)}{m_i v_i^2} \right)^{1/2} \right], \quad (39)$$

where the dependence on  $\tilde{v}_e$  (resp.  $\tilde{v}_i$ ) has been removed in favour of  $v_e$  (resp.  $v_i$ ).

**4(e)** For  $m_e v_e^2 = m_i v_i^2 \equiv k_B T$ , the pseudo-potential becomes

$$V(\phi) = -\frac{n_0}{\varepsilon_0} k_B T \left[ \left( 1 + \frac{2e\phi(x)}{k_B T} \right)^{1/2} + \left( 1 - \frac{2e\phi(x)}{k_B T} \right)^{1/2} \right] \quad (40)$$

and is plotted versus  $\phi$  in Fig. 1.

With an initial choice of the *energy* between  $-(2)^{1/2} n_0 k_B T / \varepsilon_0$  and  $-2 n_0 k_B T / \varepsilon_0$ , the pseudo-particle oscillates indefinitely in the well, resulting in a spatially periodic potential that oscillates between  $-\phi_0$  and  $+\phi_0$ .

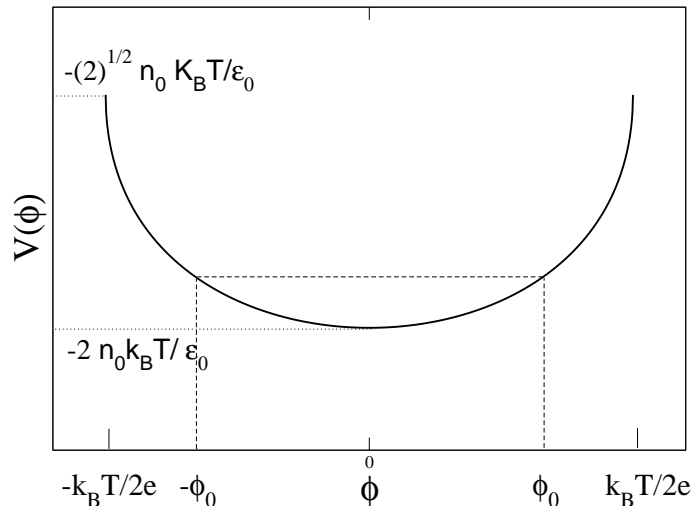


Figure 1: The pseudo-potential (40) which arises when studying the BGK modes, under the simplifying hypothesis  $m_e v_e^2 = m_i v_i^2 \equiv k_B T$ .

**4(f)** If  $\phi_0 \ll k_B T / (2e)$ , one can expand the square roots in Eq. (40) to get for Eq. (38)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{2n_0 e^2}{\varepsilon_0 k_B T} \phi = 0 \quad (41)$$

whose solution reads

$$\phi(x) = \phi_0 \sin(\sqrt{2} x / \lambda_D) \quad (42)$$

where  $\lambda_D = \sqrt{\varepsilon_0 k_B T / n_0 e^2}$  is the Debye length. Notice that in the linearized version (42) the function  $\phi(x)$  is sinusoidal, while this is not the case in the general setting considered above.

**4(g)** In practice, we have here proved that, within a self-consistent picture where particles influence the fields and vice-versa, there is a spatially periodic potential. The ion beam gets accelerated through regions of large negative potential, and thus reduces its density. Conversely the electrons are decelerated when they meet a region of negative potential. The net result is clearly a negative net charge in regions of large negative potential. The opposite holds in regions where the potential is instead positive.

The important point is that this process works not only in the linear regime (see Eq. (42)) but also in the non linear one, as governed by Eq. (38).

Interestingly, it turns out that almost any potential  $\phi(x)$  can be constructed by choosing appropriate distributions of (un-)trapped electrons and ions.

There are many practical applications of the BGK modes, including the non linear stage of the Landau damped Langmuir waves, theories of shock waves and double layers and many more domains which fall outside the scope of this book.

# B. Ising model with long and short-range interactions

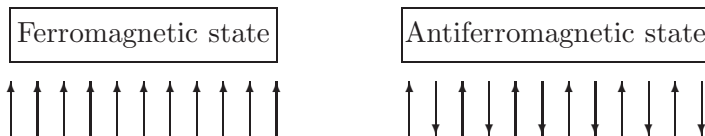
## 5. Introduction

5(a) In this 1D spin chains, the first term has an infinite-range and corresponds to the Curie-Weiss Hamiltonian. This term is responsible for the non additive properties of the model.

The prefactor in front of the first term has been set to one by an appropriate renormalization of the energy. This choice amounts to focus on the ferro coupling, with no loss of generality.

On the contrary, the second term of (1) corresponds to an interaction between nearest neighbours along a one-dimensional lattice with periodic boundary conditions. The coupling constant  $K$  might be either positive or negative, and both cases will be hereafter discussed.

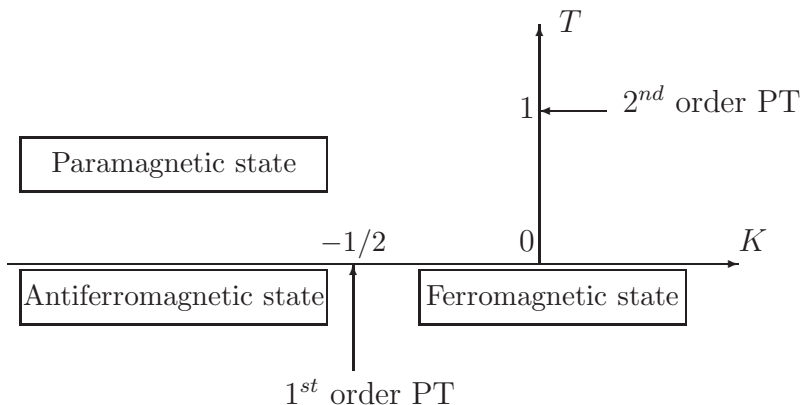
5(b) At  $T = 0$ , one can determine if the model has a phase transition at some value of  $K$  by comparing the energy of the ferromagnetic state with that of the antiferromagnetic one, since only the energy term of the free energy matters.



The ferromagnetic state with all spins up ( $S_i = 1, \forall i$ ) or down ( $S_i = -1, \forall i$ ) has a negative energy  $E_F = -N/2$ . For the antiferromagnetic state with alternate signs of nearest neighbour spins, the first term of (1) gives a vanishing contribution to the energy. The total energy reads therefore  $E_A = KN$ . By imposing  $E_A = E_F$  one gets the phase transition value  $K_t = -1/2$  at which a discontinuity of the order parameter is found, from  $m = 0$  to  $m = 1$ . Therefore we have a *first order* phase transition.

5(c) For non zero temperature, one has to take into account the entropic term of the free energy, which measures disorder. When the coupling constant  $K$  vanishes, one fully recovers the Curie-Weiss Hamiltonian, which exhibits a *second order* phase transition at  $T = 1$  (see lecture).

5(d) One therefore expects that the  $(T, K)$  phase diagram displays a transition line which is first order at low  $T$  and second order at high  $T$ , dividing the  $(T, K)$  plane in different regions as sketched in Fig. 7. The transition line separates a ferromagnetic from a paramagnetic state, although exactly at  $T = 0$  and  $K < -1/2$  the state is antiferromagnetic.



## 6. The solution in the canonical ensemble

6(a) The partition function can be written as follows

$$Z(\beta, N) = \sum_{\{S_1, \dots, S_N\}} e^{-\beta H} = \sum_{\{S_1, \dots, S_N\}} \exp \left[ \frac{\beta}{2N} \left( \sum_{i=1}^N S_i \right)^2 + \frac{\beta K}{2} \sum_{i=1}^N (S_i S_{i+1} - 1) \right]. \quad (43)$$

6(b) To get rid of the quadratic term, we use the Hubbard-Stratonovich transformation

$$\exp \left[ \frac{\beta}{2N} \left( \sum_{i=1}^N S_i \right)^2 \right] = \sqrt{\frac{\beta N}{2\pi}} \int_{-\infty}^{+\infty} dx \exp \left[ -\frac{\beta N}{2} x^2 + \beta x \sum_{i=1}^N S_i \right], \quad (44)$$

so that the partition function (43) can be rewritten as

$$Z(\beta, N) = \sqrt{\frac{\beta N}{2\pi}} \int_{-\infty}^{+\infty} dx e^{-\frac{\beta N}{2} x^2} \sum_{\{S_1, \dots, S_N\}} \left[ e^{\beta x \sum_{i=1}^N S_i + \frac{\beta K}{2} \sum_{i=1}^N (S_i S_{i+1} - 1)} \right] \quad (45)$$

$$= \sqrt{\frac{\beta N}{2\pi}} \int_{-\infty}^{+\infty} dx e^{-N\beta \tilde{f}(\beta, x)}, \quad (46)$$

if one introduces the free energy-like function  $\tilde{f}(\beta, x)$  as

$$\tilde{f}(\beta, x) = \frac{1}{2} x^2 + f_0(\beta, x), \quad (47)$$

where  $f_0(\beta, x)$  is the free energy of the nearest-neighbour Ising model with an external field  $x$ .

6(c) One has to use the transfer integral method:  $\lambda_{\pm}$  are the two eigenvalues of the transfer matrix.

6(d) As  $\lambda_+$  is greater than  $\lambda_-$  for all values of  $x$ , only the larger eigenvalue  $\lambda_+$  is relevant in the limit  $N \rightarrow \infty$ . One thus finally gets the following expression

$$\tilde{\phi}(\beta, x) = \beta \tilde{f}(\beta, x) = \frac{\beta}{2} x^2 - \ln \left[ e^{\beta K/2} \cosh(\beta x) + \sqrt{e^{\beta K} \sinh^2(\beta x) + e^{-\beta K}} \right] \quad (48)$$

which is plotted in Figs. 2 for different values of the nearest-neighbour coupling  $K$ .

In the large  $N$ -limit, the application of the saddle point method to Eq. (46) finally leads to the free energy  $f(\beta)$ , which is obtained by taking the value of  $x$  which minimizes  $\tilde{\phi}(\beta, x)$  in formula (48).

From the knowledge of the free energy, as anticipated, one gets either a second (see left panel of Fig. 2) or a first order (see right panel of Fig. 2) phase transition depending on the value of the coupling constant  $K$ .

6(e) As usual, the expansion of  $\tilde{f}(\beta, x)$  in power of  $x$  is the appropriate procedure to define the critical lines and points. One gets here

$$\tilde{f}(\beta, x) = -\ln 2 \cosh \frac{\beta K}{2} + \frac{\beta}{2} x^2 (1 - \beta e^{\beta K}) + \frac{\beta^4}{24} e^{\beta K} (3e^{2\beta K} - 1) x^4 + O(x^6). \quad (49)$$

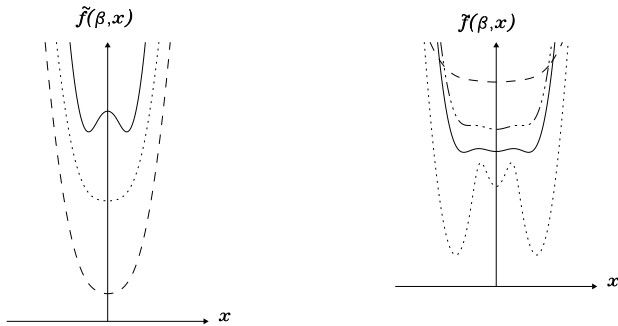


Figure 2: Evolution of  $\tilde{f}(\beta, x)$  for different values of the inverse temperature. Left panel: *Second* order phase transition:  $\beta = 1.1$  (dashed line), 1.4 (dotted), 2.5 (solid) when  $K = -0.25$ . Right panel: *First* order phase transition:  $\beta = 10$  (dotted), 2.4 (solid), 2.35 (dash-triple dot), 2 (dashed) when  $K = -0.4$ . Note that the different curves have been vertically shifted for readability purposes.

The critical point of the second order transition is obtained for each  $K$  by computing the value  $\beta_c$  at which the quadratic term of the expansion (49) vanishes provided the coefficient of the fourth order term is positive: we get therefore  $\beta_c = \exp(-\beta_c K)$ .

When also the fourth order coefficient vanishes, which means when  $3 \exp(2\beta K) = 1$ , one gets the canonical tricritical point (CTP). This condition corresponds to particular values of the local coupling constant  $K$  and of the inverse temperature  $\beta$

$$K_{CTP} = -\frac{\ln 3}{2\sqrt{3}} \simeq -0.317 \quad \text{with} \quad \beta_{CTP} = \sqrt{3}, \quad (50)$$

in which  $\beta_{CTP}$  corresponds to the canonical tricritical inverse temperature.

**6(f)** One gets finally the first order line, numerically, by requiring that  $f(\beta, 0) = f(\beta, x^*)$ , where  $x^*$  is one of the two symmetric local minima. Figure 3 represents the phase diagram in both the canonical and the microcanonical ensemble. The features of this phase diagram are very close to those of the BEG model.

**6(g)** The phase diagram in the canonical ensemble is plotted in Fig. 3.

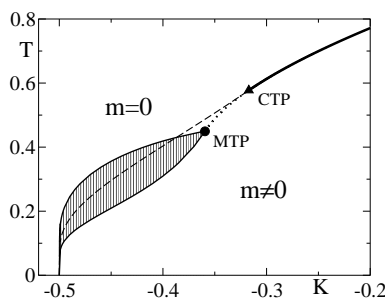


Figure 3: The canonical and microcanonical  $(K, T)$  phase diagram. In the canonical ensemble, the large  $K$  transition is continuous (bold solid line) down to the tricritical point CTP where it becomes first order (dashed line). In the microcanonical ensemble, the continuous transition coincides with the canonical one at large  $K$  (bold line). It persists at lower  $K$  (dotted line) down to the tricritical point MTP where it turns first order, with a branching of the transition line (solid lines). The shaded area is not accessible in the microcanonical ensemble.

## 7. The solution in the microcanonical ensemble

7(a) In the microcanonical ensemble, one can proceed with a simple counting method. The first term of the Hamiltonian (1) can be straightforwardly rewritten as  $-M^2/(2N)$ . Concerning the second term of the Hamiltonian, since two identical neighbouring spins do not contribute to it ( $S_i S_{i+1} - 1$  being equal to zero), while two different adjacent spins give a contribution equal to  $K$ , the total contribution associated with the second term is  $K\mathcal{U}$ . Therefore, the total energy can be written as  $E = -M^2/(2N) + K\mathcal{U}$ .

7(b) The number of microstates in a chain of  $N$  spins corresponding to an energy  $E$  and a total magnetization  $M$  can be written as

$$\Omega(E, M; N) = \Omega(N_+, N_-, \mathcal{U}) \simeq \binom{N_+}{\mathcal{U}/2} \binom{N_-}{\mathcal{U}/2}, \quad (51)$$

where  $N_+$ ,  $N_-$  and  $\mathcal{U}$  are related to  $N = N_+ + N_-$ ,  $E$  and  $M$  by the above relations. The formula is derived by taking into account that we have to distribute  $N_+$  spins among  $\mathcal{U}/2$  groups and  $N_-$  among the remaining  $\mathcal{U}/2$ . Each of these distributions gives a binomial term, and, since they are independent, the total number of states is the product of the two binomials. The expression is not exact because we are on a ring, but corrections are however of order  $N$  and do not affect the entropy. A slight correction to formula (51) is present for small  $N_+$ ,  $N_-$  and  $\mathcal{U}/2$ , and all these numbers should be indeed reduced by a unity.

7(c) Introducing  $m = M/N$ ,  $u = \mathcal{U}/N$  and  $\varepsilon = E/N = -m^2/2 + Ku$ , one gets the entropy

$$\begin{aligned} \tilde{s}(\varepsilon, m) &= \frac{1}{N} \ln \Omega = \frac{1}{2}(1+m) \ln(1+m) + \frac{1}{2}(1-m) \ln(1-m) - u \ln u \\ &\quad - \frac{1}{2}(1+m-u) \ln(1+m-u) - \frac{1}{2}(1-m-u) \ln(1-m-u), \end{aligned} \quad (52)$$

where  $u$  can be expressed as a function of  $\varepsilon$  and  $m$ .

As shown in Fig. 4 for different values of the energy  $\varepsilon$ , the shape of this function strongly depends on the value of the nearest-neighbour coupling  $K$ .

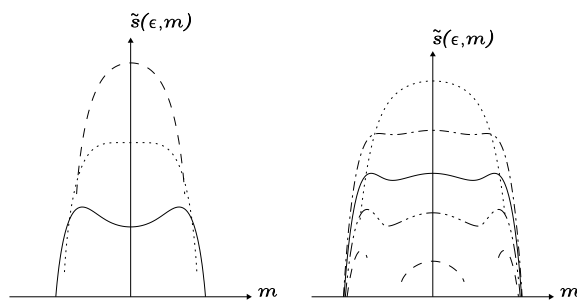


Figure 4: Evolution of the entropy  $\tilde{s}(\varepsilon, m)$  for different values of the energy. Right panel: *Second* order phase transition.  $\varepsilon = -0.1$  (dashed line),  $-0.15$  (dotted),  $-0.2$  (solid) when  $K = -0.25$ . Left Panel: *First* order phase transition.  $\varepsilon = -0.25$  (dotted),  $-0.305$  (dash-dotted),  $-0.3138$  (solid),  $-0.32$  (dash-triple dot),  $-0.33$  (dashed) when  $K = -0.4$ . The gaps present in the lower dashed curve are related to ergodicity breaking. Note that the different curves have been vertically shifted for readability purposes.

7(d) In the large  $N$ -limit, the last step is to maximize entropy  $\tilde{s}(\varepsilon, m)$  with respect to magnetization  $m$ , leading to the entropy  $s(\varepsilon) = \tilde{s}(\varepsilon, m^*)$ , where  $m^*$  is the equilibrium value. As anticipated, one gets either a second or a first order phase transition depending on the value of the coupling constant  $K$ .

7(e) As usual, and analogously to what has been done in the canonical ensemble, the expansion of  $\tilde{s}(\varepsilon, m)$  in powers of  $m$  is the appropriate procedure to define the critical lines and points. One gets here

$$\tilde{s}(\varepsilon, m) = s_0(\varepsilon) + A_{mc} m^2 + B_{mc} m^4 + O(m^4), \quad (53)$$

with the paramagnetic zero magnetization entropy

$$s_0(\varepsilon) = -\frac{\varepsilon}{K} \ln \frac{\varepsilon}{K} - \left(1 - \frac{\varepsilon}{K}\right) \ln \left(1 - \frac{\varepsilon}{K}\right) \quad (54)$$

and the expansion coefficients

$$A_{mc} = \frac{1}{2} \left[ \frac{1}{K} \ln \frac{K - \varepsilon}{\varepsilon} - \frac{\varepsilon}{K - \varepsilon} \right] \quad (55)$$

$$B_{mc} = \frac{\varepsilon^3}{12(\varepsilon - K)^3} - \frac{K^2 + K}{4(\varepsilon - K)^2} + \frac{1}{8K\varepsilon}. \quad (56)$$

Using these expressions, it is straightforward to find the second order phase transition line by requiring that  $A_{mc} = 0$  ( $B_{mc} < 0$ ), finding  $\beta_c = \exp(-\beta_c K)$ , which is the same equation found for the canonical ensemble. Again, as far as second order phase transitions are concerned, the two ensembles are equivalent. The tricritical point is obtained by the condition  $A_{mc} = B_{mc} = 0$ , which leads to

$$K_{MTP} \simeq -0.359 \quad \text{and} \quad \beta_{MTP} \simeq 2.21, \quad (57)$$

which are close, but definitely different, from  $K_{CTP} \simeq -0.317$  and  $\beta_{CTP} = \sqrt{3}$  found in Eq. (50).

The microcanonical first order phase transition line is obtained numerically by equating the entropies of the ferromagnetic and paramagnetic phases. At a given transition energy, there are two temperatures, thus giving a *temperature jump* as for the BEG and the generalized HMF model. In conclusion, similarly to the mean-field models discussed before, this model, that combines short and long-range interactions, exhibits a region of negative specific heat when the phase transition is first order in the canonical ensemble. This striking feature is therefore not restricted to mean-field models.